Forum of Mathematics, Sigma (2021), **Vol. 9**:e37 1–31
doi:10.1017/fms.2021.32

CAMBRIDGE UNIVERSITY PRESS

RESEARCH ARTICLE

Automorphic vector bundles on the stack of *G*-zips

Naoki Imai[1] and Jean-Stefan Koskivirta[2]

[1]Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo, 153-8914, Japan; E-mail: naoki@ms.u-tokyo.ac.jp.
[2]Department of Mathematics, Faculty of Science, Saitama University, 255 Shimo-Okubo, Sakura-ku, Saitama City, Saitama 338-8570, Japan; E-mail: jeanstefan.koskivirta@gmail.com.

Received: 9 September 2020; **Accepted:** 1 March 2021

2020 Mathematics Subject Classification: *Primary* – 14G35; *Secondary* – 20G40

Abstract
For a connected reductive group G over a finite field, we study automorphic vector bundles on the stack of G-zips. In particular, we give a formula in the general case for the space of global sections of an automorphic vector bundle in terms of the Brylinski-Kostant filtration. Moreover, we give an equivalence of categories between the category of automorphic vector bundles on the stack of G-zips and a category of admissible modules with actions of a 0-dimensional algebraic subgroup a Levi subgroup and monodromy operators.

1. Introduction

The stack of G-zips was introduced by Pink-Wedhorn-Ziegler [PWZ11] and [PWZ15] based on the notion of F-zip defined in the work of Moonen-Wedhorn ([MW04]). In this article, we investigate vector bundles on the stack of G-zips. Let G be a connected reductive group over a finite field $\mathbb{F}_q$ and let k denote an algebraic closure of $\mathbb{F}_q$. For a cocharacter $\mu\colon \mathbb{G}_{\mathrm{m},k} \to G_k$, Pink-Wedhorn-Ziegler have defined a smooth finite stack $G\text{-}\mathtt{Zip}^{\mu}$ over k, called the stack of G-zips of type μ. Many authors have shown that it is a useful tool to study the geometry of Shimura varieties in characteristic p. For example, let $\mathrm{Sh}(\mathbf{G},\mathbf{X})_K$ be a Shimura variety of Hodge type over a number field $\mathbf{E}$ with good reduction at a prime p. Kisin [Kis10] and Vasiu [Vas99] have constructed an integral model $\mathcal{S}_K$ over $\mathcal{O}_{\mathbf{E}_v}$ at all places $v|p$ in $\mathbf{E}$. Denote by S_K the geometric special fibre of $\mathcal{S}_K$ and by G the special fibre over $\mathbb{F}_p$ of $\mathbf{G}$ (in the context of Shimura varieties, we take $q = p$). Let μ be the cocharacter attached naturally to $\mathbf{X}$. Then Zhang [Zha18] has shown that there exists a smooth morphism of stacks $\zeta\colon S_K \to G\text{-}\mathtt{Zip}^{\mu}$, which is also surjective. The second author and Wedhorn have used the stack $G\text{-}\mathtt{Zip}^{\mu}$ to construct μ-ordinary Hasse invariants in [KW18], and this result was later generalised to all Ekedahl-Oort strata with Goldring [GK19a].

In [Kos19], the second author studied the space of global sections of the family of vector bundles $(\mathcal{V}_I(\lambda))_{\lambda\in X^*(T)}$. To explain what these vector bundles are, first recall that the cocharacter μ yields a parabolic subgroup $P \subset G_k$ as well as a Levi subgroup $L \subset P$, which is equal to the centraliser of μ (see Subsection 2.2.2 for details). Then for any algebraic P-representation (V,ρ) over k, there is a naturally attached vector bundle $\mathcal{V}(\rho)$ of rank $\dim(V)$ on $G\text{-}\mathtt{Zip}^{\mu}$ modelled on (V,ρ) (see Subsection 2.4). We call $\mathcal{V}(\rho)$ an automorphic vector bundle on $G\text{-}\mathtt{Zip}^{\mu}$ (cf. [Mil90, Chapter III, §2]).

The vector bundle $\mathcal{V}_I(\lambda)$ (for $\lambda \in X^*(T)$ a character of a maximal torus $T \subset G$) is by definition the vector bundle attached to the P-representation $V_I(\lambda) = \mathrm{Ind}_B^P(\lambda)$, where $B \subset P$ is a Borel subgroup (containing T and appropriately chosen), Ind denotes induction and I denotes the set of simple roots of L. For a k-algebraic group H, we write $\mathrm{Rep}(H)$ for the category of finite-dimensional algebraic representations of H over k. The natural projection $P \to L$ modulo the unipotent radical induces a fully faithful functor $\mathrm{Rep}(L) \to \mathrm{Rep}(P)$. In particular, all representations of the form $V_I(\lambda)$ lie in the full subcategory $\mathrm{Rep}(L)$. In the case when G is split over $\mathbb{F}_p$, we showed in a previous work [Kos19, Theorem 1] that $H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}_I(\lambda))$ can be expressed as

$$H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}_I(\lambda)) = V_I(\lambda)^{L(\mathbb{F}_p)} \cap V_I(\lambda)_{\leq 0}, \tag{1.0.1}$$

where $V_I(\lambda)^{L(\mathbb{F}_p)}$ denotes the $L(\mathbb{F}_p)$-invariant subspace of $V_I(\lambda)$ and $V_I(\lambda)_{\leq 0} \subset V_I(\lambda)$ is defined as follows: It is the direct sum of the T-weight spaces $V_I(\lambda)_\nu$ for the weights ν satisfying $\langle \nu, \alpha^\vee \rangle \leq 0$ for any simple root α outside of L.

In this article, we vastly generalise the formula (1.0.1) to the most general case. We do not assume that G is split over $\mathbb{F}_q$ and, more important, we consider arbitrary representations in the larger category $\mathrm{Rep}(P)$ as opposed to the subcategory $\mathrm{Rep}(L)$. In the context of Shimura varieties, there are many interesting vector bundles other than the family $(V_I(\lambda))_\lambda$, which may not always arise from representations in $\mathrm{Rep}(L)$. For example, in [Urb14], nearly holomorphic modular forms of weight k and order $\leq r$ are defined as sections of the vector bundle $\omega^{\otimes(k-r)} \otimes \mathrm{Sym}^r(\mathcal{H}^1_{\mathrm{dR}})$ on the modular curve $X(N)$ for some level $N \geq 1$. Here, $\mathcal{H}^1_{\mathrm{dR}}$ is the sheaf of relative de Rham cohomology of the universal elliptic curve $\mathscr{E} \to X(N)$, and $0 \subset \omega \subset \mathcal{H}^1_{\mathrm{dR}}$ is the usual Hodge filtration. In this context, the group G is GL_2, $P = B$ is a Borel subgroup of G. The vector bundle $\mathcal{H}^1_{\mathrm{dR}}$ is attached to the dual of the standard representation of GL_2 (viewed by restriction as a representation of P). Similarly, $\mathrm{Sym}^r(\mathcal{H}^1_{\mathrm{dR}})$ is attached to the rth symmetric power of that representation. More generally, on the Siegel-type Shimura variety $\mathscr{A}_g$ (which parametrise principally polarised abelian varieties of rank g), the universal abelian scheme yields a rank $2g$ vector bundle $\mathcal{H}^1_{\mathrm{dR}}$ on $\mathscr{A}_g$. One can extend the definition of $\mathcal{H}^1_{\mathrm{dR}}$ to Hodge-type Shimura varieties after choosing a Siegel embedding. Furthermore, it extends to a vector bundle on the integral model $\mathscr{S}_K$ of Kisin and Vasiu. This example shows that it is desirable to also understand vector bundles that arise from general representations of P. In this article, we determine the space $H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}(\rho))$ for any cocharacter datum (G, μ) (for the definition of cocharacter datum, see Subsection 2.2.2) and for any representation $(V, \rho) \in \mathrm{Rep}(P)$. By Zhang's smooth surjective map $\zeta : S_K \to G\text{-}\mathtt{Zip}^\mu$, this determines a natural Hecke-equivariant subspace

$$H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}(\rho)) \xhookrightarrow[\zeta^*]{} H^0(S_K, \mathcal{V}(\rho)). \tag{1.0.2}$$

In particular, we obtain Hecke-equivariant sections of $\mathcal{V}(\rho)$ on S_K. Furthermore, we can potentially study sections on Ekedahl-Oort strata by the same method, as demonstrated in [GK19a]. Another motivation for describing sections on $G\text{-}\mathtt{Zip}^\mu$ is that we would like to determine which weights λ admit nonzero automorphic forms. Specifically, let C_K denote the set of $\lambda \in X^*(T)$ such that $H^0(S_K, \mathcal{V}_I(\lambda)) \neq 0$. Similarly, let C_{zip} be the set of λ such that $H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}_I(\lambda)) \neq 0$ (one can show that they are cones in $X^*(T)$). The inclusion (1.0.2) shows that $C_{\mathrm{zip}} \subset C_K$. Denote by $(-)_{\mathbb{Q}_{>0}}$ the generated $\mathbb{Q}_{>0}$-cones. Then one can see [Kos19, Corollary 1.5.3] that $C_{K,\mathbb{Q}_{>0}}$ is independent of K, and we conjecture [GK18, Conjecture 2.1.6] that it coincides with $C_{\mathrm{zip},\mathbb{Q}_{>0}}$. Goldring and the second author proved this conjecture in some case in [GK18, Theorem D].

We show that the space $H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}(\rho))$ is given by the intersection of the L_φ-invariants of V with a generalised Brylinski-Kostant filtration (where $L_\varphi \subset L$ is a certain 0-dimensional group, see (3.2.1)). For the general statement, see Theorem 3.4.1. For the sake of brevity, we give a simplified statement in this introduction. Assume here that P is defined over $\mathbb{F}_q$ (in this case, $L_\varphi = L(\mathbb{F}_q)$). Let $\wp^* \colon X^*(T)_{\mathbb{R}} \to X^*(T)_{\mathbb{R}}$ be the map induced by the Lang torsor $\wp \colon T \to T$; $g \mapsto g\varphi(g)^{-1}$, where $\varphi \colon G \to G$ denotes the qth power Frobenius homomorphism. Let $V = \bigoplus_\nu V_\nu$ be the weight decomposition of V. For $\chi \in X^*(T)_{\mathbb{R}}$, let $\mathrm{Fil}^P_\chi V_\nu$ be the Brylinski-Kostant filtration of V_ν (see (3.4.2)).

https://doi.org/10.1017/fms.2021.32 Published online by Cambridge University Press

Theorem 1 (Corollary 3.4.2). Assume that P is defined over $\mathbb{F}_q$. For any $(V, \rho) \in \mathrm{Rep}(P)$, we have

$$H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}(\rho)) = V^{L(\mathbb{F}_q)} \cap \bigoplus_{\nu \in X^*(T)} \mathrm{Fil}^P_{\wp^{*-1}(\nu)} V_\nu.$$

In the more simple case of [Kos19], the space $V_I(\lambda)_{\leq 0}$ appearing in equation (1.0.1) is a sum of weight spaces of V. In the general case, $H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}(\rho))$ cannot be written as an intersection of $V^{L(\mathbb{F}_q)}$ with a sum of weight spaces of V (see Examples 4.3.2 for a counterexample). We include examples of concrete computations of the space $H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}(\rho))$ in Section 6.

Our second result concerns the category $\mathfrak{VB}(G\text{-}\mathtt{Zip}^\mu)$ of vector bundles on $G\text{-}\mathtt{Zip}^\mu$. As explained above, there is a natural functor $\mathcal{V}\colon \mathrm{Rep}(P) \to \mathfrak{VB}(G\text{-}\mathtt{Zip}^\mu)$. Denote by $\mathfrak{VB}_P(G\text{-}\mathtt{Zip}^\mu)$ the full subcategory that is equal to the essential image of $\mathcal{V}$. We give an explicit description of the category $\mathfrak{VB}_P(G\text{-}\mathtt{Zip}^\mu)$ of automorphic vector bundles. We define the category of L_φ-modules with Δ^P-monodromy (see Definition 5.2.2). Its objects are L_φ-modules W endowed with a set of monodromy operators indexed by Δ^P (where Δ^P denotes the set of simple roots outside the parabolic P). There is a natural functor $F_{\mathrm{MN}}\colon \mathrm{Rep}(P) \to L_\varphi\text{-}\mathrm{MN}_{\Delta^P}$ (see (5.2.1)). An L_φ-module with Δ^P-monodromy is called admissible if it lies in the essential image of F_{MN}. The category of admissible L_φ-modules Δ^P-monodromy is denoted by $L_\varphi\text{-}\mathrm{MN}^{\mathrm{adm}}_{\Delta^P}$.

Theorem 2 (Theorem 5.1.5). The functor $\mathcal{V}\colon \mathrm{Rep}(P) \to \mathfrak{VB}(G\text{-}\mathtt{Zip}^\mu)$ factors through the functor $F_{\mathrm{MN}}\colon \mathrm{Rep}(P) \to L_\varphi\text{-}\mathrm{MN}^{\mathrm{adm}}_{\Delta^P}$ and induces an equivalence of categories

$$L_\varphi\text{-}\mathrm{MN}^{\mathrm{adm}}_{\Delta^P} \longrightarrow \mathfrak{VB}_P(G\text{-}\mathtt{Zip}^\mu).$$

In particular, we deduce the following. Let S_K denote again the good reduction special fibre of a Hodge-type Shimura variety. Similarly, there is a natural functor $\mathrm{Rep}(P) \to \mathfrak{VB}(S_K)$, where $\mathfrak{VB}(S_K)$ denotes the category of vector bundles on S_K. Write again $\mathfrak{VB}_P(S_K)$ for the essential image of $\mathrm{Rep}(P)$. In this context, we have the following.

Corollary 3 (Corollary 5.1.6). The functor $\mathcal{V}\colon \mathrm{Rep}(P) \to \mathfrak{VB}_P(S_K)$ factors as

$$\mathrm{Rep}(P) \xrightarrow{F_{\mathrm{MN}}} L_\varphi\text{-}\mathrm{MN}^{\mathrm{adm}}_{\Delta^P} \xrightarrow{\zeta^*} \mathfrak{VB}_P(S_K).$$

The results of this article will be used in the follow-up articles [IK21] and [GIK21], where we study partial Hasse invariants for Shimura varieties of Hodge type.

2. Vector bundles on the stack of G-zips

2.1. *Notation*

Throughout the article, p is a prime number, q is a power of p and $\mathbb{F}_q$ is the finite field with q elements. We write $k = \overline{\mathbb{F}}_q$ for an algebraic closure of $\mathbb{F}_q$. Write $\sigma \in \mathrm{Gal}(k/\mathbb{F}_q)$ for the qth power Frobenius. For a k-scheme X and $m \in \mathbb{Z}$, we write $X^{(q^m)}$ for the base change of X by σ^m and $\varphi\colon X^{(q^m)} \to X^{(q^{m+1})}$ for the relative qth power Frobenius morphism. For an algebraic representation (V, ρ) of an algebraic group H over k, let $(V^{(q)}, \rho^{(q)})$ denote the representation $\rho \circ \varphi\colon H^{(q^{-1})} \to H \to \mathrm{GL}(V)$.

The notation G will denote a connected reductive group over $\mathbb{F}_q$. We will always write (B, T) for a Borel pair defined over $\mathbb{F}_q$; that is, $T \subset B \subset G_k$ are a maximal torus and a Borel subgroup defined over $\mathbb{F}_q$. Let B^+ be the Borel subgroup of G_k opposite to B with respect to T (i.e., the unique Borel subgroup of G such that $B^+ \cap B = T$). We will use the following notations:

- As usual, $X^*(T)$ (respectively $X_*(T)$) denotes the group of characters (respectively cocharacters) of T. The group $\mathrm{Gal}(k/\mathbb{F}_q)$ acts naturally on these groups. Let $W = W(G_k, T)$ be the Weyl group of

https://doi.org/10.1017/fms.2021.32 Published online by Cambridge University Press

G_k. Similarly, $\mathrm{Gal}(k/\mathbb{F}_q)$ acts on W. Furthermore, the actions of $\mathrm{Gal}(k/\mathbb{F}_q)$ and W on $X^*(T)$ and $X_*(T)$ are compatible in a natural sense.

- $\Phi \subset X^*(T)$: the set of T-roots of G.
- $\Phi_+ \subset \Phi$: the system of positive roots with respect to B^+ (i.e., $\alpha \in \Phi_+$ when the α-root group U_α is contained in B^+). This convention may differ from other authors. We use it to match the conventions of [Jan03, Chapter II, §1.8] and previous publications [GK19a], [Kos19].
- $\Delta \subset \Phi_+$: the set of simple roots.
- For $\alpha \in \Phi$, let $s_\alpha \in W$ be the corresponding reflection. The system $(W, \{s_\alpha\}_{\alpha\in\Delta})$ is a Coxeter system. Write $\ell\colon W \to \mathbb{N}$ for the length function. Hence, $\ell(s_\alpha) = 1$ for all $\alpha \in \Phi$. Let w_0 denote the longest element of W.
- For a subset $K \subset \Delta$, let W_K denote the subgroup of W generated by $\{s_\alpha\}_{\alpha\in K}$. Write $w_{0,K}$ for the longest element in W_K.
- Let ${}^KW$ denote the subset of elements $w \in W$ that have minimal length in the coset $W_K w$. Then ${}^KW$ is a set of representatives of $W_K\backslash W$. The longest element in the set ${}^KW$ is $w_{0,K}w_0$.
- $X^*_+(T)$ denotes the set of dominant characters – that is, characters $\lambda \in X^*(T)$ – such that $\langle \lambda, \alpha^\vee\rangle \geq 0$ for all $\alpha \in \Delta$.
- For a subset $I \subset \Delta$, let $X^*_{+,I}(T)$ denote the set of characters $\lambda \in X^*(T)$ such that $\langle \lambda, \alpha^\vee\rangle \geq 0$ for all $\alpha \in I$. We call them I-dominant characters.

Definition 2.1.1. Let $P \subset G_k$ be a parabolic subgroup containing B and let $L \subset P$ be the unique Levi subgroup of P containing T. Then we define a subset $I_P \subset \Delta$ as the unique subset such that $W(L,T) = W_{I_P}$. For an arbitrary parabolic subgroup $P \subset G_k$ containing T, we put $I_P = I_{P'} \subset \Delta$, where P' is the unique conjugate of P containing B.

- For a parabolic $P \subset G_k$, we put $\Delta^P = \Delta \setminus I_P$.

2.2. The stack of G-zips

In this section, we recall some facts about the stack of G-zips of Pink-Wedhorn-Ziegler.

2.2.1. Zip datum

Let G be a connected reductive group over $\mathbb{F}_q$. In this article, a zip datum is a tuple $\mathcal{Z} = (G, P, L, Q, M, \varphi)$ consisting of the following objects:

(i) $P \subset G_k$ and $Q \subset G_k$ are parabolic subgroups of G_k.
(ii) $L \subset P$ and $M \subset Q$ are Levi subgroups such that $L^{(q)} = M$. In particular, the q-power Frobenius isogeny induces an isogeny $\varphi\colon L \to M$.

If H is an algebraic group, denote by $R_{\mathrm{u}}(H)$ the unipotent radical of H. For $x \in P$, we can write uniquely $x = \bar{x}u$ with $\bar{x} \in L$ and $u \in R_{\mathrm{u}}(P)$. This defines a projection map $\theta^P_L\colon P \to L;\ x \mapsto \bar{x}$. Similarly, we have a projection $\theta^Q_M\colon Q \to M$. The zip group is the subgroup of $P \times Q$ defined by

$$E := \{(x, y) \in P \times Q \mid \varphi(\theta^P_L(x)) = \theta^Q_M(y)\}. \tag{2.2.1}$$

In other words, E is the subgroup of $P \times Q$ generated by $R_{\mathrm{u}}(P) \times R_{\mathrm{u}}(Q)$ and elements of the form $(a, \varphi(a))$ with $a \in L$. Let $G \times G$ act on G_k by $(a, b) \cdot g := agb^{-1}$, and let E act on G by restricting this action to E. The stack of G-zips of type $\mathcal{Z}$ can be defined as the quotient stack

$$G\text{-}\mathtt{Zip}^{\mathcal{Z}} = [E\backslash G_k]\,.$$

Although the above definition of $G\text{-}\mathtt{Zip}^{\mathcal{Z}}$ may be the most concise one, there is a more useful, equivalent definition in terms of torsors: By [PWZ15, §3C and 3D], the stack $G\text{-}\mathtt{Zip}^{\mathcal{Z}}$ is the stack over k such that for all k-scheme S, the groupoid $G\text{-}\mathtt{Zip}(S)$ is the category of tuples $\underline{\mathcal{I}} = (\mathcal{I}, \mathcal{I}_P, \mathcal{I}_Q, \iota)$, where $\mathcal{I}$

is a G_k-torsor over S, $\mathcal{I}_P \subset \mathcal{I}$ and $\mathcal{I}_Q \subset \mathcal{I}$ are a P-subtorsor and a Q-subtorsor of $\mathcal{I}$ respectively and $\iota\colon (\mathcal{I}_P/R_{\mathrm{u}}(P))^{(p)} \to \mathcal{I}_Q/R_{\mathrm{u}}(Q)$ is an isomorphism of M-torsors.

