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SOME NEW PRODUCT THEOREMS IN SUMMABILITY

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ABSTRACT. Let A, B denote sequence-to-sequence matrix methods of summability and $A \cdot B$ the "dot" or iteration product defined by $(A \cdot B)x = A(Bx)$ for all sequences x for which this exists. Some inclusion relations are given involving the methods A, B, $A \cdot B$, $B \cdot A$ and the method defined by the matrix product AB. We take A, B to be of certain types whose products have not been studied extensively before, e.g. $H^* \cdot C_k$ or $C_k \cdot H^*$ where H^* is quasi-Hausdorff (and hence upper triangular) and C_k is a Cesàro matrix (which is lower triangular). The investigations show also a link between the "Product Property" $A \subset A \cdot B$ and the translativity properties of A and B.

Section 1. In what follows, k will always denote an integer $\geqq 0$, and C_k will denote the Cesàro matrix of order k. For $0 < \alpha < 1$, the Taylor matrix T_α and the Meyer-König matrix S_α are defined by

$$(1) \qquad [T_\alpha]_{nm} = \binom{m}{n}(1 - \alpha)^{n+1}\alpha^{m-n}$$

$$(2) \qquad [S_\alpha]_{nm} = \binom{n + m}{m}(1 - \alpha)^{n+1}\alpha^{m}$$

for n, $m = 0, 1, \ldots$. If $\{\mu_n\}_{n=0}^{\infty}$ is a sequence of numbers, then the quasi-Hausdorff matrix $H^* = (H^*, \mu)$ and the Meyer-König-Ramanujan matrix $S^* = (S^*, \mu)$ are defined by

$$(3) \qquad [H^*]_{nm} = \begin{cases} \binom{m}{n}\Delta^{m-n}\mu_n & \text{if } m \geqq n, \\ 0 & \text{if } m < n \end{cases}$$

and

$$(4) \qquad [S^*]_{nm} = \binom{n + m}{n}\Delta^{m}\mu_n$$

respectively, for n, $m = 0, 1, \ldots$. Here $\Delta\mu_n = \mu_n - \mu_{n+1}$, $\Delta^0\mu_n = \mu_n$ and

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$\Delta^{m+1}\mu_n = \Delta(\Delta^m\mu_n)$. If we take $\mu_n = (1 - \alpha)^{n+1}$, then (3) yields (1) and (4) yields (2).

Given a sequence $s = \{s_n\}_{n=0}^{\infty}$ and an integer $k \geqq 0$, we define the kth left translate L_ks and the kth right translate R_ks of s as follows:

$$L_ks = \{s_{n+k}\}_{n=0}^{\infty} \text{ and } R_ks = \{s_{n-k}\}_{n=0}^{\infty} \text{ with } s_i = 0 \text{ if } i < 0.$$

A summability method is said to be left-translative [right translative] if it sums L_1s [respectively R_1s] whenever it sums the sequence s; the method is said to be translative if it is both left and right translative.

2. **Lemmas and Theorems.**

We begin