2.2.2. Cocharacter datum

A convenient way to give a zip datum is using cocharacters. A *cocharacter datum* is a pair (G, μ) where G is a reductive connected group over $\mathbb{F}_q$ and $\mu\colon \mathbb{G}_{\mathrm{m},k} \to G_k$ is a cocharacter. There is a natural way to attach to (G, μ) a zip datum $\mathcal{Z}_\mu$, defined as follows. First, denote by $P_+(\mu)$ (respectively $P_-(\mu)$) the unique parabolic subgroup of G_k such that $P_+(\mu)(k)$ (respectively $P_-(\mu)(k)$) consists of the elements $g \in G(k)$ satisfying that the map

$$\mathbb{G}_{\mathrm{m},k} \to G_k;\ t \mapsto \mu(t)g\mu(t)^{-1} \quad (\text{respectively } t \mapsto \mu(t)^{-1}g\mu(t))$$

extends to a morphism of varieties $\mathbb{A}^1_k \to G_k$. This construction yields a pair of parabolics $(P_+(\mu), P_-(\mu))$ in G_k such that the intersection $P_+(\mu) \cap P_-(\mu) = L(\mu)$ is the centraliser of μ. It is a common Levi subgroup of $P_+(\mu)$ and $P_-(\mu)$. Set $P = P_-(\mu)$, $Q = (P_+(\mu))^{(q)}$, $L = L(\mu)$ and $M = (L(\mu))^{(q)}$. Then the tuple $\mathcal{Z}_\mu := (G, P, L, Q, M, \varphi)$ is a zip datum, which we call the zip datum attached to the cocharacter datum (G, μ). We write simply $G\text{-}\mathtt{Zip}^\mu$ for $G\text{-}\mathtt{Zip}^{\mathcal{Z}_\mu}$. For simplicity, we will always consider zip data arising in this way from a cocharacter datum.

2.2.3. Frames

In this article, given a zip datum $\mathcal{Z} = (G, P, L, Q, M, \varphi)$, a frame for $\mathcal{Z}$ is a triple (B, T, z) where (B, T) is a Borel pair of G_k defined over $\mathbb{F}_q$ satisfying the following conditions:

(i) One has the inclusion $B \subset P$.
(ii) $z \in W$ is an element satisfying the conditions

$$^{z}B \subset Q \quad \text{and} \quad B \cap M = {}^{z}B \cap M.$$

Remark 2.2.1. Let (B, T) be a Borel pair defined over $\mathbb{F}_q$ such that $B \subset P$. Then we can find $z \in W$ such that (B, T, z) is a frame. This follows from the proof of [PWZ11, Proposition 3.7].

A frame may not always exist. However, if (G, μ) is a cocharacter datum and $\mathcal{Z}_\mu$ is the associated zip datum (Subsection 2.2.2), then we can find a $G(k)$-conjugate $\mu' = \mathrm{ad}(g)\circ\mu$ (with $g \in G(k)$) such that $\mathcal{Z}_{\mu'}$ admits a frame. This follows easily from Remark 2.2.1 and the fact that G is quasi-split over $\mathbb{F}_q$. Hence, it is harmless to assume that a frame exists, and we will only consider a zip datum that admits a frame.

Remark 2.2.2. If the cocharacter μ is defined over $\mathbb{F}_q$, then so are P and Q. In particular, we have in this case $L = M$ and P, Q are opposite parabolic subgroups with common Levi subgroup L.

For a zip datum (G, P, L, Q, M, φ), we put $I = I_P \subset \Delta$. Note that $\Delta^P = \Delta \setminus I$.

Lemma 2.2.3 ([GK19b, Lemma 2.3.4]). *Let $\mu\colon \mathbb{G}_{m,k} \to G_k$ be a cocharacter, and let $\mathcal{Z}_\mu$ be the attached zip datum. Assume that (B, T) is a Borel pair defined over $\mathbb{F}_q$ such that $B \subset P$. We put $z = \sigma(w_{0,I})w_0$. Then (B, T, z) is a frame for $\mathcal{Z}_\mu$.*

2.2.4. Parametrisation of E-orbits

Recall that the group E from (2.2.1) acts on G_k. We review below the parametrisation of E-orbits following [PWZ11].

Assume that $\mathcal{Z}$ has a frame (B, T, z). For $w \in W$, fix a representative $\dot{w} \in N_G(T)$, such that $(w_1w_2)^{\cdot} = \dot{w}_1\dot{w}_2$ whenever $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$ (this is possible by choosing a Chevalley system, [ABD+66, Exposé XXIII, §6]). For $w \in W$, define G_w as the E-orbit of $\dot{w}\dot{z}^{-1}$. We note that G_w is independent of the choices of $\dot{w}$ and a frame by [PWZ11, Proposition 5.8]. If no confusion occurs, we write w instead of $\dot{w}$. Define a twisted order on ${}^IW$ as follows. For $w, w' \in {}^IW$, write $w' \preccurlyeq w$ if there exists $w_1 \in W_L$ such that $w' \leq w_1 w\sigma(w_1)^{-1}$. This defines a partial order on ${}^IW$ [PWZ11, Corollary 6.3].

https://doi.org/10.1017/fms.2021.32 Published online by Cambridge University Press

Theorem 2.2.4 ([PWZ11, Theorem 6.2, Theorem 7.5]). *The map $w \mapsto G_w$ restricts to a bijection*

$$ {}^I W \longrightarrow \{E\text{-orbits in } G_k\}. \tag{2.2.2}$$

For $w \in {}^I W$, one has $\dim(G_w) = \ell(w) + \dim(P)$. Furthermore, for $w \in {}^I W$, the Zariski closure of G_w is

$$\overline{G}_w = \bigsqcup_{w' \in {}^I W,\ w' \preccurlyeq w} G_{w'}.$$

Each E-orbit is locally closed in G_k. Because E is smooth over k, all E-orbits are also smooth over k. However, the Zariski closure $\overline{G}_w$ of G_w may have highly complicated singularities; see [Kos18] for a description of the normalisation of $\overline{G}_w$. The closure of an E-orbit is a union of E-orbits; hence we obtain a stratification of G.

In particular, there is a unique open E-orbit $U_{\mathcal{Z}} \subset G_k$ corresponding to the longest element $w_{0,I} w_0 \in {}^I W$ via (2.2.2). For an E-orbit G_w (with $w \in {}^I W$), we write $\mathcal{X}_w := [E \backslash G_w]$ for the corresponding locally closed substack of $G\text{-}\mathtt{Zip}^{\mathcal{Z}} = [E \backslash G_k]$.

If $\mathcal{Z}$ arises from a cocharacter datum (Subsection 2.2.2), we write U_μ for $U_{\mathcal{Z}_\mu}$. Using the terminology pertaining to the theory of Shimura varieties, we call U_μ the *μ-ordinary stratum* of $G\text{-}\mathtt{Zip}^\mu$. The corresponding substack $\mathcal{U}_\mu := [E \backslash U_\mu]$ is called the *μ-ordinary locus*. It corresponds to the μ-ordinary locus in the good reduction of Shimura varieties, studied, for example, in [Wor13], [Moo04]. For more details about Shimura varieties, we refer to Subsection 2.5.

2.3. *Reminders about representation theory*

If H is an algebraic group over a field K, denote by $\mathrm{Rep}(H)$ the category of algebraic representations of H on finite-dimensional K-vector spaces. We will denote such a representation by (V, ρ) or sometimes simply ρ or V.

Let H be a split connected reductive K-group and choose a Borel pair (B_H, T) defined over K. Irreducible representations of H are in one-to-one correspondence with dominant characters $X^*_+(T)$. This bijection is given by the highest weight of a representation. For $\lambda \in X^*_+(T)$, let $\mathcal{L}_\lambda$ be the line bundle attached to λ on the flag variety H/B_H by the usual associated sheaf construction [Jan03, §5.8]. Define an H-representation $V_H(\lambda)$ by

$$V_H(\lambda) := H^0(H/B_H, \mathcal{L}_\lambda). \tag{2.3.1}$$

In other words, $V_H(\lambda)$ is the induced representation $\mathrm{Ind}_{B_H}^H \lambda$. Then $V_H(\lambda)$ is a representation of highest weight λ. We view elements of $V_H(\lambda)$ as functions $f \colon H \longrightarrow \mathbb{A}^1$ satisfying the relation

$$f(hb) = \lambda(b^{-1}) f(h), \quad \forall h \in H,\ \forall b \in B_H. \tag{2.3.2}$$

For dominant characters λ, λ', there is a natural surjective map

$$V_H(\lambda) \otimes V_H(\lambda') \to V_H(\lambda + \lambda'). \tag{2.3.3}$$

In the description given by (2.3.2), this map is simply given by mapping $f \otimes f'$ (where $f \in V_H(\lambda)$, $f' \in V_H(\lambda')$) to the function $ff' \in V_H(\lambda + \lambda')$.

Denote by $W_H := W(H, T)$ the Weyl group and $w_{0,H} \in W_H$ the longest element. Then $V_H(\lambda)$ has a unique B_H-stable line, which is a weight space for the weight $w_{0,H}\lambda$.

https://doi.org/10.1017/fms.2021.32 Published online by Cambridge University Press

2.4. *Vector bundles on the stack of G-zips*

2.4.1. General theory

For an algebraic stack $\mathcal{X}$, write $\mathfrak{VB}(\mathcal{X})$ for the category of vector bundles on $\mathcal{X}$. Let X be a k-scheme and H an affine k-group scheme acting on X. If $\rho\colon H \to \mathrm{GL}(V)$ is a finite-dimensional algebraic representation of H, it gives rise to a vector bundle $\mathcal{V}_{H,X}(\rho)$ on the stack $[H\backslash X]$. This vector bundle can be defined geometrically as $[H\backslash(X \times_k V)]$, where H acts diagonally on $X \times_k V$. We obtain a functor

$$\mathcal{V}_{H,X}\colon\ \mathrm{Rep}(H) \to \mathfrak{VB}([H\backslash X]).$$

In particular, similar to the usual associated sheaf constrution [Jan03, Chapter I, §5.8, Equation (1)], the space of global sections $H^0([H\backslash X], \mathcal{V}_{H,X}(\rho))$ is identified with

$$H^0([H\backslash X], \mathcal{V}_{H,X}(\rho)) = \{f\colon X \to V \mid f(h\cdot x) = \rho(h)f(x), \quad \forall h \in H,\ \forall x \in X\}. \tag{2.4.1}$$

2.4.2. Automorphic vector bundles on $G\text{-}\mathtt{Zip}^{\mathcal{Z}}$

Fix a zip datum

$$\mathcal{Z} = (G, P, L, Q, M, \varphi)$$

and a frame (B, T, z) as usual. By the previous paragraph, we obtain a functor $\mathcal{V}_{E,G}\colon \mathrm{Rep}(E) \to \mathfrak{VB}(G\text{-}\mathtt{Zip}^{\mathcal{Z}})$, which we simply denote by $\mathcal{V}$. For $(V,\rho) \in \mathrm{Rep}(E)$, the space of global sections of $\mathcal{V}(\rho)$ is

$$H^0(G\text{-}\mathtt{Zip}^{\mathcal{Z}}, \mathcal{V}(\rho)) = \{f\colon G_k \to V \mid f(\epsilon\cdot g) = \rho(\epsilon)f(g), \quad \forall \epsilon \in E,\ \forall g \in G_k\}.$$

One has the following easy lemma, which follows from the fact that G_k admits an open dense E-orbit (see discussion below Theorem 2.2.4).

Lemma 2.4.1 ([Kos19, Lemma 1.2.1]). *Let (V,ρ) be an E-representation. Then we have* $\dim H^0(G\text{-}\mathtt{Zip}^{\mathcal{Z}}, \mathcal{V}(\rho)) \le \dim(V)$.

The first projection $p_1\colon E \to P$ induces a functor $p_1^*\colon \mathrm{Rep}(P) \to \mathrm{Rep}(E)$. If $(V,\rho) \in \mathrm{Rep}(P)$, we write again $\mathcal{V}(\rho)$ for $\mathcal{V}(p_1^*(\rho))$. Let $\mathfrak{VB}_P(G\text{-}\mathtt{Zip}^{\mathcal{Z}})$ be the essentail image of $\mathcal{V}\colon \mathrm{Rep}(P) \to \mathfrak{VB}(G\text{-}\mathtt{Zip}^{\mathcal{Z}})$. We call $\mathfrak{VB}_P(G\text{-}\mathtt{Zip}^{\mathcal{Z}})$ the category of automorphic vector bundles (cf. [Mil90, Chapter III, Remark 2.3]). The goal of this article is to study the vector bundles $\mathcal{V}(\rho)$ on $G\text{-}\mathtt{Zip}^{\mathcal{Z}}$ and determine their properties for $\rho \in \mathrm{Rep}(P)$. In particular, we seek to understand the properties of $\mathcal{V}(\rho)$ in terms of the representation (V,ρ) defining it.

2.4.3. L-representations

Let $\theta^P_L\colon P \to L$ denote again the natural projection modulo the unipotent radical $R_{\mathrm{u}}(P)$, as in Subsection 2.2.1. It induces by composition a functor

$$(\theta^P_L)^*\colon\ \mathrm{Rep}(L) \to \mathrm{Rep}(P).$$

It is easy to see that $(\theta^P_L)^*$ is a fully faithful functor, and its image is the full subcategory of $\mathrm{Rep}(P)$ of P-representations that are trivial on $R_{\mathrm{u}}(P)$. Hence, we view $\mathrm{Rep}(L)$ as a full subcategory of $\mathrm{Rep}(P)$. If $(V,\rho) \in \mathrm{Rep}(L)$, we write again $\mathcal{V}(\rho) := \mathcal{V}((\theta^P_L)^*(\rho))$. For $\lambda \in X^*_{+,I}(T)$, write $B_L := B \cap L$ and define an L-representation as

$$V_I(\lambda) = \mathrm{Ind}_{B_L}^L(\lambda).$$

This is the representation defined in (2.3.1) for $H = L$ and $B_H = B_L$. Denote by $\mathcal{V}_I(\lambda)$ the vector bundle on $G\text{-}\mathtt{Zip}^{\mathcal{Z}}$ attached to $V_I(\lambda)$. We call $\mathcal{V}_I(\lambda)$ the *automorphic vector bundle associated to the weight* λ on $G\text{-}\mathtt{Zip}^{\mathcal{Z}}$. This terminology stems from Shimura varieties (see Subsection 2.5 for further details). Note that if $\lambda \in X^*(T)$ is not L-dominant, then $V_I(\lambda) = 0$ and hence $\mathcal{V}_I(\lambda) = 0$. In [Kos19], the

https://doi.org/10.1017/fms.2021.32 Published online by Cambridge University Press

second author studied the vector bundles $\mathcal{V}_I(\lambda)$ on $G\text{-}\mathtt{Zip}^{\mathcal{Z}}$. In particular, he investigated the question of determining the set C_{zip} of characters $\lambda \in X^*_{+,I}(T)$ such that the space $H^0(G\text{-}\mathtt{Zip}^{\mathcal{Z}}, \mathcal{V}_I(\lambda))$ is nonzero. In a work in progress [GIK21] with Goldring, we completely determine C_{zip} under the condition that P is defined over $\mathbb{F}_q$ and the Frobenius σ acts on I by $-w_{0,I}$.

2.5. Shimura varieties

In this subsection, we explain the link between the stack of G-zips and Shimura varieties. Let $(\mathbf{G}, \mathbf{X})$ be a Shimura datum [Del79, §2.1.1]. In particular, $\mathbf{G}$ is a connected reductive group over $\mathbb{Q}$. Furthermore, $\mathbf{X}$ provides a well-defined $\mathbf{G}(\overline{\mathbb{Q}})$-conjugacy class $\{\mu\}$ of cocharacters of $\mathbf{G}_{\overline{\mathbb{Q}}}$. Write $\mathbf{E} = E(\mathbf{G}, \mathbf{X})$ for the reflex field of $(\mathbf{G}, \mathbf{X})$ (i.e., the field of definition of $\{\mu\}$) and $\mathcal{O}_{\mathbf{E}}$ for its ring of integers. Given an open compact subgroup $K \subset \mathbf{G}(\mathbf{A}_f)$, write $\mathrm{Sh}(\mathbf{G}, \mathbf{X})_K$ for the canonical model at level K over $\mathbf{E}$ (cf. [Del79, §2.2]). For K small enough in $\mathbf{G}(\mathbb{A}_f)$, $\mathrm{Sh}(\mathbf{G}, \mathbf{X})_K$ is a smooth, quasi-projective scheme over $\mathbf{E}$. For a small enough K, every inclusion $K' \subset K$ induces a finite étale projection $\pi_{K'/K}\colon \mathrm{Sh}(\mathbf{G}, \mathbf{X})_{K'} \to \mathrm{Sh}(\mathbf{G}, \mathbf{X})_K$.

Let $g \geq 1$ and let (V, ψ) be a $2g$-dimensional, nondegenerate symplectic space over $\mathbb{Q}$. Write $\mathrm{GSp}(2g) = \mathrm{GSp}(V, \psi)$ for the group of symplectic similitudes of (V, ψ). Write $\mathbf{X}_g$ for the double Siegel half-space [Del79, §1.3.1]. The pair $(\mathrm{GSp}(2g), \mathbf{X}_g)$ is called the Siegel-Shimura datum and has reflex field $\mathbb{Q}$. Recall that $(\mathbf{G}, \mathbf{X})$ is of Hodge type if there exists an embedding of Shimura data $\iota\colon (\mathbf{G}, \mathbf{X}) \hookrightarrow (\mathrm{GSp}(2g), \mathbf{X}_g)$ for some $g \geq 1$. Henceforth, assume that $(\mathbf{G}, \mathbf{X})$ is of Hodge type.

Fix a prime number p and assume that the level K is of the form $K = K_p K^p$ where $K_p \subset \mathbf{G}(\mathbb{Q}_p)$ is a hyperspecial subgroup and $K^p \subset \mathbf{G}(\mathbb{A}_f^p)$ is an open compact subgroup. Recall that a hyperspecial subgroup of $\mathbf{G}(\mathbb{Q}_p)$ exists if and only if $\mathbf{G}_{\mathbb{Q}_p}$ is unramified and is of the form $K_p = \mathscr{G}(\mathbb{Z}_p)$ where $\mathscr{G}$ is a reductive group over $\mathbb{Z}_p$ such that $\mathscr{G} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq \mathbf{G}_{\mathbb{Q}_p}$ and $\mathscr{G} \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ is connected.

We assume that $p > 2$. For any place v above p in $\mathbf{E}$, Kisin [Kis10] and Vasiu [Vas99] constructed a family of smooth $\mathcal{O}_{\mathbf{E}_v}$-schemes $\mathscr{S} = (\mathscr{S}_K)_{K^p}$, where $K = K_p K^p$ and K^p is a small enough compact open subgroup of $\mathbf{G}(\mathbb{A}_f^p)$. For $K'^p \subset K^p$, one has again a finite étale projection $\pi_{K'/K}\colon \mathscr{S}_{K_p K'^p} \to \mathscr{S}_{K_p K^p}$, where $K = K_p K^p$ and $K' = K_p K'^p$ and the tower $\mathscr{S} = (\mathscr{S}_K)_{K^p}$ is an $\mathcal{O}_{\mathbf{E}_v}$-model of the tower $(\mathrm{Sh}(\mathbf{G}, \mathbf{X})_K)_{K^p}$. We write S_K for the geometric special fibre of $\mathscr{S}_K$.

We take a representative $\mu \in \{\mu\}$ defined over $\mathbf{E}_v$ by [Kot84, (1.1.3) Lemma (a)]. We can also assume that μ extends to $\mu\colon \mathbb{G}_{\mathrm{m},\mathcal{O}_{\mathbf{E}_v}} \to \mathscr{G}_{\mathcal{O}_{\mathbf{E}_v}}$ [Kim18, Corollary 3.3.11]. Denote by $\mathbf{L} \subset \mathbf{G}_{\mathbf{E}_v}$ the centraliser of the cocharacter μ. We take a parabolic sugroups $\mathbf{P}$ of $\mathbf{G}_{\mathbf{E}_v}$, which has $\mathbf{L}$ as a Levi subgroup. Since $\mathbf{G}_{\mathbb{Q}_p}$ is unramified, it is quasi-split, hence we can choose a Borel subgroup $\mathbf{B} \subset \mathbf{G}_{\mathbb{Q}_p}$ and a maximal torus $\mathbf{T} \subset \mathbf{B}$. There is $g \in \mathbf{G}(\mathbf{E}_v)$ such that $\mathbf{B}_{\mathbf{E}_v} \subset g\mathbf{P}g^{-1}$. Write $g = bg_0$ with $b \in B(\mathbf{E}_v)$ and $g_0 \in \mathscr{G}(\mathcal{O}_{\mathbf{E}_v})$ by the Iwasawa decomposition. Then replacing μ by its conjugate by g_0, we may assume that $\mathbf{B}_{\mathbf{E}_v} \subset \mathbf{P}$.

By properness of the scheme of parabolic subgroups of $\mathscr{G}$ [ABD+66, Exposé XXVI, Corollaire 3.5], the subgroups $\mathbf{B}$ and $\mathbf{P}$ extend uniquely to subgroups $\mathscr{B} \subset \mathscr{G}$ over $\mathbb{Z}_p$ and $\mathscr{P} \subset \mathscr{G}_{\mathcal{O}_{\mathbf{E}_v}}$ over $\mathcal{O}_{\mathbf{E}_v}$ respectively. Let $\mathscr{L} \subset \mathscr{P}$ be the centraliser of $\mu\colon \mathbb{G}_{\mathrm{m},\mathcal{O}_{\mathbf{E}_v}} \to \mathscr{G}_{\mathcal{O}_{\mathbf{E}_v}}$. We take a Borel subgroup $\mathbf{B}^{\mathrm{op}}$ of $\mathbf{G}_{\mathbb{Q}_p}$ such that $\mathbf{T} = \mathbf{B} \cap \mathbf{B}^{\mathrm{op}}$. The subgroup $\mathbf{B}^{\mathrm{op}}$ extends uniquely to a subgroup $\mathscr{B}^{\mathrm{op}} \subset \mathscr{G}$ over $\mathbb{Z}_p$. We put $\mathscr{T} = \mathscr{B} \cap \mathscr{B}^{\mathrm{op}}$. Set $G = \mathscr{G} \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ and denote by B, T, P, L the geometric special fibre of $\mathscr{B}, \mathscr{T}, \mathscr{P}, \mathscr{L}$ respectively. By slight abuse of notation, we denote again by μ its mod p reduction $\mu\colon \mathbb{G}_{\mathrm{m},k} \to G_k$. Then (G, μ) is a cocharacter datum, and it yields a zip datum (G, P, L, Q, M, φ) as in Subsection 2.2.2 (because G is defined over $\mathbb{F}_p$, in the context of Shimura varieties, we always take $q = p$; hence, φ is the pth power Frobenius).

By a result of Zhang [Zha18, 4.1], there exists a natural smooth morphism

$$\zeta\colon S_K \to G\text{-}\mathtt{Zip}^{\mu}.$$

This map is also surjective by [SYZ19, Corollary 3.5.3(1)]. The map ζ amounts to the existence of a universal G-zip $\underline{\mathcal{I}} = (\mathcal{I}, \mathcal{I}_P, \mathcal{I}_Q, \iota)$ over S_K, using the description of $G\text{-}\mathtt{Zip}^{\mu}$ provided at the end of Subsection 2.2.1. In the construction of Zhang, the G_k-torsor $\mathcal{I}$ and the P-torsor $\mathcal{I}_P$ over S_K are actually the reduction of a $\mathscr{G}$-torsor and a $\mathscr{P}$-torsor over $\mathscr{S}_K$, which we denote by $\mathscr{I}$ and $\mathscr{I}_{\mathscr{P}}$ respectively.

Example 2.5.1. We explain the example of the Siegel-type Shimura variety. In this case, one has $\mathbf{G} = \mathrm{GSp}(V, \psi)$ for a symplectic space (V, ψ) of dimension $2g$ $(g \geq 1)$ over $\mathbb{Q}$. The $\mathbb{Z}_p$-model $\mathcal{G} = \mathrm{GSp}(\Lambda, \psi)$ is given by a self-dual $\mathbb{Z}_p$-lattice $\Lambda \subset V_{\mathbb{Q}_p}$; that is, a lattice satisfying $\Lambda^\vee = \Lambda$, where $\Lambda^\vee := \{x \in V_{\mathbb{Q}_p} \mid \forall y \in \Lambda,\ \psi(x, y) \in \mathbb{Z}_p\}$. The cocharacter $\mu\colon \mathbb{G}_{\mathrm{m},\mathbb{Z}_p} \to \mathbf{G}_{\mathbb{Z}_p}$ induces a decomposition $\Lambda = \Lambda_0 \oplus \Lambda_1$, where Λ_0, Λ_1 are free $\mathbb{Z}_p$-modules of rank g. Here $z \in \mathbb{G}_\mathrm{m}$ acts via μ on Λ_i by the character $z \mapsto z^i$ for $i \in \{0, 1\}$. Define two filtrations

$$\begin{aligned} \mathrm{Fil}_0(\Lambda)\colon\ & 0 \subset \Lambda_0 \subset \Lambda \quad \text{and} \\ \mathrm{Fil}_1(\Lambda)\colon\ & 0 \subset \Lambda_1 \subset \Lambda. \end{aligned}$$

Then $\mathcal{P}$ can be defined as the parabolic subgroup of $\mathcal{G}$ stabilising $\mathrm{Fil}_0(\Lambda)$. The scheme $\mathcal{S}_K$ (with $K = K_p K^p$ and $K_p = \mathcal{G}(\mathbb{Z}_p)$ as above) is a moduli space classifying triples $(A, \xi, \eta K^p)$ where A is an abelian variety of rank g endowed with a principal polarisation ξ and a K^p-level structure ηK^p. Here η is a symplectic isomorphism $H^1(A, \mathbb{A}^p) \simeq V \otimes \mathbb{A}^p$ and ηK^p is its K^p-coset in the set of such isomorphisms.

Let $\mathcal{A} \to \mathcal{S}_K$ denote the universal abelian scheme. Then

$$\mathcal{H} := H^1_{\mathrm{dR}}(\mathcal{A}/\mathcal{S}_K)$$

is a rank $2g$ vector bundle on $\mathcal{S}_K$, and the principal polarisation ξ induces on $\mathcal{H}$ a perfect, symplectic pairing, which we denote by ψ_ξ. The vector bundle $\mathcal{H}$ also carries a natural Hodge filtration (which we denote by $\mathrm{Fil}_{\mathrm{Hdg}}$):

$$0 \subset \Omega_{\mathcal{A}/\mathcal{S}_K} \subset \mathcal{H},$$

where $\Omega_{\mathcal{A}/\mathcal{S}_K}$ is the push-forward of the sheaf of relative Kähler differentials $\Omega^1_{\mathcal{A}/\mathcal{S}_K}$ by the structural morphism $f\colon \mathcal{A} \to \mathcal{S}_K$. It is a rank g-subbundle of $\mathcal{H}$. We obtain a $\mathcal{G}$-torsor $\mathcal{I}$ and a $\mathcal{P}$-torsor $\mathcal{I}_\mathcal{P}$ over $\mathcal{S}_K$ as follows: For an $\mathcal{S}_K$-scheme S, we define $\mathcal{I}(S)$ by

$$\underline{\mathrm{Isom}}_{\mathcal{O}_S}\left((\Lambda \otimes \mathcal{O}_S, \psi), (\mathcal{H} \otimes_{\mathcal{O}_{\mathcal{S}_K}} \mathcal{O}_S, \psi_\xi)\right)$$

and $\mathcal{I}_\mathcal{P}(S)$ by

$$\underline{\mathrm{Isom}}_{\mathcal{O}_S}\left((\Lambda \otimes \mathcal{O}_S, \psi, \mathrm{Fil}_0(\Lambda) \otimes \mathcal{O}_S), (\mathcal{H} \otimes_{\mathcal{O}_{\mathcal{S}_K}} \mathcal{O}_S, \psi_\xi, \mathrm{Fil}_{\mathrm{Hdg}} \otimes_{\mathcal{O}_{\mathcal{S}_K}} \mathcal{O}_S)\right).$$

This defines two fppf sheaves on $\mathcal{S}_K$. Furthermore, $\mathcal{G}$ acts naturally on $\mathcal{I}$ via its action on Λ. Furthermore, because the parabolic group $\mathcal{P} \subset \mathcal{G}$ stabilises $\mathrm{Fil}_0(\Lambda)$, the group $\mathcal{P}$ acts naturally on $\mathcal{I}_\mathcal{P}$. This defines respectively a $\mathcal{G}$-torsor and a $\mathcal{P}$-torsor on $\mathcal{S}_K$.

Over $S_K = \mathcal{S}_K \otimes \mathbb{F}_p$, the G-zip $\underline{\mathcal{I}} = (\mathcal{I}, \mathcal{I}_P, \mathcal{I}_Q, \iota)$ is defined as follows. First define $\mathcal{I}$ and $\mathcal{I}_P$ to be the base change to S_K of $\mathcal{I}$ and $\mathcal{I}_\mathcal{P}$. To define the Q-torsor $\mathcal{I}_Q$, recall that $H := H^1_{\mathrm{dR}}(A/S_K)$ admits a conjugate filtration $\mathrm{Fil}_{\mathrm{conj}} \subset H$: Let $f\colon A \to S_K$ denote the universal abelian scheme (with $A := \mathcal{A} \otimes_{\mathcal{S}_K} S_K$), then there is a conjugate spectral sequence $E_2^{ab} = R^a f_*(\mathcal{H}^b(\Omega^\bullet_{A/S_K})) \Rightarrow H^{a+b}_{\mathrm{dR}}(A/S_K)$. For abelian varieties, this spectral sequence degenerates and gives the filtration $\mathrm{Fil}_{\mathrm{conj}}$ on $H^1_{\mathrm{dR}}(A/S_K)$. Note that the conjugate filtration only exists on the special fibre of $\mathcal{S}_K$, contrary to the Hodge filtration. For an S_K-scheme S, we put

$$\mathcal{I}_Q(S) = \underline{\mathrm{Isom}}_{\mathcal{O}_S}\left((\Lambda \otimes \mathcal{O}_S, \psi, \mathrm{Fil}_1(\Lambda) \otimes \mathcal{O}_S), (H \otimes_{\mathcal{O}_{S_K}} \mathcal{O}_S, \psi_\xi, \mathrm{Fil}_{\mathrm{conj}} \otimes_{\mathcal{O}_{S_K}} \mathcal{O}_S)\right).$$

Because Q stabilises the filtration $\mathrm{Fil}_1(\Lambda) \otimes \mathbb{F}_p$, it acts naturally on $\mathcal{I}_Q$, and again we obtain a Q-torsor on S_K. Finally, the isomorphism $\iota\colon (\mathcal{I}_P/R_\mathrm{u}(P))^{(p)} \to \mathcal{I}_Q/R_\mathrm{u}(Q)$ is naturally induced by the Frobenius and Verschiebung homomorphisms (or, more generally, the Cartier isomorphism; see [MW04, 7.3]).

https://doi.org/10.1017/fms.2021.32 Published online by Cambridge University Press

For each $\mathbf{L}$-dominant character $\lambda \in X^*(\mathbf{T})$, we have the unique irreducible representation $\mathbf{V}_I(\lambda)$ of $\mathbf{P}$ over $\overline{\mathbb{Q}}_p$ of highest weight λ. Because we are in characteristic zero, $\mathbf{V}_I(\lambda)$ coincides with $H^0(\mathbf{P}/\mathbf{B}, \mathcal{L}_\lambda)$, as defined in (2.3.1) in Subsection 2.3. It admits a natural model over $\overline{\mathbb{Z}}_p$, namely,

$$\mathbf{V}_I(\lambda)_{\overline{\mathbb{Z}}_p} := H^0(\mathscr{P}/\mathscr{B}, \mathcal{L}_\lambda),$$

where $\mathcal{L}_\lambda$ is the line bundle attached to λ viewed as a character of $\mathscr{T}$. Its reduction modulo p is the P-representation $V_I(\lambda) = H^0(P/B, \mathcal{L}_\lambda)$ over $k = \overline{\mathbb{F}}_p$. Because $\mathscr{S}_K$ is endowed naturally with a $\mathscr{P}$-torsor $\mathscr{I}_{\mathscr{P}}$, we obtain a vector bundle $\mathscr{V}_I(\lambda)$ on $\mathscr{S}_K$ by applying the $\mathscr{P}$-representation $\mathbf{V}_I(\lambda)_{\overline{\mathbb{Z}}_p}$ to $\mathscr{I}_{\mathscr{P}}$. The vector bundle $\mathscr{V}_I(\lambda)$ for $\lambda \in X^*(\mathbf{T})_{+,I}$ is called *the automorphic vector bundle associated to the weight* λ. For an $\mathcal{O}_{\mathbf{E}_v}$-algebra R, the space $H^0(\mathscr{S}_K \otimes_{\mathcal{O}_{\mathbf{E}_v}} R, \mathscr{V}_I(\lambda))$ may be called the space of automorphic forms of level K and weight λ with coefficients in R. More generally, by the same formalism, we have a commutative diagram of functors

$$\begin{tikzcd}
\mathrm{Rep}_{\overline{\mathbb{Z}}_p}(\mathscr{P}) \arrow[r, "\mathscr{V}"] \arrow[d] & \mathfrak{VB}(\mathscr{S}_K) \arrow[d] \\
\mathrm{Rep}_{\overline{\mathbb{F}}_p}(P) \arrow[r, "\mathcal{V}"] & \mathfrak{VB}(S_K)
\end{tikzcd}$$

where the vertical arrows are reduction modulo p and the horizontal arrows are obtained by applying the $\mathscr{P}$-torsor $\mathscr{I}_{\mathscr{P}}$ and the P-torsor $\mathcal{I}_P$ respectively. The vector bundles obtained in this way on $\mathscr{S}_K$ and S_K are called *automorphic vector bundles* following [Mil90, Chapter III, Remark 2.3].

Furthermore, the map $\zeta\colon S_K \to G\text{-}\mathtt{Zip}^\mu$ induces a factorization of the lower horizontal arrow of the above diagram as

$$\mathrm{Rep}_{\overline{\mathbb{F}}_p}(P) \xrightarrow{\ \mathcal{V}\ } \mathfrak{VB}(G\text{-}\mathtt{Zip}^\mu) \xrightarrow{\ \zeta^*\ } \mathfrak{VB}(S_K). \tag{2.5.1}$$

Note also that for any P-representation (V, ρ), the map $\zeta\colon S_K \to G\text{-}\mathtt{Zip}^\mu$ induces by pullback a natural injective morphism

$$H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}(\rho)) \to H^0(S_K, \mathcal{V}(\rho)).$$

In Section 3, we determine the space $H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}(\rho))$ in all generality (i.e., even for cocharacter data (G, μ) that are not attached to Shimura varieties). For general pairs (G, μ) with μ minuscule (but not necessarily attached to Shimura varieties), one has the following remark.

Remark 2.5.2. Let F be a local field with ring of integers $\mathcal{O}$ and residue field $\mathbb{F}_q$. Let G be an unramified reductive group over $\mathcal{O}$. Let (B, T) be a Borel pair of G, and let μ be a dominant cocharacter of G. Then Xiao-Zhu define the moduli of local shtukas $\mathrm{Sht}_\mu^{\mathrm{loc}}$ classifying modifications bounded by μ of a G-torsor and its Frobenius twist (see [XZ17, Definition 5.2.1]). Similarly, there is a moduli $\mathrm{Sht}_\mu^{\mathrm{loc}(m,n)}$ of restricted local shtuka [XZ17, §5.3], with a natural projection $\mathrm{Sht}_\mu^{\mathrm{loc}} \to \mathrm{Sht}_\mu^{\mathrm{loc}(m,n)}$. In the case when μ is minuscule, Xiao-Zhu show in [XZ17, Lemma 5.3.6] that there exists a natural perfectly smooth morphism $\mathrm{Sht}_\mu^{\mathrm{loc}(2,1)} \to G\text{-}\mathtt{Zip}^{\mu,\mathrm{pf}}$, where pf denotes the perfection and the special fibre of G is again denoted by G (see Subsection 3.5 for further details).

3. The space of global sections $H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}(\rho))$

3.1. Adapted morphisms

To determine the space $H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}(\rho))$ (for (V, ρ) a P-representation), we use a similar method as in [Kos19, §3.2], where we studied representations of the type $V_I(\lambda)$. We review some of the notions introduced in [Kos19, §3.2].

https://doi.org/10.1017/fms.2021.32 Published online by Cambridge University Press

Let X be an irreducible normal k-variety and let $U \subset X$ be an open subset such that $S = X \setminus U$ is irreducible of codimension 1. For $f \in H^0(U, \mathcal{O}_X)$, denote by $Z_U(f) \subset U$ the vanishing locus of f in U and let $\overline{Z_U(f)}$ be its Zariski closure in X. We endow all locally closed subsets of schemes with the reduced structure. Let Y be an irreducible k-variety and $\psi\colon Y \to X$ be a k-morphism.

Definition 3.1.1. We say that ψ is adapted to f (with respect to U) if

(i) $\psi(Y) \cap U \neq \emptyset$ and
(ii) $\psi(Y) \cap S$ is not contained in $\overline{Z_U(f)}$.

Lemma 3.1.2. *If $\psi(Y)$ intersects U and $\psi(Y) \cap S$ is dense in S, then ψ is adapted to any nonzero section $f \in H^0(U, \mathcal{O}_X)$.*

Proof. We need to show that condition (ii) is satisfied. We may assume that $Z_U(f) \neq \emptyset$. Then, the closed subset $\overline{Z_U(f)}$ has codimension 1 in X and intersects U; hence, $\overline{Z_U(f)} \cap S$ has codimension ≥ 1 in S, so it cannot contain $\psi(Y) \cap S$. □

Lemma 3.1.3 ([Kos19, Lemma 3.2.2]). *Let $\psi\colon Y \to X$ be a morphism adapted to $f \in H^0(U, \mathcal{O}_X)$. Then f extends to X if and only if $\psi^*(f) \in H^0(\psi^{-1}(U), \mathcal{O}_Y)$ extends to Y. In this case, f vanishes along S if and only if $\psi^*(f)$ vanishes along $\psi^{-1}(S)$.*

We apply the above notions to the following situation. From now on, let (G, μ) be a cocharacter datum, with attached zip datum $\mathcal{Z} = (G, P, L, Q, M, \varphi)$ as in Subsection 2.2.2. Assume that (B, T) is a Borel pair defined over $\mathbb{F}_q$ such that $B \subset P$. We take a frame (B, T, z) as in Lemma 2.2.3. Consider the variety G_k and the open subset $U_\mu \subset G_k$ (the μ-ordinary stratum, defined after Theorem 2.2.4). The complement of U_μ in G_k is not irreducible in general, so in order to apply the previous results, we slightly modify the problem. Recall the parametrisation of E-orbits in G_k (2.2.2). Using Theorem 2.2.4, we have

$$G_k \setminus U_\mu = \bigcup_{\alpha \in \Delta^P} Z_\alpha, \quad Z_\alpha = \overline{E \cdot s_\alpha}, \tag{3.1.1}$$

where $E \cdot s_\alpha$ denotes the E-orbit of s_α and the bar denotes the Zariski closure. Indeed, by (2.2.2), the E-orbits of codimension 1 in G_k are the E-orbits of wz^{-1} where $w \in {}^I W$ is an element of length $\ell(w_{0,I} w_0) - 1$. These elements are of the form $w_{0,I} s_\alpha w_0$ for $\alpha \in \Delta^P$. Because $z = \sigma(w_{0,I}) w_0$, the element wz^{-1} has the form $w_{0,I} s_\alpha \sigma(w_{0,I})$. Because $(w_{0,I}, \sigma(w_{0,I})) \in E$, this element generates the same E-orbit as s_α. This proves the decomposition (3.1.1) above. For any $\alpha \in \Delta^P$, define an open subset

$$X_\alpha := G_k \setminus \bigcup_{\beta \in \Delta^P,\ \beta \neq \alpha} Z_\beta.$$

Clearly, $U_\mu \subset X_\alpha$ and one has $X_\alpha \setminus U_\mu = E \cdot s_\alpha$. In particular, $X_\alpha \setminus U_\mu$ is irreducible. We define a morphism that satisfies the conditions of Definition 3.1.1 for the pair (X_α, U_μ).

We take an isomorphism $u_\alpha\colon \mathbb{G}_a \to U_\alpha$ for $\alpha \in \Phi$ so that $(u_\alpha)_{\alpha \in \Phi}$ is a realisation in the sense of [Spr98, §8.1.4]. In particular, we have

$$t u_\alpha(x) t^{-1} = u_\alpha(\alpha(t) x) \tag{3.1.2}$$

for $x \in \mathbb{G}_a$ and $t \in T$. For $\alpha \in \Phi$, there is a unique homomorphism

$$\phi_\alpha\colon \mathrm{SL}_{2,k} \to G_k$$

such that

$$\phi_\alpha\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) = u_\alpha(x), \quad \phi_\alpha\left(\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\right) = u_{-\alpha}(x)$$

as in [Spr98, §9.2.2]. Also note that $\phi_\alpha(\mathrm{diag}(t,t^{-1})) = \alpha^\vee(t)$.

Let $\alpha \in \Delta^P$. Set $Y = E \times \mathbb{A}^1$ and

$$\psi_\alpha\colon Y \to G_k;\ ((x,y),t) \mapsto x\phi_\alpha(A(t))\,y^{-1} \quad \text{where } A(t) = \begin{pmatrix} t & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{SL}_{2,k}\,.$$

Note that $\phi_\alpha(A(0)) = s_\alpha$ in W. The following identity will be crucial for later purposes:

$$A(t) = \begin{pmatrix} 1 & 0 \\ -t^{-1} & 1 \end{pmatrix}\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\begin{pmatrix} 1 & t^{-1} \\ 0 & 1 \end{pmatrix}. \tag{3.1.3}$$

Let $\wp\colon T \to T;\ g \mapsto g\varphi(g)^{-1}$ be the Lang torsor. Then $\wp$ induces the isomorphism

$$\wp_*\colon X_*(T)_{\mathbb{R}} \xrightarrow{\sim} X_*(T)_{\mathbb{R}};\ \delta \mapsto \wp\circ\delta = \delta - q\sigma(\delta).$$

We put $\delta_\alpha = \wp_*^{-1}(\alpha^\vee)$. Recall that σ denotes the qth power Frobenius action on Δ. We put

$$m_\alpha = \min\{m \geq 1 \mid \sigma^{-m}(\alpha) \notin I\} \tag{3.1.4}$$

and $t_\alpha = t^{-1}\alpha(\varphi(\delta_\alpha(t)))^{-1} = t\alpha(\delta_\alpha(t))^{-1} \in t^{\mathbb{Q}}$, where t is an indeterminate.

Proposition 3.1.4. *The following properties hold:*

(1) *The image of ψ_α is contained in X_α.*

(2) *For any $(x,y) \in E$ and $t \in \mathbb{A}^1$, one has $\psi_\alpha((x,y),t) \in U_\mu \iff t \neq 0$.*

(3) *For all $(x,y) \in E$, we have $\psi_\alpha((x,y),0) \in E\cdot s_\alpha$.*

Proof. It suffices to show (2) and (3). If $t = 0$, we have $\phi_\alpha(A(0)) = s_\alpha$ in W. Hence, $\psi_\alpha((x,y),0) \in E\cdot s_\alpha$. Assume that $t \neq 0$. We put

$$u_{t,\alpha} = \prod_{i=1}^{m_\alpha-1} \phi_{\sigma^{-i}(\alpha)}\left(\begin{pmatrix} 1 & -t_\alpha^{\frac{1}{q^i}} \\ 0 & 1 \end{pmatrix}\right),$$

where the products are taken in the increasing order of indices. By (3.1.3) and the definitions of δ_α, t_α and $u_{t,\alpha}$, we have

$$\begin{aligned} \phi_\alpha(A(t)) &= \phi_\alpha\left(\begin{pmatrix} 1 & 0 \\ -t^{-1} & 1 \end{pmatrix}\right)\delta_\alpha(t)\varphi(\delta_\alpha(t))^{-1}\phi_\alpha\left(\begin{pmatrix} 1 & t^{-1} \\ 0 & 1 \end{pmatrix}\right) \\ &= \phi_\alpha\left(\begin{pmatrix} 1 & 0 \\ -t^{-1} & 1 \end{pmatrix}\right)\delta_\alpha(t)\phi_\alpha\left(\begin{pmatrix} 1 & t_\alpha \\ 0 & 1 \end{pmatrix}\right)\varphi(\delta_\alpha(t))^{-1} \\ &= \phi_\alpha\left(\begin{pmatrix} 1 & 0 \\ -t^{-1} & 1 \end{pmatrix}\right)\delta_\alpha(t)u_{t,\alpha}\left(\varphi(\delta_\alpha(t))\phi_\alpha\left(\begin{pmatrix} 1 & -t_\alpha \\ 0 & 1 \end{pmatrix}\right)u_{t,\alpha}\right)^{-1}. \end{aligned} \tag{3.1.5}$$

We have

$$\left(\phi_\alpha\left(\begin{pmatrix} 1 & 0 \\ -t^{-1} & 1 \end{pmatrix}\right)\delta_\alpha(t)u_{t,\alpha}, \varphi(\delta_\alpha(t))\phi_\alpha\left(\begin{pmatrix} 1 & -t_\alpha \\ 0 & 1 \end{pmatrix}\right)u_{t,\alpha}\right) \in E \tag{3.1.6}$$

because

$$\phi_\alpha\left(\begin{pmatrix} 1 & 0 \\ -t^{-1} & 1 \end{pmatrix}\right) \in R_{\mathrm{u}}(P), \quad \phi_{\sigma^{-(m_\alpha-1)}(\alpha)}\left(\begin{pmatrix} 1 & -t_\alpha^{\frac{1}{q^{m_\alpha-1}}} \\ 0 & 1 \end{pmatrix}\right) \in R_{\mathrm{u}}(Q)$$

by $\alpha \notin I$ and $\sigma^{-(m_\alpha-1)}(\alpha) \notin \sigma(I)$. Hence, we have $\psi_\alpha((x,y),t) \in U_\mu$ if $t \neq 0$. □

Set $Y_0 := E \times \mathbb{G}_{\mathrm{m}} \subset Y$. We obtain a map $\psi_\alpha\colon Y_0 \to U_\mu$.

https://doi.org/10.1017/fms.2021.32 Published online by Cambridge University Press

Corollary 3.1.5. *Let $f\colon U_\mu \to \mathbb{A}^n$ be a regular map. Then f extends to a regular map $G_k \to \mathbb{A}^n$ if and only if for all $\alpha \in \Delta^P$, the map $f \circ \psi_\alpha\colon Y_0 \to \mathbb{A}^n$ extends to a map $Y \to \mathbb{A}^n$.*

Proof. Applying Lemma 3.1.2 and Lemma 3.1.3 to the coordinate functions of f, we can extend f to $\bigcup_{\alpha\in\Delta^P} X_\alpha$. Because the complement of $\bigcup_{\alpha\in\Delta^P} X_\alpha$ in G has codimension ≥ 2, we can extend f to G by normality. □

3.2. *The space of μ-ordinary sections*

Recall that $\mathcal{U}_\mu = [E\backslash U_\mu] \subset G\text{-}\mathtt{Zip}^\mu$ denotes the μ-ordinary locus (see Subsection 2.2.4). The open substack $\mathcal{U}_\mu \subset G\text{-}\mathtt{Zip}^\mu$ is dense and hence induces an obvious injective map

$$H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}(\rho)) \to H^0(\mathcal{U}_\mu, \mathcal{V}(\rho))$$

for any $(V,\rho) \in \mathrm{Rep}(P)$. This will give an upper bound approximation of the space $H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}(\rho))$. We claim that $1 \in U_\mu$. Indeed, by Theorem 2.2.4, U_μ coincides with the E-orbit of the element $w_{0,I}w_0z^{-1}$. Because $z = \sigma(w_{0,I})w_0$, we obtain $w_{0,I}w_0z^{-1} = w_{0,I}\sigma(w_{0,I})$. This element is in the same E-orbit as 1, because $(w_{0,I}, \sigma(w_{0,I})) \in E$. This proves the claim.

We denote by $L_\varphi \subset E$ the scheme-theoretical stabiliser of the element 1. Note that

$$L_\varphi = E \cap \{(x,x) \mid x \in G_k\} \tag{3.2.1}$$

is a 0-dimensional algebraic group. In general it is nonsmooth. Denote by $L_0 \subset L$ the largest algebraic subgroup defined over $\mathbb{F}_q$. In other words,

$$L_0 = \bigcap_{n\geq 0} L^{(q^n)}.$$

In view of (3.2.1), it is clear that the restriction of the first projection $E \to P$ induces a closed immersion $L_\varphi \to P$. Hence, we will identify L_φ with its image and view it as a subgroup of P.

Lemma 3.2.1 ([KW18, Lemma 3.2.1]).
(1) *One has $L_\varphi \subset L$.*
(2) *The group L_φ can be written as a semidirect product*

$$L_\varphi = L_\varphi^\circ \rtimes L_0(\mathbb{F}_q)$$

where L_φ° is the identity component of L_φ. Furthermore, L_φ° is a finite unipotent algebraic group.
(3) *Assume that P is defined over $\mathbb{F}_q$. Then $L_0 = L$ and $L_\varphi = L(\mathbb{F}_q)$, viewed as a constant algebraic group.*

Proposition 3.2.2. *The stack $\mathcal{U}_\mu$ is isomorphic to $B(L_\varphi) = [1/L_\varphi]$, the classifying stack of L_φ.*

Proof. The action map $E \to U_\mu$, $e \mapsto e\cdot 1$ induces an isomorphism $E/L_\varphi \simeq U_\mu$. Hence, $\mathcal{U}_\mu = [E\backslash U_\mu] \simeq [E\backslash(E/L_\varphi)] \simeq [1/L_\varphi]$. □

Corollary 3.2.3. *The category of vector bundles on $\mathcal{U}_\mu$ is equivalent to the category $\mathrm{Rep}(L_\varphi)$ of representations of L_φ. Furthermore, for all $(V,\rho) \in \mathrm{Rep}(L_\varphi)$, the space of global sections of the attached vector bundle $\mathcal{V}(\rho)$ on $\mathcal{U}_\mu$ identifies with the space of L_φ-invariants of V:*

$$H^0(\mathcal{U}_\mu, \mathcal{V}(\rho)) = V^{L_\varphi}. \tag{3.2.2}$$

Furthermore, this identification is functorial in (V,ρ).

The identity (3.2.2) can be seen as an isomorphism between two functors $\mathrm{Rep}(L_\varphi) \to \mathrm{Vec}_k$. The notation V^{L_φ} for the space of invariants is to be understood in a scheme-theoretical way as the set of

$v \in V$ such that for any k-algebra R, one has $\rho(x)v = v$ in $V \otimes_k R$ for all $x \in L_\varphi(R)$. In particular, if $(V,\rho) \in \mathrm{Rep}(P)$ and $\mathcal{V}(\rho)$ is the attached vector bundle on $G\text{-}\mathtt{Zip}^\mu$, the restriction of $\mathcal{V}(\rho)$ to $\mathcal{U}_\mu$ is attached to the restriction of ρ to L_φ, and the formula (3.2.2) applies similarly.

By (2.4.1), any $f \in V^{L_\varphi} = H^0(\mathcal{U}_\mu, \mathcal{V}(\rho))$ corresponds bijectively to a unique function

$$\tilde{f}\colon U_\mu \to V \tag{3.2.3}$$

satisfying $\tilde{f}(1) = f$ and $\tilde{f}(axb^{-1}) = \rho(a)\tilde{f}(x)$ for all $(a,b) \in E$ and all $x \in U_\mu$. The strategy to determine the space $H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}(\rho))$ will be to characterise which of these functions extends to a function $G_k \to V$. We will use Corollary 3.1.5 for this purpose. As another preliminary, we introduce (a generalisation of) the Brylinski-Kostant filtration in the next section.

3.3. Brylinski-Kostant filtration

Lemma 3.3.1. *Let $\alpha \in \Phi$. Let V be a finite-dimensional algebraic representation of TU_α. Let $v \in V_\nu$ for $\nu \in X^*(T)$. Then we have*

$$u_\alpha(x)(v) - v = \sum_{j=1}^{\infty} x^j v_j,$$

where $v_j \in V_{\nu+j\alpha}$.

Proof. This is proved in the proof of [Don85, Proposition 3.3.2]. We recall the argument. We write $u_\alpha(x)v$ as $\sum_{j\geq 0} x^j v_j$ for some $v_j \in V$. We note that $v_0 = v$. By (3.1.2), we have $v_j \in V_{\nu+j\alpha}$. □

For $\alpha \in \Phi$, we define $E_\alpha^{(j)}\colon V \to V$ by

$$u_\alpha(x)v = \sum_{j\geq 0} x^j E_\alpha^{(j)}(v)$$

for $j \geq 0$ and put $E_\alpha^{(j)} = 0$ if $j < 0$. By Lemma 3.3.1, we have $E_\alpha^{(j)}(v) \in V_{\nu+j\alpha}$ for $v \in V_\nu$.

Let $\Xi = (\alpha_1, \ldots, \alpha_m) \in \Phi^m$. Let H be a closed subgroup scheme of G contaning T and U_{α_i} for $1 \leq i \leq m$. Let V be a finite-dimensional algebraic representation of H. Let $\mathbf{a} = (a_1, \ldots, a_m) \in (k^\times)^m$ and $\mathbf{r} = (r_1, \ldots, r_m) \in \mathbb{R}^m$. We put

$$(\mathbb{Z}^m)_{\mathbf{r}} = \left\{ (n_1, \ldots, n_m) \in \mathbb{Z}^m \;\middle|\; \sum_{i=1}^m n_i r_i = 0 \right\},$$
$$\Lambda_{\Xi,\mathbf{r}} = \left\{ \sum_{i=1}^m n_i \alpha_i \;\middle|\; (n_1, \ldots, n_m) \in (\mathbb{Z}^m)_{\mathbf{r}} \right\}.$$

For $[\nu] \in X^*(T)/\Lambda_{\Xi,\mathbf{r}}$, we put

$$V_{[\nu]} = \bigoplus_{\nu \in [\nu]} V_\nu.$$

We use the notation $\mathbf{j}$ for $(j_1, \ldots, j_m) \in \mathbb{Z}^m$. For $[\mathbf{j}] \in \mathbb{Z}^m/(\mathbb{Z}^m)_{\mathbf{r}}$ and $[\nu] \in X^*(T)/\Lambda_{\Xi,\mathbf{r}}$, we put

$$[\mathbf{j}] \cdot \mathbf{r} = \sum_{i=1}^m j_i r_i \in \mathbb{R},$$
$$[\nu] + [\mathbf{j}] \cdot \Xi = \left[\nu + \sum_{i=1}^m j_i \alpha_i \right] \in X^*(T)/\Lambda_{\Xi,\mathbf{r}},$$

https://doi.org/10.1017/fms.2021.32 Published online by Cambridge University Press

which are well defined. For $[\nu] \in X^*(T)/\Lambda_{\Xi,\mathbf{r}}$ and a function $\delta\colon X^*(T) \to \mathbb{R}$, we define $\mathrm{Fil}_{\delta}^{\Xi,\mathbf{a},\mathbf{r}} V_{[\nu]}$ by

$$\bigcap_{[\mathbf{j}] \in \mathbb{Z}^m/(\mathbb{Z}^m)_{\mathbf{r}}} \bigcap_{\substack{\chi \in [\nu]+[\mathbf{j}]\cdot\Xi,\\ [\mathbf{j}]\cdot\mathbf{r} > \delta(\chi)}} \mathrm{Ker}\left(\sum_{\mathbf{j} \in [\mathbf{j}]} \mathrm{pr}_{\chi} \circ a_1^{j_1} E_{\alpha_1}^{(j_1)} \circ \cdots \circ a_m^{j_m} E_{\alpha_m}^{(j_m)} \colon V_{[\nu]} \to V_{\chi} \right),$$

where $\mathrm{pr}_{\chi}\colon V_{[\nu]+[\mathbf{j}]\cdot\Xi} \to V_{\chi}$ denotes the projection.

Example 3.3.2. Assume that $\Xi = (\alpha) \in \Phi$, $r_1 = 1$ and δ is a constant function $c \in \mathbb{R}$. Then $\Lambda_{\Xi,\mathbf{r}} = 0$ and $V_{[\nu]} = V_{\nu}$ for $\nu \in X^*(T)$. In this case,

$$\mathrm{Fil}_c^{\Xi,\mathbf{a},\mathbf{r}} V_{\nu} = \bigcap_{j>c} \mathrm{Ker}\left(E_{\alpha}^{(j)} \colon V_{\nu} \to V_{\nu+j\alpha} \right), \tag{3.3.1}$$

which we simply write $\mathrm{Fil}_c^{\alpha} V_{\nu}$. This is a Brylinski-Kostant filtration (cf. [XZ19, (3.3.2)]).

3.4. Main result

We now investigate the space of global sections over $G\text{-}\mathtt{Zip}^{\mu}$ of the vector bundle $\mathcal{V}(\rho)$ for $(V, \rho) \in \mathrm{Rep}(P)$. By (3.2.2), this space is contained in $V^{L_{\varphi}}$. Conversely, the problem is to determine which $f \in V^{L_{\varphi}}$ correspond to sections of $\mathcal{V}(\rho)$ that extend from $\mathcal{U}_{\mu}$ to $G\text{-}\mathtt{Zip}^{\mu}$. Equivalently, we ask for which $f \in V^{L_{\varphi}}$ the regular function $\tilde{f}\colon U_{\mu} \to V$ defined in (3.2.3) extends to a regular function $G_k \to V$.

Recall the definition of the integer m_{α} in (3.1.4) for each $\alpha \in \Delta^P$. For example, if P is defined over $\mathbb{F}_q$, then $m_{\alpha} = 1$ for all $\alpha \in \Delta^P$. We put $\mathbf{a}_{\alpha} = (-1, \ldots, -1) \in (k^{\times})^{m_{\alpha}}$. For $\alpha \in \Delta^P$, we put $\Xi_{\alpha} = (-\alpha, \sigma^{-1}(\alpha), \ldots, \sigma^{-(m_{\alpha}-1)}(\alpha))$ and $\mathbf{r}_{\alpha} = (r_{\alpha,1}, \ldots, r_{\alpha,m_{\alpha}})$, where $r_{\alpha,1} = 1 - \langle \alpha, \delta_{\alpha} \rangle$ and

$$r_{\alpha,i} = \frac{\langle \alpha, \delta_{\alpha} \rangle - 1}{q^{i-1}}$$

for $2 \le i \le m_{\alpha}$. We view δ_{α} as a function $X^*(T) \to \mathbb{R}$ by $\chi \mapsto \langle \chi, \delta_{\alpha} \rangle$.

Theorem 3.4.1. *Let $(V, \rho) \in \mathrm{Rep}(P)$. Via the inclusion $H^0(G\text{-}\mathtt{Zip}^{\mu}, \mathcal{V}(\rho)) \subset V^{L_{\varphi}}$ (see Corollary 3.2.3) one has an identification*

$$H^0(G\text{-}\mathtt{Zip}^{\mu}, \mathcal{V}(\rho)) = V^{L_{\varphi}} \cap \bigcap_{\alpha \in \Delta^P} \bigoplus_{[\nu] \in X^*(T)/\Lambda_{\Xi_{\alpha},\mathbf{r}_{\alpha}}} \mathrm{Fil}_{\delta_{\alpha}}^{\Xi_{\alpha},\boldsymbol{a}_{\alpha},\mathbf{r}_{\alpha}} V_{[\nu]}. \tag{3.4.1}$$

Proof. Let $f \in V^{L_{\varphi}}$, and let $\tilde{f}\colon U_{\mu} \to V$ be the function defined in (3.2.3). It suffices to show that $\tilde{f}$ extends to G if and only if

$$f \in \bigoplus_{[\nu] \in X^*(T)/\Lambda_{\Xi_{\alpha},\mathbf{r}_{\alpha}}} \mathrm{Fil}_{\delta_{\alpha}}^{\Xi_{\alpha},\mathbf{a}_{\alpha},\mathbf{r}_{\alpha}} V_{[\nu]}$$

for all $\alpha \in \Delta^P$. By Corollary 3.1.5, $\tilde{f}$ extends to G_k if and only if $\tilde{f} \circ \psi_{\alpha}\colon Y_0 \to V$ extends to a function $Y \to V$. We now give an explicit formula for $\tilde{f} \circ \psi_{\alpha}((x, y), t)$. Using (3.1.5) and (3.1.6), the element $\psi_{\alpha}((x, y), t) \in U$ can be written as $x_1 x_2^{-1}$ with $(x_1, x_2) \in E$ and

$$x_1 = x\phi_{\alpha}\left(\begin{pmatrix} 1 & 0 \\ -t^{-1} & 1 \end{pmatrix}\right)\delta_{\alpha}(t)u_{t,\alpha}, \quad x_2 = y\varphi(\delta_{\alpha}(t))\phi_{\alpha}\left(\begin{pmatrix} 1 & -t_{\alpha} \\ 0 & 1 \end{pmatrix}\right)u_{t,\alpha}.$$

It follows that

$$(\tilde{f} \circ \psi_{\alpha})((x, y), t) = \tilde{f}(x_1 x_2^{-1}) = \rho(x_1) f = \rho(x)\rho\left(\phi_{\alpha}\left(\begin{pmatrix} 1 & 0 \\ -t^{-1} & 1 \end{pmatrix}\right)\delta_{\alpha}(t)u_{t,\alpha}\right) f.$$

https://doi.org/10.1017/fms.2021.32 Published online by Cambridge University Press

Hence, the function $\tilde{f} \circ \psi_\alpha$ extends to Y if and only if the function

$$F_\alpha \colon t \mapsto \rho\left(\phi_\alpha\left(\begin{pmatrix} 1 & 0 \\ -t^{-1} & 1 \end{pmatrix}\right)\delta_\alpha(t)u_{t,\alpha}\right) f$$

lies in $k[t] \otimes V$. Write $f = \sum_{\nu \in X^*(T)} f_\nu$ by the weight decomposition of f. We put

$$f^{\mathbf{j}}_{\nu,\Xi_\alpha} = E^{(j_1)}_{-\alpha} E^{(j_2)}_{\sigma^{-1}(\alpha)} \cdots E^{(j_{m_\alpha})}_{\sigma^{-(m_\alpha-1)}(\alpha)} f_\nu \in V_{\nu+\mathbf{j}\cdot\Xi_\alpha}$$

for $\mathbf{j} = (j_1, \ldots, j_{m_\alpha}) \in \mathbb{Z}^{m_\alpha}$ and $\nu \in X^*(T)$. We obtain

$$\begin{aligned} F_\alpha(t) &= \rho\left(\delta_\alpha(t)\phi_\alpha\left(\begin{pmatrix} 1 & 0 \\ -\alpha(\delta_\alpha(t))t^{-1} & 1 \end{pmatrix}\right)u_{t,\alpha}\right) f \\ &= \sum_\nu \rho\left(\delta_\alpha(t)\phi_\alpha\left(\begin{pmatrix} 1 & 0 \\ -t^{\langle\alpha,\delta_\alpha\rangle-1} & 1 \end{pmatrix}\right)\prod_{i=2}^{m_\alpha}\phi_{\sigma^{-(i-1)}(\alpha)}\left(\begin{pmatrix} 1 & -t_\alpha^{\frac{1}{q^{i-1}}} \\ 0 & 1 \end{pmatrix}\right)\right) f_\nu \\ &= \sum_\nu \rho\left(\delta_\alpha(t)\right) \sum_{\mathbf{j}\in\mathbb{Z}^{m_\alpha}} \left((-t^{\langle\alpha,\delta_\alpha\rangle-1})^{j_1}\prod_{i=2}^{m_\alpha}(-t_\alpha^{\frac{1}{q^{i-1}}})^{j_i}\right) f^{\mathbf{j}}_{\nu,\Xi_\alpha} \\ &= \sum_\nu \sum_{\mathbf{j}\in\mathbb{Z}^{m_\alpha}} t^{\langle\nu+\mathbf{j}\cdot\Xi_\alpha,\delta_\alpha\rangle}\left((-t^{\langle\alpha,\delta_\alpha\rangle-1})^{j_1}\prod_{i=2}^{m_\alpha}(-t_\alpha^{\frac{1}{q^{i-1}}})^{j_i}\right) f^{\mathbf{j}}_{\nu,\Xi_\alpha}. \end{aligned}$$

For fixed $\chi \in X^*(T)$, let $F_{\alpha,\chi}(t)$ be the V_χ-component of $F_\alpha(t)$. Then we have

$$\begin{aligned} F_{\alpha,\chi}(t) &= \sum_{\mathbf{j}\in\mathbb{Z}^{m_\alpha}} t^{\langle\chi,\delta_\alpha\rangle}\left((-t^{\langle\alpha,\delta_\alpha\rangle-1})^{j_1}\prod_{i=2}^{m_\alpha}(-t_\alpha^{\frac{1}{q^{i-1}}})^{j_i}\right) f^{\mathbf{j}}_{\chi-\mathbf{j}\cdot\Xi_\alpha,\Xi_\alpha} \\ &= \sum_{[\mathbf{j}]\in\mathbb{Z}^{m_\alpha}/(\mathbb{Z}^{m_\alpha})_{\mathbf{r}_\alpha}} \sum_{\mathbf{j}\in[\mathbf{j}]} t^{\langle\chi,\delta_\alpha\rangle-\mathbf{j}\cdot\mathbf{r}_\alpha}(-1)^{\sum_{i=1}^{m_\alpha} j_i} f^{\mathbf{j}}_{\chi-\mathbf{j}\cdot\Xi_\alpha,\Xi_\alpha}. \end{aligned}$$

The exponents of t in two terms in the last expression are equal if and only if the indices belong to the same coset in $\mathbb{Z}^{m_\alpha}/(\mathbb{Z}^{m_\alpha})_{\mathbf{r}_\alpha}$. Therefore, $F_{\alpha,\chi}(t)$ lies in $k[t]\otimes V_\chi$ for all $\chi \in X^*(T)$ if and only if we have

$$\sum_{\mathbf{j}\in[\mathbf{j}]}(-1)^{\sum_{i=1}^{m_\alpha} j_i} f^{\mathbf{j}}_{\chi-\mathbf{j}\cdot\Xi_\alpha,\Xi_\alpha} = 0$$

for all $\chi \in X^*(T)$ and $[\mathbf{j}] \in \mathbb{Z}^{m_\alpha}/(\mathbb{Z}^{m_\alpha})_{\mathbf{r}_\alpha}$ such that $\mathbf{j}\cdot\mathbf{r}_\alpha > \langle\chi,\delta_\alpha\rangle$. This condition is equivalent to that f belongs to $\bigoplus_{[\nu]\in X^*(T)/\Lambda_{\Xi_\alpha,\mathbf{r}_\alpha}} \mathrm{Fil}^{\Xi_\alpha,\mathbf{a}_\alpha,\mathbf{r}_\alpha}_{\delta_\alpha} V_{[\nu]}$. Hence, the claim follows. □

We now give some corollaries of Theorem 3.4.1 in that case where the formula (3.4.1) becomes simpler. For $\nu \in X^*(T)$ and $\chi \in X^*(T)_\mathbb{R}$, we put

$$\mathrm{Fil}^P_\chi V_\nu = \bigcap_{\alpha\in\Delta^P} \mathrm{Fil}^{-\alpha}_{\langle\chi,\alpha^\vee\rangle} V_\nu \tag{3.4.2}$$

where $\mathrm{Fil}^{-\alpha}_{\langle\chi,\alpha^\vee\rangle} V_\nu$ was defined in Example 3.3.2. The morphism $\wp\colon T \to T$ induces the isomorphism

$$\wp^*\colon X^*(T)_\mathbb{R} \xrightarrow{\sim} X^*(T)_\mathbb{R};\ \lambda \mapsto \lambda\circ\wp = \lambda - q\sigma^{-1}(\lambda).$$

https://doi.org/10.1017/fms.2021.32 Published online by Cambridge University Press

Corollary 3.4.2. *Assume that P is defined over $\mathbb{F}_q$. Let $(V,\rho) \in \operatorname{Rep}(P)$. Via the inclusion $H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}(\rho)) \subset V^{L(\mathbb{F}_q)}$ one has*

$$H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}(\rho)) = V^{L(\mathbb{F}_q)} \cap \bigoplus_{\nu \in X^*(T)} \operatorname{Fil}^P_{\wp^{*-1}(\nu)} V_\nu.$$

Proof. For $\alpha \in \Delta^P$ and $\nu \in X^*(T)$, we have

$$\operatorname{Fil}^{\Xi_\alpha, \mathbf{a}_\alpha, \mathbf{r}_\alpha}_{\delta_\alpha} V_{[\nu]} = \operatorname{Fil}^{-\alpha}_{\langle \nu, \delta_\alpha \rangle} V_\nu = \operatorname{Fil}^{-\alpha}_{\langle \wp^{*-1}(\nu), \alpha^\vee \rangle} V_\nu.$$

Hence, the claim follows from Lemma 3.2.1(3) and Theorem 3.4.1. □

Assume again that P is defined over $\mathbb{F}_q$. To simplify further, assume that $(V,\rho) \in \operatorname{Rep}(P)$ is trivial on the unipotent radical $R_{\mathrm{u}}(P)$. Then we have $E^{(j)}_{-\alpha} = 0$ for all $\alpha \in \Delta^P$ and all $j > 0$. It follows that $\operatorname{Fil}^{-\alpha}_c V_\nu = V_\nu$ for $c \geq 0$ and $\operatorname{Fil}^{-\alpha}_c V_\nu = 0$ for $c < 0$. We obtain that for all $\chi \in X^*(T)_{\mathbb{R}}$, one has

$$\operatorname{Fil}^P_\chi V_\nu = \begin{cases} V_\nu & \text{if for all } \alpha \in \Delta^P \text{ one has } \langle \chi, \alpha^\vee \rangle \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Define a subspace $V^{\Delta^P}_{\geq 0} \subset V$ as follows:

$$V^{\Delta^P}_{\geq 0} = \bigoplus_{\langle \nu, \delta_\alpha \rangle \geq 0,\ \forall \alpha \in \Delta^P} V_\nu. \tag{3.4.3}$$

For example, if T is split over $\mathbb{F}_q$, then $\delta_\alpha = -\alpha^\vee/(q-1)$, and therefore $V^{\Delta^P}_{\geq 0}$ is the direct sum of the weight spaces V_ν for those $\nu \in X^*(T)$ satisfying $\langle \nu, \alpha^\vee \rangle \leq 0$ for all $\alpha \in \Delta^P$.

Corollary 3.4.3. *Assume that P is defined over $\mathbb{F}_q$ and furthermore that $(V,\rho) \in \operatorname{Rep}(P)$ is trivial on the unipotent radical $R_u(P)$. Then one has an equality*

$$H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}(\rho)) = V^{L(\mathbb{F}_q)} \cap V^{\Delta^P}_{\geq 0}.$$

This formula recovers the result [Kos19, Theorem 1] (with slightly different notation). In [Kos19, Theorem 1], only the special case when G is split over $\mathbb{F}_p$ and V is of the form $V_I(\lambda)$ was considered.

3.5. Perfection

As noted in Remark 2.5.2, the perfection of the stack of G-zips appears in connection with the moduli of local shtukas. In [XZ17, Lemma 5.3.6], the zip datum that appears satisfies that P is defined over $\mathbb{F}_q$. We do not make this assumption here. For a scheme X over k, define the perfection of X as the projective limit

$$X^{\mathrm{pf}} := \varprojlim_{\varphi_X} X$$

where φ_X denotes the absolute qth power Frobenius endomorphism of X. There is a natural map $X^{\mathrm{pf}} \longrightarrow X$. We have an isomorphism

$$X^{\mathrm{pf}} \simeq \varprojlim \left(\cdots \xrightarrow{\varphi} X^{(q^{-2})} \xrightarrow{\varphi} X^{(q^{-1})} \xrightarrow{\varphi} X \right),$$

where φ denotes the relative qth power Frobenius endomorphism. The perfection of $G\text{-}\mathtt{Zip}^\mu$ is then given by

$$G\text{-}\mathtt{Zip}^{\mu,\mathrm{pf}} = [E^{\mathrm{pf}} \backslash G^{\mathrm{pf}}_k].$$

https://doi.org/10.1017/fms.2021.32 Published online by Cambridge University Press

Similar to Proposition 3.2.2, the perfection of the μ-ordinary locus $\mathcal{U}_\mu^{\mathrm{pf}}$ is isomorphic to $[1/L_\varphi^{\mathrm{pf}}]$. Because $L_\varphi = L_\varphi^\circ \rtimes L_0(\mathbb{F}_q)$ by Lemma 3.2.1(2), we obtain

$$\mathcal{U}_\mu^{\mathrm{pf}} = [1/L_0(\mathbb{F}_q)]. \tag{3.5.1}$$

If (V,ρ) is a P-representation, then we obtain a P^{pf}-representation by pullback, which we denote by ρ^{pf}. This yields a vector bundle $\mathcal{V}(\rho^{\mathrm{pf}})$ on $G\text{-}\mathtt{Zip}^{\mu,\mathrm{pf}}$, which also coincides with the pullback of $\mathcal{V}(\rho)$ under the natural map

$$G\text{-}\mathtt{Zip}^{\mu,\mathrm{pf}} \to G\text{-}\mathtt{Zip}^{\mu}.$$

By equation (3.5.1), we see that the space $H^0(G\text{-}\mathtt{Zip}^{\mu,\mathrm{pf}}, \mathcal{V}(\rho^{\mathrm{pf}}))$ is naturally a subspace of $V^{L_0(\mathbb{F}_q)}$.

Corollary 3.5.1. *Let* $(V,\rho) \in \mathrm{Rep}(P)$. *We have*

$$H^0(G\text{-}\mathtt{Zip}^{\mu,\mathrm{pf}}, \mathcal{V}(\rho^{\mathrm{pf}})) = V^{L_0(\mathbb{F}_q)} \cap \bigcap_{\alpha\in\Delta^P} \bigoplus_{[\nu]\in X^*(T)/\Lambda_{\Xi_\alpha,\mathbf{r}_\alpha}} \mathrm{Fil}_{\delta_\alpha}^{\Xi_\alpha,\mathbf{a}_\alpha,\mathbf{r}_\alpha} V_{[\nu]}.$$

Proof. Let d be the smallest positive integer such that μ is defined over $\mathbb{F}_{q^d}$. We show that $H^0(G\text{-}\mathtt{Zip}^{\mu,\mathrm{pf}}, \mathcal{V}(\rho^{\mathrm{pf}}))$ is given by the subspace of elements $f \in V$ such that there exists $n \geq 1$ with $f \in H^0(G\text{-}\mathtt{Zip}^{\mu}, \mathcal{V}(\rho^{(q^{nd})}))$. Indeed, such a section is given by a map $f\colon G_k^{\mathrm{pf}} \to V$ satisfying an E^{pf}-equivariance condition with respect to ρ^{pf}. Because V is a scheme of finite type, such a map is given by a map $f_n\colon G_k \to V$ at a finite level of the system $(\cdots \xrightarrow{\varphi^d} G_k \xrightarrow{\varphi^d} G_k)$. We have

$$\mathrm{Fil}_{\delta_\alpha}^{\Xi_\alpha,\mathbf{a}_\alpha,\mathbf{r}_\alpha} V_{[q^{nd}\nu]}^{(q^{nd})} = \mathrm{Fil}_{\delta_\alpha}^{\Xi_\alpha,\mathbf{a}_\alpha,\mathbf{r}_\alpha} V_{[\nu]}.$$

Hence, changing ρ to $\rho^{(q^n)}$ only affects V^{L_φ}. The result follows. □

3.6. L-semisimplification

If $\rho\colon P \to \mathrm{GL}(V)$ is an arbitrary representation, we can attach a P-representation $(V, \rho^{L\text{-ss}})$ that is trivial on $R_{\mathrm{u}}(P)$. The representation $\rho^{L\text{-ss}}$ is defined as the composition

$$\rho^{L\text{-ss}}\colon P \xrightarrow{\theta_L^P} L \xrightarrow{\rho} \mathrm{GL}(V),$$

where $\theta_L^P\colon P \to L$ is the natural projection map whose kernel is $R_{\mathrm{u}}(P)$, as defined in Subsection 2.2.1. We call $\rho^{L\text{-ss}}$ the L-semisimplification of ρ. We sometimes write $V^{L\text{-ss}}$ to denote this representation (even though the underlying vector space is the same as V).

One obvious property of $V^{L\text{-ss}}$ is $(V^{L\text{-ss}})^{L_\varphi} = V^{L_\varphi}$ because $L_\varphi \subset L$ by Lemma 3.2.1(1). In particular, by Corollary 3.2.3, we have for all $(V,\rho) \in \mathrm{Rep}(P)$ the equality

$$H^0(\mathcal{U}_\mu, \mathcal{V}(\rho^{L\text{-ss}})) = H^0(\mathcal{U}_\mu, \mathcal{V}(\rho)). \tag{3.6.1}$$

Note that this identification is somewhat indirect: It is not induced by a morphism between the sheaves $\mathcal{V}(\rho)$ and $\mathcal{V}(\rho^{L\text{-ss}})$. For $f \in H^0(\mathcal{U}_\mu, \mathcal{V}(\rho))$, we will write $f^{L\text{-ss}}$ for its image under the identification (3.6.1) and call it the L-semisimplification of f. As an element of V, $f^{L\text{-ss}}$ is the same as f, but we want to emphasise the fact that the representation has changed.

We now give another interpretation of L-semisimplification when P is defined over $\mathbb{F}_q$. Write again $U_\mu \subset G_k$ for the unique open E-orbit and recall that $1 \in U_\mu$ (see Subsection 3.2).

Lemma 3.6.1. *Assume that P is defined over $\mathbb{F}_q$. There exists a unique regular map $\Theta\colon U_\mu \to L$ such that for any $(a,b) \in E$, one has*

$$\Theta(ab^{-1}) = \theta_L^P(a)\theta_L^Q(b)^{-1}. \tag{3.6.2}$$

Furthermore, we have $L \subset U_\mu$ and the inclusion $L \subset U_\mu$ is a section of Θ.

Proof. First, note that because P is defined over $\mathbb{F}_q$, one has $L = M$; hence, the formula (3.6.2) makes sense. The unicity of Θ is obvious. For the existence, consider the map $\tilde{\Theta}\colon E \to L;\ (a,b) \mapsto \theta_L^P(a)\theta_L^Q(b)^{-1}$. Because P is defined over $\mathbb{F}_q$, one has $L_\varphi = L(\mathbb{F}_q)$ (Lemma 3.2.13). For all $(a,b) \in E$ and all $x \in L(\mathbb{F}_q)$, one has $\tilde{\Theta}(ax, bx) = \tilde{\Theta}(a,b)$. Hence, $\tilde{\Theta}$ factors to a map $\Theta\colon E/L(\mathbb{F}_q) \simeq U_\mu \to L$. This proves the first result. Now, if $x \in L$, we can write $x = a\varphi(a)^{-1}$ with $a \in L$ by Lang's theorem. Hence, $x \in U_\mu$ and $\Theta(x) = a\varphi(a)^{-1} = x$, so the second statement is proved. □

Example 3.6.2. Consider the case $G = \mathrm{Sp}(2n)_{\mathbb{F}_q}$ for $n \geq 1$. We write an element of G_k as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with A, B, C, D square matrices of size $n \times n$. Let $P \subset G_k$ be the parabolic subgroup defined by the condition $B = 0$ and $Q \subset G_k$ the parabolic subgroup defined by the condition $C = 0$. We put $L = P \cap Q$. This gives a zip datum (G, P, L, Q, L, φ). The Zariski open subset $U_\mu \subset G_k$ is the set of matrices in G_k for which A is invertible. The map $\Theta\colon U_\mu \to L$ is given by

$$\Theta\colon \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{pmatrix}.$$

Proposition 3.6.3. *Assume that P is defined over $\mathbb{F}_q$. Let $(V, \rho) \in \mathrm{Rep}(P)$ and let $f \in V^{L(\mathbb{F}_q)}$. Let $\tilde{f}$ be the corresponding function $U_\mu \to V$ defined in (3.2.3). Then the function $\widetilde{f^{L\text{-}ss}}\colon U_\mu \to V$ that corresponds to the L-semisimplification $f^{L\text{-}ss}$ is the composition*

$$U_\mu \xrightarrow{\Theta} L \hookrightarrow U_\mu \xrightarrow{\tilde{f}} V.$$

Proof. Put $f' = \tilde{f} \circ \Theta$. For $(a,b) \in E$ and $g \in U_\mu$ such that $g = ab^{-1}$, we have

$$\begin{aligned} f'(g) &= f'(ab^{-1}) = \tilde{f}(\Theta(ab^{-1})) \\ &= \tilde{f}(\theta_L^P(a)\theta_L^Q(b)^{-1}) = \rho(\theta_L^P(a))f = \rho^{L\text{-ss}}(a)f = \widetilde{f^{L\text{-ss}}}(g). \end{aligned}$$

Hence, $f' = \widetilde{f^{L\text{-ss}}}$. □

Let $f \in H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}(\rho))$ be a global section. We may view its restriction $f|_{\mathcal{U}_\mu}$ as a section of $\mathcal{V}(\rho^{L\text{-ss}})$ over $\mathcal{U}_\mu$ by the identification (3.6.1). It is thus natural to ask whether $(f|_{\mathcal{U}_\mu})^{L\text{-ss}}$ extends to a global section over $G\text{-}\mathtt{Zip}^\mu$. We prove that this holds when P is defined over $\mathbb{F}_q$ in the following proposition.

Proposition 3.6.4. *Assume that P is defined over $\mathbb{F}_q$. The identification (3.6.1) extends to a commutative diagram*

$$\begin{tikzcd} H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}(\rho)) \arrow[r, hook] \arrow[d, hook] & H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}(\rho^{L\text{-}ss})) \arrow[d, hook] \\ H^0(\mathcal{U}_\mu, \mathcal{V}(\rho)) \arrow[r, "="] & H^0(\mathcal{U}_\mu, \mathcal{V}(\rho^{L\text{-}ss})). \end{tikzcd}$$

Proof. Let $f \in H^0(G\text{-}\mathtt{Zip}^{\mu}, \mathcal{V}(\rho))$. Because P is defined over $\mathbb{F}_q$, we can apply Corollary 3.4.2 to the representation (V, ρ). Furthermore, because $R_{\mathrm{u}}(P)$ acts trivially on $(V^{L\text{-ss}}, \rho^{L\text{-ss}})$, we can apply Corollary 3.4.3 to $(V^{L\text{-ss}}, \rho^{L\text{-ss}})$. Therefore, it suffices to show that for each $\nu \in X^*(T)$,

$$V^{L(\mathbb{F}_q)} \cap \bigoplus_{\nu \in X^*(T)} \mathrm{Fil}^{P}_{\wp^{*-1}(\nu)} V_\nu \quad \subset \quad V^{L(\mathbb{F}_q)} \cap V^{\Delta^P}_{\geq 0}.$$

By (3.4.3), it suffices to show the following: For any fixed $\nu \in X^*(T)$, if $\mathrm{Fil}^{P}_{\wp^{*-1}(\nu)} V_\nu \neq 0$, then $\langle \wp^{*-1}(\nu), \alpha^\vee \rangle \geq 0$ for all $\alpha \in \Delta^P$. More generally, using (3.4.2), it suffices to show that for any $\alpha \in \Delta^P$ and any integer $c \in \mathbb{Z}$ such that $\mathrm{Fil}^{-\alpha}_c V_\nu \neq 0$, one has $c \geq 0$. This is trivial by (3.3.1) because $E^{(0)}_{-\alpha}$ is the identity map. □

Remark 3.6.5. Proposition 3.6.4 does not hold in general without the assumption that P is defined over $\mathbb{F}_q$, as an example in Subsection 6.2 shows.

4. The case of $G = \mathrm{SL}_{2,\mathbb{F}_q}$

4.1. Notation for SL_2

Let B_2 and B_2^+ be the lower-triangular and upper-triangular Borel subgroup of $\mathrm{SL}_{2,k}$. Let T_2 be the diagonal torus of $\mathrm{SL}_{2,k}$. We put

$$u_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \in B_2(k).$$

For $r \in \mathbb{Z}$, let χ_r be the character of B_2 defined by

$$\begin{pmatrix} x & 0 \\ z & x^{-1} \end{pmatrix} \mapsto x^r.$$

Let $\mathrm{Std}\colon \mathrm{SL}_{2,k} \to \mathrm{GL}_{2,k}$ be the standard representation. Restrictions of χ_r and Std to subgroups are denoted by the same notations.

4.2. Zip datum

Let $G = \mathrm{SL}_{2,\mathbb{F}_q}$ and $\mu\colon \mathbb{G}_{\mathrm{m},k} \to G_k;\ x \mapsto \mathrm{diag}(x, x^{-1})$. Let $\mathcal{Z}_\mu = (G, P, L, Q, M, \varphi)$ be the associated zip datum. We have $P = B_2$, $Q = B_2^+$ and $L = M = T_2$. We take $(B, T) = (B_2, T_2)$ as a Borel pair and take a frame as in Lemma 2.2.3. Denote by α the unique element of Δ. In our convention of positivity, $\alpha = \chi_2$. Note that $I = \emptyset$ and $\Delta^P = \{\alpha\}$. Identify $X^*(T) = \mathbb{Z}$ such that $r \in \mathbb{Z}$ corresponds to the character χ_r. The zip group E is equal to

$$\left\{ \left(\begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix}, \begin{pmatrix} a^q & b \\ 0 & a^{-q} \end{pmatrix} \right) \in B_2 \times B_2^+ \right\}.$$

The unique open E-orbit $U_\mu \subset G_k$ is given by

$$U_\mu = \left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathrm{SL}_{2,k} \;\middle|\; x \neq 0 \right\}.$$

https://doi.org/10.1017/fms.2021.32 Published online by Cambridge University Press

4.3. *The space* $H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}(\rho))$

Let $\rho\colon B \to \mathrm{GL}(V)$ be a representation. We write the weight decomposition as $V = \bigoplus_{i\in\mathbb{Z}} V_i$ where T acts on V_i by the character χ_i for all $i \in \mathbb{Z}$. We have

$$H^0(\mathcal{U}_\mu, \mathcal{V}(\rho)) = V^{L(\mathbb{F}_q)} = \bigoplus_{i\in(q-1)\mathbb{Z}} V_i$$

by Corollary 3.2.3. Because in this case the parabolic $P = B$ is defined over $\mathbb{F}_q$, we can apply Corollary 3.4.2 to compute the space of global section

$$H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}(\rho)).$$

Also, because T is split over $\mathbb{F}_q$, the map $\wp^*$ is given by $\nu \mapsto -(q-1)\nu$; hence, $\wp^{*-1}(\nu) = \frac{-\nu}{q-1}$. We obtain

$$H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}(\rho)) = V^{L(\mathbb{F}_q)} \cap \bigoplus_{i\in\mathbb{Z}} \mathrm{Fil}^P_{\frac{-\chi_i}{q-1}} V_i = \bigoplus_{i\in-(q-1)\mathbb{N}} \mathrm{Fil}^{-\alpha}_{\frac{-i}{q-1}} V_i,$$

where we used that $\mathrm{Fil}^{-\alpha}_{\frac{-i}{q-1}} V_i = 0$ for $i > 0$. In particular, $H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}(\rho))$ is stable by T and is entirely determined by its weight spaces $\mathrm{Fil}^{-\alpha}_{\frac{-i}{q-1}} V_i \subset V_i$ for $i \in -(q-1)\mathbb{N}$. Let $(V, \rho) \in \mathrm{Rep}(B)$ and set $n = \dim(V)$. Set $V_{\leq i} = \bigoplus_{j\leq i} V_j$ and $V_{\geq i} = \bigoplus_{j\geq i} V_j$. Then using Lemma 3.3.1, we have a B-stable filtration

$$\cdots \subset V_{\leq i-1} \subset V_{\leq i} \subset V_{\leq i+1} \subset \cdots.$$

For all $i \in -(q-1)\mathbb{N}$, we have

$$H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}(\rho))_i = \left\{ f \in V_i \;\middle|\; \rho(u_2) f \in V_{\geq \frac{(q+1)i}{q-1}} \right\} \tag{4.3.1}$$

by the definition of $\mathrm{Fil}^{-\alpha}_{\frac{-i}{q-1}} V_i$.

Lemma 4.3.1. *Let* $(V, \rho) \in \mathrm{Rep}(B)$ *and* $m \in \mathbb{Z}$ *be the smallest weight of* ρ. *Then one has an inclusion*

$$\bigoplus_{\substack{i\in-(q-1)\mathbb{N},\\ (q+1)i\leq(q-1)m}} V_i \subset H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}(\rho)). \tag{4.3.2}$$

Proof. Let $f \in V_i$ with $i \in -(q-1)\mathbb{N}$ and $(q+1)i \leq (q-1)m$. Then we have $V_{\geq \frac{(q+1)i}{q-1}} = V$, so we have $f \in H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}(\rho))_i$. □

The following example shows that $H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}(\rho))$ is not a sum of weight spaces of V in general.

Example 4.3.2. For $i \in \{1, -1\}$, let e_i be a nonzero vector of weight i of Std. Consider $\rho := \mathrm{Std} \otimes \mathrm{Std}$ with basis $e_i \otimes e_j$ for $i, j \in \{1, -1\}$. The weights of ρ are $\{2, 0, -2\}$, and $\dim(V_2) = \dim(V_{-2}) = 1$, $\dim(V_0) = 2$. Then we have

$$H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}(\rho))_0 = \mathrm{Span}(e_1 \otimes e_{-1} - e_{-1} \otimes e_1).$$

4.4. *Property (P)*

Proposition 4.4.1. *Let* $\rho\colon B \to \mathrm{GL}(V)$ *be an algebraic representation. Let* $m_1, \ldots, m_n$ *be the weights of* V *ordered so that* $m_1 > m_2 > \cdots > m_n$. *The following properties are equivalent:*

(i) *The subspace* $V^{R_u(B)}$ *is one-dimensional (and hence is equal to* V_{m_n}*).*

https://doi.org/10.1017/fms.2021.32 Published online by Cambridge University Press

(ii) *The intersection of all nonzero B-subrepresentations in V is nonzero.*
(iii) *For all $1 \leq i \leq n$, we have $\dim(V_{m_i}) = 1$ and for any $v \in V_{m_i} \setminus \{0\}$, the projection of $\rho(u_2)v$ onto V_{m_n} is nonzero.*

Proof. We show (i) $\Rightarrow$ (ii). If $W \subset V$ is a nonzero B-subrepresentation, then $W^{R_{\mathrm{u}}(B)} \subset V^{R_{\mathrm{u}}(B)}$. Because $W^{R_{\mathrm{u}}(B)} \neq 0$, we have $W^{R_{\mathrm{u}}(B)} = V^{R_{\mathrm{u}}(B)}$ and hence $V^{R_{\mathrm{u}}(B)} \subset W$.

We show (ii) $\Rightarrow$ (iii). We show that for any nonzero $v \in V_{m_i}$ the projection of $\rho(u_2)v$ onto V_{m_n} is nonzero. For a contradiction, assume that it is zero. Because $B = R_{\mathrm{u}}(B)T$, the B-subrepresentation generated by v is generated by v as an $R_{\mathrm{u}}(B)$-representation. Hence, this representation does not have a nontrivial intersection with V_{m_n} by Lemma 3.3.1. This contradicts (ii). Hence, the claim follows. We note that $\dim V_{m_n} = 1$ by (ii). Assume that $\dim V_{m_i} \geq 2$ for some i. Then there is a nonzero $v \in V_{m_i}$ such that the projection to V_{m_n} of $\rho(u_2)v$ is zero. This is a contradiction.

We show (iii) $\Rightarrow$ (i). Assume $\dim V^{R_{\mathrm{u}}(B)} \geq 2$. Then $V^{R_{\mathrm{u}}(B)}$ contains V_{m_i} for some $i \neq n$. For any nonzero $v \in V_{m_i} \subset V^{R_{\mathrm{u}}(B)}$, the projection of $\rho(u_2)v$ onto V_{m_n} is zero. This is a contradiction. □

We say that $(V, \rho) \in \mathrm{Rep}(B)$ satisfies the property (P) if the equivalent conditions of Proposition 4.4.1 are satisfied.

Example 4.4.2. For $\lambda \in X^*_+(T)$, the restriction to B of $\mathrm{Ind}_B^{G_k}(\lambda)$ satisfies the property (P) by the last sentence of Subsection 2.3.

Proposition 4.4.3. *Assume that $(V, \rho) \in \mathrm{Rep}(B)$ satisfies the property (P). Then the inclusion (4.3.2) is an equality; that is,*

$$H^0(G\text{-}\mathtt{Zip}^{\mu}, \mathcal{V}(\rho)) = \bigoplus_{\substack{i \in -(q-1)\mathbb{N}, \\ (q+1)i \leq (q-1)m}} V_i.$$

Proof. In this case, the element $\rho(u_2)f$ in equation (4.3.1) has a nonzero projection onto V_m by Proposition 4.4.1(iii). Thus, if $f \in H^0(G\text{-}\mathtt{Zip}^{\mu}, \mathcal{V}(\rho))_i$, then we must have $m \geq \frac{(q+1)i}{q-1}$. This shows that (4.3.2) is an equality. □

5. Category of automorphic vector bundles on $G\text{-}\mathtt{Zip}^{\mu}$

5.1. *The category* $\mathfrak{VB}_P(G\text{-}\mathtt{Zip}^{\mu})$

Recall the functor $\mathrm{Rep}(P) \to \mathfrak{VB}_P(G\text{-}\mathtt{Zip}^{\mu})$ (Subsection 2.4.2). This functor is not fully faithful even after restricting to the full subcategory $\mathrm{Rep}(L) \subset \mathrm{Rep}(P)$ (see Subsection 2.4.3). Indeed, consider the following example.

Example 5.1.1. Assume that P is defined over $\mathbb{F}_q$. Let $\mathbf{1} \in \mathrm{Rep}(L)$ be the trivial L-representation and $(V, \rho) \in \mathrm{Rep}(L)$. Then $\mathrm{Hom}_{\mathrm{Rep}(L)}(\mathbf{1}, V) = V^L$, whereas we have

$$\mathrm{Hom}_{\mathfrak{VB}(G\text{-}\mathtt{Zip}^{\mu})}(\mathcal{V}(\mathbf{1}), \mathcal{V}(\rho)) = H^0(G\text{-}\mathtt{Zip}^{\mu}, \mathcal{V}(\rho)) = V^{L(\mathbb{F}_q)} \cap V_{\geq 0}^{\Delta^P}$$

by Corollary 3.4.3.

To overcome the problem, we introduce L_{φ}-modules with additional structures.

Definition 5.1.2. An L_{φ}-module with Δ^P-monodromy is a pair $((\tau, V), \mathcal{N})$ where $\tau \colon L_{\varphi} \to \mathrm{GL}_k(V)$ is a finite-dimensional representation of L_{φ} with a decomposition $V = \bigoplus_{\nu \in X^*(T)} V_{\nu}$ as k-vector spaces and $\mathcal{N} = \{N_{\alpha'}^{(j)}\}_{\alpha \in \Delta^P,\, \alpha' \in \Xi_{\alpha},\, j \in \mathbb{Z}}$ is a set of k-linear endmorphisms of V such that $N_{\alpha'}^{(j)}(V_{\nu}) \subset V_{\nu + j\alpha'}$, $N_{\alpha'}^{(0)} = \mathrm{Id}$ and $N_{\alpha'}^{(j)} = 0$ for $j < 0$.

Morphisms are given as follows: Let $((\tau, V), \mathcal{N})$ and $((\tau', V'), \mathcal{N}')$ be two L_{φ}-modules with Δ^P-monodromy. Then a morphism $((\tau, V), \mathcal{N}) \to ((\tau', V'), \mathcal{N}')$ is a k-linear map $f \colon V \to V'$ that satisfies the following:

(1) f is an L_φ-equivariant morphism.
(2) For $\alpha \in \Delta^P$, $[\mathbf{j}] \in \mathbb{Z}^{m_\alpha}/(\mathbb{Z}^{m_\alpha})_{r_\alpha}$ and $\chi \in X^*(T)$ such that $[\mathbf{j}] \cdot \mathbf{r}_\alpha > \delta_\alpha(\chi)$, we have

$$\sum_{\mathbf{j} \in [\mathbf{j}]} \sum_{\mathbf{j}' \in \mathbb{Z}^{m_\alpha}} (-1)^{\sum_{i=1}^{m_\alpha} j'_i} \operatorname{pr}_\chi \left(N'^{(j'_1)}_{\alpha_1} \cdots N'^{(j'_{m_\alpha})}_{\alpha_{m_\alpha}} f N^{(j_{m_\alpha} - j'_{m_\alpha})}_{\alpha_{m_\alpha}} \cdots N^{(j_1 - j'_1)}_{\alpha_1} \right) = 0,$$

where pr_χ denotes the projection

$$\operatorname{pr}_\chi \colon \operatorname{Hom}(V, V') \simeq \bigoplus_{\nu, \nu' \in X^*(T)} \operatorname{Hom}(V_\nu, V'_{\nu'}) \to \bigoplus_{\nu \in X^*(T)} \operatorname{Hom}(V_\nu, V'_{\nu+\chi}).$$

We denote by $L_\varphi\text{-}\mathrm{MN}_{\Delta^P}$ the category of L_φ-modules with Δ^P-monodromy.

Remark 5.1.3. The condition (2) in Definition 5.1.2 means that f is compatible with $\mathcal{N}$ and $\mathcal{N}'$ in some sense. Assume that P is defined over $\mathbb{F}_q$. Then the condition (2) in Definition 5.1.2 is simplified as follows: For $\alpha \in \Delta^P$, $\chi \in X^*(T)$ and $j \in \mathbb{N}$ such that $j r_{\alpha,1} > \delta_\alpha(\chi)$, we have

$$\operatorname{pr}_\chi \left(\sum_{0 \le j' \le j} (-1)^{j'} N'^{(j')}_{-\alpha} f N^{(j-j')}_{-\alpha} \right) = 0.$$

The morphism $N^{(j)}_{-\alpha}$ is an analogue of $N^j/j!$ for a monodromy operator N in characteristic zero. In this sense,

$$f \mapsto \sum_{0 \le j' \le j} (-1)^{j'} N'^{(j')}_{-\alpha} f N^{(j-j')}_{-\alpha}$$

is an analogue of jth iterate of

$$f \mapsto fN - N'f$$

divided by $j!$ for monodromy operators N and N' in characteristic zero.

We have the functor

$$F_{\mathrm{MN}} \colon \operatorname{Rep}(P) \to L_\varphi\text{-}\mathrm{MN}_{\Delta^P};\ (V, \rho) \mapsto \left((V, \rho|_{L_\varphi}), \{E^{(j)}_{\alpha'}\}_{\alpha \in \Delta^P,\, \alpha' \in \Xi_\alpha,\, j \in \mathbb{Z}} \right)$$

where we equip V with the natural T-weight decomposition $V = \bigoplus_\nu V_\nu$.

Definition 5.1.4. An L_φ-module with Δ^P-monodromy is called admissible if it is in the essential image of F_{MN}. We denote by $L_\varphi\text{-}\mathrm{MN}^{\mathrm{adm}}_{\Delta^P}$ the category of admissible L_φ-modules with Δ^P-monodromy.

Theorem 5.1.5. *The functor* $\mathcal{V} \colon \operatorname{Rep}(P) \to \mathfrak{BB}(G\text{-}\mathtt{Zip}^\mu)$ *factors through the functor* $F_{\mathrm{MN}} \colon \operatorname{Rep}(P) \to L_\varphi\text{-}\mathrm{MN}^{\mathrm{adm}}_{\Delta^P}$ *and induces an equivalence of categories*

$$L_\varphi\text{-}\mathrm{MN}^{\mathrm{adm}}_{\Delta^P} \longrightarrow \mathfrak{BB}_P(G\text{-}\mathtt{Zip}^\mu).$$

Proof. For two P-representations (V, ρ) and (V', ρ'), one has

$$\begin{aligned}
\operatorname{Hom}_{\mathfrak{BB}(G\text{-}\mathtt{Zip}^\mu)}(\mathcal{V}(\rho), \mathcal{V}(\rho')) &= \operatorname{Hom}_{\mathfrak{BB}(G\text{-}\mathtt{Zip}^\mu)}(\mathcal{V}(\mathbf{1}), \mathcal{V}(\rho)^\vee \otimes \mathcal{V}(\rho')) \\
&= \operatorname{Hom}_{\mathfrak{BB}(G\text{-}\mathtt{Zip}^\mu)}(\mathcal{V}(\mathbf{1}), \mathcal{V}(\rho^\vee \otimes \rho')) \\
&= H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}(\rho^\vee \otimes \rho')) \\
&= (V^\vee \otimes V')^{L_\varphi} \cap \bigcap_{\alpha \in \Delta^P} \bigoplus_{[\nu] \in X^*(T)/\Lambda_{\Xi_\alpha, \mathbf{r}_\alpha}} \operatorname{Fil}^{\Xi_\alpha, \mathbf{a}_\alpha, \mathbf{r}_\alpha}_{\delta_\alpha} (V^\vee \otimes V')_{[\nu]},
\end{aligned}$$

https://doi.org/10.1017/fms.2021.32 Published online by Cambridge University Press

where we used Theorem 3.4.1 in the last line. We can see from the definition that this space coincides with the space of homomorphisms $F_{\mathrm{MN}}(V,\rho) \to F_{\mathrm{MN}}(V',\rho')$ using that the action of $u_{\alpha'}(x)$ on $V^\vee \otimes V'$ is given by $f \mapsto \rho'(u_{\alpha'}(x)) \circ f \circ \rho(u_{\alpha'}(-x))$ for $\alpha' \in \Xi_\alpha$. □

Let S_K denote the good reduction special fibre of a Hodge-type Shimura variety, with the same notations and assumptions as in Subsection 2.5. Recall that there is a functor $\mathcal{V}\colon \mathrm{Rep}(P) \to \mathfrak{VB}(S_K)$ (see (2.5.1)), which induces functors

$$\mathrm{Rep}(P) \xrightarrow{\mathcal{V}} \mathfrak{VB}_P(G\text{-}\mathtt{Zip}^\mu) \xrightarrow{\zeta^*} \mathfrak{VB}_P(S_K),$$

where $\mathfrak{VB}_P(S_K)$ also denotes the essential image of $\mathrm{Rep}(P)$ in $\mathfrak{VB}(S_K)$. We obtain the following corollary in the context of Shimura varieties.

Corollary 5.1.6. *The functor* $\mathcal{V}\colon \mathrm{Rep}(P) \to \mathfrak{VB}_P(S_K)$ *factors as*

$$\mathrm{Rep}(P) \xrightarrow{F_{\mathrm{MN}}} L_\varphi\text{-}\mathrm{MN}^{\mathrm{adm}}_{\Delta^P} \xrightarrow{\zeta^*} \mathfrak{VB}_P(S_K).$$

5.2. *The category* $\mathfrak{VB}_L(G\text{-}\mathtt{Zip}^\mu)$

We assume that P is defined over $\mathbb{F}_q$. Hence, in what follows, we have $L_\varphi = L(\mathbb{F}_q)$.

Definition 5.2.1. Let $\mathfrak{VB}_L(G\text{-}\mathtt{Zip}^\mu)$ denote the full subcategory of $\mathfrak{VB}(G\text{-}\mathtt{Zip}^\mu)$, which is equal to the essential image of the functor $\mathrm{Rep}(L) \to \mathfrak{VB}(G\text{-}\mathtt{Zip}^\mu)$. We call it the category of L-vector bundles on $G\text{-}\mathtt{Zip}^\mu$.

For example, the automorphic vector bundles $(\mathcal{V}(\lambda))_{\lambda \in X^*(T)}$ (see Subsection 2.4.3) lie in the subcategory of L-vector bundles on $G\text{-}\mathtt{Zip}^\mu$.

Definition 5.2.2. A Δ^P-filtered L_φ-module is a pair $((\tau,V),\mathcal{F})$ where $\tau\colon L_\varphi \to \mathrm{GL}_k(V)$ is a finite-dimensional representation of L_φ and $\mathcal{F} = \{V^\alpha_{\geq \bullet}\}_{\alpha \in \Delta^P}$ is a set of filtrations on V. Here, $V^\alpha_{\geq\bullet}$ denotes a descending filtration $(V^\alpha_{\geq r})_{r \in \mathbb{R}}$.

Morphisms are given as follows. Let $((\tau,V),\mathcal{F})$ and $((\tau',V'),\mathcal{F}')$ be two Δ^P-filtered L_φ-modules. Then a morphism $((\tau,V),\mathcal{F}) \to ((\tau',V'),\mathcal{F}')$ is a k-linear map $f\colon V \to V'$ that satisfies the following:

(1) f is an L_φ-equivariant morphism.
(2) For each $\alpha \in \Delta^P$, the map f is compatible with the filtrations $V^\alpha_{\geq\bullet}$ and $V'^\alpha_{\geq\bullet}$ in the sense that $f(V^\alpha_{\geq r}) \subset V'^\alpha_{\geq r}$ for any $r \in \mathbb{R}$.

We denote by $L_\varphi\text{-}\mathrm{MF}^{\mathrm{adm}}_{\Delta^P}$ the category of Δ^P-filtered L_φ-modules.

Let $((\tau,V),\mathcal{N}) \in L_\varphi\text{-}\mathrm{MN}_{\Delta^P}$. For $\alpha \in \Delta^P$, define the α-filtration $(V^\alpha_{\geq\bullet})$ of V as follows: Let $V = \bigoplus_\nu V_\nu$ be the weight decomposition of V. For all $r \in \mathbb{R}$, let $V^\alpha_{\geq r}$ be the direct sum of V_ν for all ν satisfying $\langle \nu, \delta_\alpha\rangle \geq r$. We call $V^\alpha_{\geq\bullet}$ the α-filtration of V. Thus, we have a functor $L_\varphi\text{-}\mathrm{MN}_{\Delta^P} \to L_\varphi\text{-}\mathrm{MF}_{\Delta^P}$. Taking composition, we obtain

$$F_{\mathrm{MF}}\colon \mathrm{Rep}(L) \to \mathrm{Rep}(P) \xrightarrow{F_{\mathrm{MN}}} L_\varphi\text{-}\mathrm{MN}_{\Delta^P} \to L_\varphi\text{-}\mathrm{MF}_{\Delta^P}. \tag{5.2.1}$$

Definition 5.2.3. A Δ^P-filtered L_φ-module is called admissible if it is in the essential image of F_{MF}. We denote by $L_\varphi\text{-}\mathrm{MF}^{\mathrm{adm}}_{\Delta^P}$ the category of admissible Δ^P-filtered L_φ-modules.

Theorem 5.2.4. *The functor* $\mathcal{V}\colon \mathrm{Rep}(L) \to \mathfrak{VB}(G\text{-}\mathtt{Zip}^\mu)$ *factors through the functor* $F_{MF}\colon \mathrm{Rep}(L) \to L_\varphi\text{-}MF^{\mathrm{adm}}_{\Delta^P}$ *and induces an equivalence of categories*

$$L_\varphi\text{-}MF^{\mathrm{adm}}_{\Delta^P} \longrightarrow \mathfrak{VB}_L(G\text{-}\mathtt{Zip}^\mu).$$

https://doi.org/10.1017/fms.2021.32 Published online by Cambridge University Press

Proof. By Theorem 5.1.5, it suffices to show

$$\mathrm{Hom}_{L_\varphi\text{-}\mathrm{MN}_{\Delta^P}}(F_{\mathrm{MN}}(\rho), F_{\mathrm{MN}}(\rho')) = \mathrm{Hom}_{L_\varphi\text{-}\mathrm{MF}_{\Delta^P}}(F_{\mathrm{MF}}(\rho), F_{\mathrm{MF}}(\rho'))$$

for $(V,\rho), (V',\rho') \in \mathrm{Rep}(L)$. This follows from Remark 5.1.3 and the definitions of morphisms in L_φ-MN_{Δ^P} and L_φ-MF_{Δ^P}. □

6. Examples

6.1. *The algebras R_I and R_Δ*

Fix a connected reductive group G over $\mathbb{F}_q$, a cocharacter $\mu\colon \mathbb{G}_{\mathrm{m},k} \to G_k$ and a frame (B,T,z) for $\mathcal{Z}_\mu$ (Subsection 2.2.3). For $\lambda \in X_+^*(T)$, denote by $V_\Delta(\lambda)$ the G-representation $\mathrm{Ind}_B^G(\lambda)$. We add a subscript Δ to avoid confusion with $V_I(\lambda) = \mathrm{Ind}_{B_L}^L(\lambda)$ for $\lambda \in X_{+,I}^*(T)$ (see §2.4.3). Let $\mathcal{V}_\Delta(\lambda)$ be the vector bundle on $G\text{-}\mathtt{Zip}^\mu$ attached to $V_\Delta(\lambda)$. We put

$$R_I = \bigoplus_{\lambda \in X_{+,I}^*(T)} H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}_I(\lambda)) \quad \text{and} \quad R_\Delta = \bigoplus_{\lambda \in X_+^*(T)} H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}_\Delta(\lambda)).$$

By (2.3.3), the k-vector spaces R_I and R_Δ have a natural structure of k-algebra. They capture information about all $\mathcal{V}_I(\lambda)$ and $\mathcal{V}_\Delta(\lambda)$ at once.

Remark 6.1.1. In general, we do not know whether R_I and R_Δ are finite-type algebras, but we conjecture that this is the case. The algebra R_I was studied in [Kos19]. In the case of $G = \mathrm{Sp}(4)$ with a cocharacter μ whose centraliser Levi subgroup is isomorphic to GL_2, we showed that R_I is a polynomial algebra in three indeterminates [Kos19, Theorem 5.4.1].

In this first example, we examine R_Δ in the case of $G = \mathrm{SL}_{2,\mathbb{F}_q}$ with the zip datum explained in Subsection 4.2. In this case, the algebra R_I is very simple: It is a polynomial algebra in one indeterminate, generated by the classical Hasse invariant. Let $n \in \mathbb{N}$. The representation $V_\Delta(\chi_n)$ identifies with $\mathrm{Sym}^n(\mathrm{Std})$. The weights of $V_\Delta(\chi_n)$ are $\{-n+2i \mid 0 \le i \le n\}$. By Example 4.4.2 and Proposition 4.4.3, we have

$$H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}_\Delta(\chi_n)) = \bigoplus_{\substack{i \in -(q-1)\mathbb{N},\\ (q+1)i \le -(q-1)n}} V_\Delta(\chi_n)_i \tag{6.1.1}$$

for all $n \ge 0$. Let x, y be indeterminates. Let SL_2 act on $k[x,y]$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot P = P(ax+cy, bx+dy).$$

Then $V_\Delta(\chi_n) = \mathrm{Sym}^n(\mathrm{Std})$ is the subrepresentation of $k[x,y]$ spanned by homogeneous polynomials in x, y of degree n. The highest weight vector is x^n. By (6.1.1), we have

$$H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}_\Delta(\chi_n)) = \mathrm{Span}_k\left(x^j y^{n-j} \,\middle|\, j \ge 0,\ q-1 | n-2j,\ (q+1)j \le n\right).$$

Similarly, R_Δ is the subalgebra of $k[x,y]$ generated by $x^j y^{n-j}$ for all $0 \le j \le n$ with $q-1 | n-2j$ and $(q+1)j \le n$.

Proposition 6.1.2. *The algebra R_Δ is generated by y^{q-1} and xy^q. In particular, it is a polynomial algebra in two indeterminates.*

Proof. It is clear that y^{q-1} and xy^q are elements of R_Δ. Let $n \ge 0$ and $0 \le j \le n$ such that $x^j y^{n-j} \in R_\Delta$. We can write $x^j y^{n-j} = (xy^q)^j y^{n-(q+1)j}$. Note that $n \ge (q+1)j$ and $q-1$ divides $n-(q+1)j = n-2j-(q-1)j$. It follows that $x^j y^{n-j}$ lies in the subalgebra of $k[x,y]$ generated by y^{q-1} and xy^q. □

https://doi.org/10.1017/fms.2021.32 Published online by Cambridge University Press

We give an interpretation of these sections. In the case of $G = \mathrm{SL}_{2,\mathbb{F}_q}$, recall that for an $\mathbb{F}_q$-scheme S, the groupoid $G\text{-}\mathtt{Zip}^\mu(S)$ consists of tuples $\underline{\mathcal{H}} = (\mathcal{H}, \omega, F, V)$ where

(1) $\mathcal{H}$ is a locally free $\mathcal{O}_S$-module of rank 2 with a trivialisation $\det(\mathcal{H}) \simeq \mathcal{O}_S$,
(2) $\omega \subset \mathcal{H}$ is a locally free $\mathcal{O}_S$-submodule of rank 1 such that $\mathcal{H}/\omega$ is locally free and
(3) $F\colon \mathcal{H}^{(q)} \to \mathcal{H}$ and $V\colon \mathcal{H} \to \mathcal{H}^{(q)}$ are $\mathcal{O}_S$-linear maps satisfying the conditions $\mathrm{Ker}(F) = \mathrm{Im}(V) = \omega^{(q)}$ and $\mathrm{Ker}(V) = \mathrm{Im}(F)$.

Consider the flag space $\mathcal{F}_G$ over $G\text{-}\mathtt{Zip}^\mu$ parametrising pairs $(\underline{\mathcal{H}}, \mathcal{L})$ with $\mathcal{L} \subset \mathcal{H}$ a locally free $\mathcal{O}_S$-submodule of rank 1 such that $\mathcal{H}/\mathcal{L}$ is locally free. The natural projection map $\pi_G\colon \mathcal{F}_G \to G\text{-}\mathtt{Zip}^\mu$ is a $\mathbb{P}^1$-fibration. For $n \in \mathbb{Z}$, the push-forward $\pi_{G,*}(\mathcal{L}^{-n})$ coincides with the vector bundle $\mathcal{V}_\Delta(\chi_n)$. Consider the map

$$\mathcal{L} \subset \mathcal{H} \xrightarrow{V} \mathcal{H}^{(q)} \to (\mathcal{H}/\mathcal{L})^{(q)} \simeq \mathcal{L}^{-q},$$

where we used that $\mathcal{H}/\mathcal{L} \simeq \mathcal{L}^{-1}$ by the trivialisation $\det(\mathcal{H}) \simeq \mathcal{O}_S$. We obtain a section of $\mathcal{L}^{-(q+1)}$. It corresponds to the element xy^q in Proposition 6.1.2. On the other hand, the classical Hasse invariant $Ha \in H^0(S, \omega^{q-1})$ is given by the map $V\colon \omega \to \omega^{(q)} \simeq \omega^q$. By sending Ha under the morphism

$$\omega \subset \mathcal{H} \to \mathcal{H}/\mathcal{L} \simeq \mathcal{L}^{-1},$$

we obtain a section of $\mathcal{L}^{-(q-1)}$. This section corresponds to y^{q-1} in Proposition 6.1.2.

6.2. Example on L-semisimplification

We give an example that shows that Proposition 3.6.4 does not hold in general without the assumption that P is defined over $\mathbb{F}_q$. Let $G = \mathrm{Res}_{\mathbb{F}_{q^2}/\mathbb{F}_q} \mathrm{SL}_{2,\mathbb{F}_{q^2}}$ and

$$\mu\colon \mathbb{G}_{\mathrm{m},k} \to G_k \simeq \mathrm{SL}_{2,k} \times \mathrm{SL}_{2,k};\ z \mapsto \left(\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right).$$

Let $\mathcal{Z}_\mu = (G, P, L, Q, M, \varphi)$ be the associated zip datum. We have $P = B_2 \times \mathrm{SL}_{2,k}$, $L = T_2 \times \mathrm{SL}_{2,k}$, $Q = \mathrm{SL}_{2,k} \times B_2^+$ and $M = \mathrm{SL}_{2,k} \times T_2$. We take $(B, T) = (B_2 \times B_2, T_2 \times T_2)$ as a Borel pair and take a frame as in Lemma 2.2.3. Then Δ^P consists of one root $\alpha = \chi_2 \boxtimes \chi_0$. We have

$$L_\varphi = \left\{ \left(\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}, \begin{pmatrix} x^q & y \\ 0 & x^{-q} \end{pmatrix} \right) \in L \,\middle|\, x \in \mathbb{F}_{q^2}^\times, y^q = 0 \right\}.$$

We have

$$\delta_\alpha = \frac{-\alpha^\vee - q\sigma(\alpha^\vee)}{q^2 - 1}, \quad \mathbf{r}_\alpha = \left(\frac{q^2+1}{q^2-1}, \frac{-(q^2+1)}{q(q^2-1)} \right),$$
$$(\mathbb{Z}^2)_{\mathbf{r}_\alpha} = \{(n_1, n_2) \in \mathbb{Z}^2 \mid qn_1 = n_2\}.$$

We define $\rho\colon P \to \mathrm{GL}(V)$ by

$$\left(\mathrm{Sym}^{q^2-1}(\mathrm{Std}) \otimes \chi_{q^2-1} \right) \boxtimes \mathrm{Sym}^{q^2-1}(\mathrm{Std}^{(q)}).$$

We write (V', ρ') for $(V^{L\text{-ss}}, \rho^{L\text{-ss}})$. Then we have $V^{L_\varphi} = V$ and $V'^{L_\varphi} = V'$. We put $\nu = \chi_0 \boxtimes \chi_{-q(q^2-3)}$. We have

$$V_{[\nu]} = V_\nu \oplus V_{\nu+\alpha-q\sigma(\alpha)}.$$

https://doi.org/10.1017/fms.2021.32 Published online by Cambridge University Press

We parametrise elements $[\mathbf{j}] \in \mathbb{Z}^2/(\mathbb{Z}^2)_{\mathbf{r}_\alpha}$ by classes $[(0, j)]$ with $j \in \mathbb{Z}$. Using this notation, we have

$$\mathrm{Fil}_{\delta_\alpha}^{\Xi_\alpha, \mathbf{a}_\alpha, \mathbf{r}_\alpha} V_{[\nu]} = \bigcap_{j \in \mathbb{Z}} \bigcap_{\substack{\chi \in [\nu + j\sigma(\alpha)], \\ jr_{\alpha,2} > \delta_\alpha(\chi)}} \mathrm{Ker}\left(\sum_{j_1 \in \mathbb{Z}} \mathrm{pr}_\chi \circ E_{-\alpha}^{(j_1)} \circ E_{\sigma(\alpha)}^{(j+qj_1)} : V_{[\nu]} \to V_\chi \right)$$

because $(-1)^{j_1}(-1)^{j+qj_1} = (-1)^j \in k$. We have $V_\chi \neq 0$ if and only if $\chi = \nu + i_1\alpha + qi_2\sigma(\alpha)$ for $0 \leq i_1 \leq q^2 - 1$ and $-1 \leq i_2 \leq q^2 - 2$. For $\chi = \nu + i_1\alpha + qi_2\sigma(\alpha)$, the conditions $\chi \in [\nu + j\sigma(\alpha)]$ and $jr_{\alpha,2} > \delta_\alpha(\chi)$ hold if and only if $j = q(i_1 + i_2)$ and $i_2 - i_1 > q^2 - 2 - 2/(q^2 - 1)$. Hence,

$$\chi \in [\nu + j\sigma(\alpha)],\ jr_{\alpha,2} > \delta_\alpha(\chi),\ V_\chi \neq 0 \Longleftrightarrow \chi = \nu + q(q^2 - 2)\sigma(\alpha),\ j = q(q^2 - 2).$$

We put $\chi_0 = \nu + q(q^2 - 2)\sigma(\alpha)$ and $j_0 = q(q^2 - 2)$. Then we have

$$\begin{aligned} \mathrm{Fil}_{\delta_\alpha}^{\Xi_\alpha, \mathbf{a}_\alpha, \mathbf{r}_\alpha} V_{[\nu]} &= \mathrm{Ker}\left(\mathrm{pr}_{\chi_0} \circ \left(E_{\sigma(\alpha)}^{(j_0)} + E_{-\alpha}^{(1)} \circ E_{\sigma(\alpha)}^{(j_0+q)} \right) : V_{[\nu]} \to V_{\chi_0} \right) \\ &= \left\{ (v_1, v_2) \in V_\nu \oplus V_{\nu+\alpha-q\sigma(\alpha)} \;\middle|\; E_{\sigma(\alpha)}^{(j_0)}(v_1) + (E_{-\alpha}^{(1)} \circ E_{\sigma(\alpha)}^{(j_0+q)})(v_2) = 0 \right\}. \end{aligned}$$

We note that

$$E_{\sigma(\alpha)}^{(j_0)} : V_\nu \to V_{\chi_0}, \quad E_{-\alpha}^{(1)} \circ E_{\sigma(\alpha)}^{(j_0+q)} : V_{\nu+\alpha-q\sigma(\alpha)} \to V_{\chi_0}$$

are isomorphisms. In the same way, we have

$$\mathrm{Fil}_{\delta_\alpha}^{\Xi_\alpha, \mathbf{a}_\alpha, \mathbf{r}_\alpha} V'_{[\nu]} = \mathrm{Ker}\left(\mathrm{pr}_{\chi_0} \circ E_{\sigma(\alpha)}^{(j_0)} : V'_{[\nu]} \to V'_{\chi_0} \right) = V'_{\nu+\alpha-q\sigma(\alpha)}$$

using $E_{-\alpha}^{(1)} = 0$ for (V', ρ'). Hence, $\mathrm{Fil}_{\delta_\alpha}^{\Xi_\alpha, \mathbf{a}_\alpha, \mathbf{r}_\alpha} V_{[\nu]} \not\subset \mathrm{Fil}_{\delta_\alpha}^{\Xi_\alpha, \mathbf{a}_\alpha, \mathbf{r}_\alpha} V'_{[\nu]}$. Therefore, we have $H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}(\rho)) \not\subset H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}(\rho'))$.

6.3. The case of the unitary group $U(2,1)$ *with* p *inert*

In this subsection, we examine an example that arises in the study of Picard surfaces. These are Shimura varieties of PEL type (in particular, of Hodge type) attached to unitary groups $\mathbf{G}$ over $\mathbb{Q}$ with respect to some totally imaginary quadratic extension $\mathbf{E}/\mathbb{Q}$. We impose that $\mathbf{G}_\mathbb{R} \simeq \mathrm{GU}(2, 1)$. We choose a rational prime p that is inert in $\mathbf{E}$ and consider the attached zip datum (G, P, Q, L, M, φ). Because p is inert, the parabolic P is not defined over $\mathbb{F}_p$. We study the space $H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}_I(\lambda))$. To simplify, we will work with a unitary group U, instead of a group of unitary similitudes GU. The case of GU is very similar.

Let (V, ψ) be a 3-dimensional vector space over $\mathbb{F}_{q^2}$ endowed with a nondegenerate Hermitian form $\psi : V \times V \to \mathbb{F}_{q^2}$ (in the context of Shimura varieties, take $q = p$). Write $\mathrm{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q) = \{\mathrm{Id}, \sigma\}$. We take a basis $\mathcal{B} = (v_1, v_2, v_3)$ of V where ψ is given by the matrix

$$J = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}.$$

We define a reductive group G by

$$G(R) = \{ f \in \mathrm{GL}_{\mathbb{F}_{q^2}}(V \otimes_{\mathbb{F}_q} R) \mid \psi_R(f(x), f(y)) = \psi_R(x, y),\ \forall x, y \in V \otimes_{\mathbb{F}_q} R \}$$

for any $\mathbb{F}_q$-alegebra R. One has an identification $G_{\mathbb{F}_{q^2}} \simeq \mathrm{GL}(V)$, given as follows: For any $\mathbb{F}_{q^2}$-algebra R, we have an $\mathbb{F}_{q^2}$-algebra isomorphism $\mathbb{F}_{q^2} \otimes_{\mathbb{F}_q} R \to R \times R$, $a \otimes x \mapsto (ax, \sigma(a)x)$. By tensoring with V, we obtain an isomorphism $V \otimes_{\mathbb{F}_q} R \to (V \otimes_{\mathbb{F}_{q^2}} R) \oplus (V \otimes_{\mathbb{F}_{q^2}} R)$. Then any element of $G(R)$

https://doi.org/10.1017/fms.2021.32 Published online by Cambridge University Press

stabilises this decomposition and is entirely determined by its restriction to the first summand. This yields an isomorphism as claimed. Using the basis $\mathcal{B}$, we identify $G_{\mathbb{F}_{q^2}}$ with $\mathrm{GL}_{3,\mathbb{F}_{q^2}}$. The action of σ on the set $\mathrm{GL}_3(k)$ is given as follows: $\sigma \cdot A = J\sigma({}^tA)^{-1}J$. Let T denote the maximal diagonal torus and B the lower-triangular Borel subgroup of G_k. Note that by our choice of the basis $\mathcal{B}$, the groups B and T are defined over $\mathbb{F}_q$. Identify $X^*(T) = \mathbb{Z}^3$ such that $(k_1, k_2, k_3) \in \mathbb{Z}^3$ corresponds to the character $\mathrm{diag}(x_1, x_2, x_3) \mapsto \prod_{i=1}^3 x_i^{k_i}$. The simple roots are $\Delta = \{e_1 - e_2, e_2 - e_3\}$, where (e_1, e_2, e_3) is the canonical basis of $\mathbb{Z}^3$.

Define a cocharacter $\mu\colon \mathbb{G}_{\mathrm{m},k} \to G_k$ such that μ is given by $x \mapsto \mathrm{diag}(x, x, 1)$ via the identification $G_k \simeq \mathrm{GL}_{3,k}$. Let $\mathcal{Z}_\mu = (G, P, L, Q, M, \varphi)$ be the associated zip datum. Note that P is not defined over $\mathbb{F}_q$. One has $I = \{e_1 - e_2\}$ and $\Delta^P = \{\alpha\}$ with $\alpha = e_2 - e_3$.

Lemma 6.3.1. *Let H be the function on G_k defined by*

$$H\left((x_{i,j})_{1\le i,j\le 3}\right) = x_{1,1}^q\Delta_1 - x_{2,1}^q\Delta_2 \quad \textit{with} \begin{cases}\Delta_1 = x_{1,1}x_{2,2} - x_{1,2}x_{2,1},\\ \Delta_2 = x_{1,1}x_{2,3} - x_{2,1}x_{1,3}.\end{cases}$$

The μ-ordinary stratum $U_\mu \subset G_k$ is equal to the complement of the vanishing locus of H.

Proof. In this case, there is a unique E-orbit of codimension 1 by the first part of Theorem 2.2.4. Furthermore, this E-orbit is dense in $G_k \setminus U_\mu$ by the closure relation. Hence, it suffices to show that H does not vanish on U_μ. The group E consists of pairs $(x, y) \in P \times Q$ with

$$x = \begin{pmatrix} a & b & 0\\ c & d & 0\\ e & f & g\end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} g^q & h & i\\ 0 & d^q & b^q\\ 0 & c^q & a^q\end{pmatrix}^{-1}.$$

Because $1 \in U_\mu$, the open U_μ consists of elements of the form xy^{-1}. We find

$$H(xy^{-1}) = (ag^q)^q g^q d^q(ad - bc) - (cg^q)^q g^q b^q(ad - bc) = g^{q^2+q}(ad - bc)^{q+1}.$$

This expression is nonzero, so the result is proved. □

We have

$$L_\varphi = \left\{\begin{pmatrix} a & b & \\ & d & \\ & & a^{-q}\end{pmatrix} \in L \;\middle|\; a, d \in \mathbb{F}_{q^2}^\times,\ d^{q+1} = 1,\ b^q = 0\right\}.$$

The endomorphism $\wp_*\colon X_*(T)_\mathbb{R} \to X_*(T)_\mathbb{R}$ is given by the matrix

$$\wp_* = \begin{pmatrix} 1 & & q\\ & 1+q & \\ q & & 1\end{pmatrix}.$$

Hence, it follows that $\delta_\alpha = \wp_*^{-1}(\alpha^\vee) = \frac{1}{q^2-1}(-q, q-1, 1)$. We have $m_\alpha = 2$, $\mathbf{a}_\alpha = (-1, -1)$, $\Xi_\alpha = (-\alpha, \sigma(\alpha))$ and

$$\mathbf{r}_\alpha = \left(\frac{q^2 - q + 1}{q^2 - 1}, \frac{-q^2 + q - 1}{q(q^2 - 1)}\right), \qquad (\mathbb{Z}^2)_{\mathbf{r}_\alpha} = \{(n_1, n_2) \in \mathbb{Z}^2 \mid qn_1 = n_2\}.$$

The group $\Lambda_{\Xi_\alpha, \mathbf{r}_\alpha}$ is

$$\Lambda_{\Xi_\alpha, \mathbf{r}_\alpha} = \mathbb{Z}(q, -(q + 1), 1).$$

Let $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ be an L-dominant character (i.e., $\lambda_1 \geq \lambda_2$) and consider the L-representation $V_I(\lambda)$. We simply write V for $V_I(\lambda)$ sometimes. Under the isomorphism

$$\mathrm{GL}_2 \times \mathbb{G}_{\mathrm{m}} \to L;\ (A, z) \mapsto \begin{pmatrix} A & \\ & z \end{pmatrix},$$

the representation V corresponds to the representation

$$\det\nolimits_{\mathrm{GL}_2}^{\lambda_2} \otimes \mathrm{Sym}^{\lambda_1-\lambda_2}(\mathrm{Std}_{\mathrm{GL}_2}) \otimes \xi_{\lambda_3},$$

where ξ_r is the character of $\mathrm{GL}_2 \times \mathbb{G}_{\mathrm{m}}$ given by $(A, z) \mapsto z^r$. Hence, V is a representation of dimension $\lambda_1 - \lambda_2 + 1$ and it has weights

$$\nu_i := (\lambda_1 - i, \lambda_2 + i, \lambda_3), \quad 0 \leq i \leq \lambda_1 - \lambda_2.$$

Note that the difference $\nu_i - \nu_{i'}$ of two weights is never in $\Lambda_{\Xi_\alpha, \mathbf{r}_\alpha}$ unless $i = i'$. Therefore, $V_{[\nu]} = V_\nu$ for all $\nu \in \mathbb{Z}^3$. One deduces

$$V^{L_\varphi} = \bigoplus_{\substack{q|i,\ q+1|\lambda_2+i,\\ q^2-1|\lambda_1-i-q\lambda_3}} V_{\nu_i}.$$

It remains to determine $\mathrm{Fil}_{\delta_\alpha}^{\Xi_\alpha, \mathbf{a}_\alpha, \mathbf{r}_\alpha} V_\nu$, which is either 0 or V_ν. We parametrise elements $[\mathbf{j}] \in \mathbb{Z}^2/(\mathbb{Z}^2)_{\mathbf{r}_\alpha}$ by classes $[(0, j)]$ with $j \in \mathbb{Z}$. Then, an element $\mathbf{j} \in [\mathbf{j}]$ can be written as $(0, j) + j_1(1, q)$ with $j_1 \in \mathbb{Z}$. Using this notation, we obtain

$$\mathrm{Fil}_\delta^{\Xi, \mathbf{a}, \mathbf{r}} V_\nu = \bigcap_{j \in \mathbb{Z}} \bigcap_{\substack{\chi \in [\nu + j\sigma(\alpha)],\\ jr_{\alpha,2} > \delta_\alpha(\chi)}} \mathrm{Ker}\left(\sum_{j_1 \in \mathbb{Z}} \mathrm{pr}_\chi \circ E_{-\alpha}^{(j_1)} \circ E_{\sigma(\alpha)}^{(j+qj_1)} : V_\nu \to V_\chi \right)$$

because $(-1)^{j_1}(-1)^{j+qj_1} = (-1)^j \in k$. We have $E_{-\alpha}^{(j_1)} = 0$ unless $j_1 = 0$ because $\alpha \in \Delta^P$ and V is trivial on $R_{\mathrm{u}}(P)$. Hence, in the sum appearing in the above formula, only the case $j_1 = 0$ contributes. Furthermore, $E_{\sigma(\alpha)}^{(j)}(V_\nu) \subset V_{\nu + j\sigma(\alpha)}$. Hence, we have

$$\mathrm{Fil}_\delta^{\Xi, \mathbf{a}, \mathbf{r}} V_\nu = \bigcap_{j > q\langle \nu, \delta_\alpha \rangle} \mathrm{Ker}\left(E_{e_1 - e_2}^{(j)} : V_\nu \to V_{\nu + j(e_1 - e_2)} \right).$$

Take $\nu = \nu_i$ for some $0 \leq i \leq \lambda_1 - \lambda_2$. We deduce $\mathrm{Fil}_{\delta_\alpha}^{\Xi_\alpha, \mathbf{a}_\alpha, \mathbf{r}_\alpha} V_{\nu_i} = V_{\nu_i}$ if and only if for all $j \geq 0$ such that $j > q\langle \nu_i, \delta_\alpha \rangle$, one has $E_{e_1 - e_2}^{(j)}(V_{\nu_i}) = 0$. Computing explicitly the representation V, one sees that this space is zero if and only if the binomial coefficient $\binom{i}{j}$ is divisible by p. In particular, it is never zero for $j = i$. We deduce that

$$\mathrm{Fil}_{\delta_\alpha}^{\Xi_\alpha, \mathbf{a}_\alpha, \mathbf{r}_\alpha} V_{\nu_i} = V_{\nu_i} \iff i \leq q\langle \nu_i, \delta_\alpha \rangle.$$

Furthermore, we find

$$\langle \nu_i, \delta_\alpha \rangle = \frac{i(2q-1)}{q^2-1} + \frac{1}{q^2-1}(-q\lambda_1 + (q-1)\lambda_2 + \lambda_3).$$

For $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in X_{+,I}^*(T)$, we put

$$F(\lambda) = \frac{q}{q^2 - q + 1}(q\lambda_1 - (q-1)\lambda_2 - \lambda_3).$$

We deduce the following.

https://doi.org/10.1017/fms.2021.32 Published online by Cambridge University Press

Proposition 6.3.2. *We have*

$$H^0(G\text{-}\mathtt{Zip}^{\mu}, \mathcal{V}_I(\lambda)) = \bigoplus_{\substack{q|i,\ q+1|\lambda_2+i,\\ q^2-1|\lambda_1-i-q\lambda_3,\ i\geq F(\lambda)}} V_I(\lambda)_{\nu_i}. \tag{6.3.1}$$

(1) For example, take $\lambda = (1+q, 1, q)$. Then one sees that $V_I(\lambda)^{L_\varphi} = V_I(\lambda)_{\nu_q}$, where $\nu_q = (1, 1+q, q)$. One finds $F(\lambda) = q$; hence, $H^0(G\text{-}\mathtt{Zip}^{\mu}, \mathcal{V}_I(\lambda)) = V_I(\lambda)_{\nu_q}$.
(2) Similarly, take $\lambda = (1, 0, q)$. Then we find $V_I(\lambda)^{L_\varphi} = V_I(\lambda)_{\nu_0}$, where $\nu_0 = \lambda = (1, 0, q)$. We have $F(\lambda) = 0$; hence, again $H^0(G\text{-}\mathtt{Zip}^{\mu}, \mathcal{V}_I(\lambda)) = V_I(\lambda)_{\nu_0}$.
(3) Take $\lambda = (q+1, q+1, q^2+q)$. Then $V_I(\lambda)$ is a 1-dimensional representation of L (i.e., a character), and $V_I(\lambda)^{L_\varphi} = V_I(\lambda)$. Because $F(\lambda) = -\frac{q(q^2-1)}{q^2-q+1} < 0$, we have $H^0(G\text{-}\mathtt{Zip}^{\mu}, \mathcal{V}_I(\lambda)) = V_I(\lambda)$. It is spanned by the μ-ordinary (nonclassical) Hasse invariant H given by Lemma 6.3.1, also constructed in [GN17] and [KW18].

Recall the cone $C_{\mathrm{zip}} \subset X^*_{+,I}(T)$ studied in [Kos19], [GK18], defined as the set of $\lambda \in X^*(T)$ such that $H^0(G\text{-}\mathtt{Zip}^{\mu}, \mathcal{V}_I(\lambda)) \neq 0$. In this example, we deduce that it is the set of $\lambda \in X^*_{+,I}(T)$ such that there exists $0 \leq i \leq \lambda_1 - \lambda_2$ satisfying the four conditions listed below the direct sum sign of (6.3.1). For a cone $C \subset X^*(T)$, write $\langle C\rangle$ for the saturated cone of C; that is, the set of $\lambda \in X^*(T)$ such that $N\lambda$ lies in C for some positive integer N.

Corollary 6.3.3. *We have*

$$\langle C_{\mathrm{zip}}\rangle = \left\{(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{Z}^3 \mid \lambda_1 \geq \lambda_2,\ (q-1)\lambda_1 + \lambda_2 - q\lambda_3 \leq 0\right\}.$$

Proof. Assume that $\lambda \in C_{\mathrm{zip}}$. Then, in particular, $\lambda_1 - \lambda_2 \geq F(\lambda)$, which amounts to $(q-1)\lambda_1 + \lambda_2 - q\lambda_3 \leq 0$. Conversely, assume that $\lambda \in X^*_{+,I}(T)$ satisfies $\lambda_1 - \lambda_2 \geq F(\lambda)$. Then, after changing λ to $q(q^2-1)\lambda$, we find that $i = \lambda_1 - \lambda_2$ satisfies the four conditions below the direct sum sign of (6.3.1); hence, $\lambda \in \langle C_{\mathrm{zip}}\rangle$. This terminates the proof. □

Remark 6.3.4. The two sections of weight $(1+q, 1, q)$ and $(1, 0, q)$ given in (1) and (2) are partial Hasse invariants (viewing them as a section of the stack of zip flags $G\text{-}\mathtt{ZipFlag}^{\mu}$, their vanishing locus is a single flag stratum; see [Kos19, §1.3] for details). Their weights generate the cone $\langle C_{\mathrm{Sbt}}\rangle$ defined in [Kos19, Definition 1.7.1]. The cone $\langle C_{\mathrm{zip}}\rangle$ is not spanned by these weights because G does not satisfy the equivalent conditions of [Kos19, Lemma 2.3.1]. We also refer to [GIK21] for a general study of the cone C_{zip} as well as related results.

Acknowledgements. The authors thank the referee for helpful comments and suggestions. This work was supported by JSPS KAKENHI Grant Numbers 18F18311 and 18H01109.

Conflict of Interest: None.

References

[ABD+66] M. Artin, J. E. Bertin, M. Demazure, P. Gabriel, A. Grothendieck, M. Raynaud and J.-P. Serre, SGA3: Schémas en groupes [Group schemes], Vol. 1963/64 *(Institut des Hautes Études Scientifiques*, Paris, 1965/1966).

[Del79] P. Deligne, 'Variétés de Shimura: Interprétation modulaire, et techniques de construction de modèles canoniques' [Shimura varieties: Moduli interpretation and techniques of construction of canonical models], in *Automorphic Forms, Representations and L-functions*, Part 2, ed. by A. Borel and W. Casselman, Vol. **33** of *Proc. Symp. Pure Math.* (American Mathematical Society, Providence, RI, 1979), 247–289.

[Don85] S. Donkin, *Rational Representations of Algebraic Groups: Tensor Products and Filtration*, Vol. **1140** of *Lecture Notes in Mathematics* (Springer, Berlin, 1985).

[GIK21] W. Goldring, N. Imai and J.-S. Koskivirta, 'Weights of mod p automorphic forms and partial Hasse invariants', Preprint, 2021.

[GK18] W. Goldring and J.-S. Koskivirta, 'Automorphic vector bundles with global sections on G-Zip $^{\mathcal{Z}}$-schemes', *Compositio Math.* **154** (2018), 2586–2605.

[GK19a] W. Goldring and J.-S. Koskivirta, 'Strata Hasse invariants, Hecke algebras and Galois representations', *Invent. Math.* **217**(3) (2019), 887–984.

[GK19b] W. Goldring and J.-S. Koskivirta, 'Zip stratifications of flag spaces and functoriality', *IMRN* (**2019**) (12) (2019), 3646–3682.

[GN17] W. Goldring and M.-H. Nicole, 'The μ-ordinary Hasse invariant of unitary Shimura varieties', *J. Reine Angew. Math.* **728** (2017), 137–151.

[IK21] N. Imai and J.-S. Koskivirta, 'Partial Hasse invariants for Shimura varieties of Hodge-type', Preprint, 2021.

[Jan03] J. Jantzen, *Representations of Algebraic Groups*, Vol. **107** of *Math. Surveys and Monographs*, 2nd ed. (American Mathematical Society, Providence, RI, 2003).

[Kim18] W. Kim, 'Rapoport–Zink uniformization of Shimura varieties', *Forum Math. Sigma* **6** (2018), e16.

[Kis10] M. Kisin, 'Integral models for Shimura varieties of abelian type', *J. Amer. Math. Soc.* **23**(4) (2010), 967–1012.

[Kos18] J.-S. Koskivirta, Normalization of closed Ekedahl-Oort strata, *Can. Math. Bull.* **61**3 (2018), 572–587.

[Kos19] J.-S. Koskivirta, 'Automorphic forms on the stack of G-zips', *Results Math.* **74** (2019), no. 3, Paper No. 91, 52 pp.

[KW18] J.-S. Koskivirta and T. Wedhorn, 'Generalized μ-ordinary Hasse invariants', *J. Algebra* **502** (2018), 98–119.

[Kot84] R. E. Kottwitz, 'Shimura varieties and twisted orbital integrals', *Math. Ann.* **269** (1984), 287–300.

[Mil90] J. Milne, ' *Canonical models of (mixed) Shimura varieties and automorphic vector bundles*', in *Automorphic Forms, Shimura Varieties, and L-Functions*, Vol. I, ed. by L. Clozel and J. Milne, Vol. **11** of Perspect. Math. (Academic Press, Boston, 1990), 283–414.

[Moo04] B. Moonen, 'Serre-Tate theory for moduli spaces of PEL-type', *Ann. Sci. ENS* **37**(2) (2004), 223–269.

[MW04] B. Moonen and T. Wedhorn, 'Discrete invariants of varieties in positive characteristic', *IMRN* **72** (2004), 3855–3903.

[PWZ11] R. Pink, T. Wedhorn and P. Ziegler, 'Algebraic zip data', *Doc. Math.* **16** (2011), 253–300.

[PWZ15] R. Pink, T. Wedhorn and P. Ziegler, ' F-zips with additional structure', *Pacific J. Math.* **274**(1) (2015), 183–236.

[SYZ19] X. Shen, C.-F. Yu and C. Zhang, 'EKOR strata for Shimura varieties with parahoric level structure', Preprint, 2019, arXiv:1910.07785.

[Spr98] T. Springer, *Linear Algebraic Groups*, Vol. **9** of *Progress in Math.*, 2nd edn. (Birkhauser, Boston, MA, 1998).

[Urb14] E. Urban, 'Nearly overconvergent modular forms', in *Iwasawa Theory*, Vol. 7 of *Contrib. Math. Comput. Sci.* (Springer, Heidelberg, Germany, 2014), 401–441.

[Vas99] A. Vasiu, 'Integral canonical models of Shimura varieties of preabelian type', *Asian J. Math.* **3** (1999), 401–518.

[Wor13] D. Wortmann, 'The μ-ordinary locus for Shimura varieties of Hodge type', Preprint, 2013, arXiv:1310.6444.

[XZ17] L. Xiao and X. Zhu, 'Cycles on Shimura varieties via geometric Satake', Preprint, 2017, arXiv:1707.05700.

[XZ19] L. Xiao and X. Zhu, 'On vector-valued twisted conjugation invariant functions on a group', in *Representations of Reductive Groups*, Vol. **101** of *Proc. Sympos. Pure Math.* (American Mathematical Society, Providence, RI, 2019), 361–425. With an appendix by Stephen Donkin.

[Zha18] C. Zhang, 'Ekedahl-Oort strata for good reductions of Shimura varieties of Hodge type', *Can. J. Math.* **70**(2) (2018), 451–480.

https://doi.org/10.1017/fms.2021.32 Published online by Cambridge University Press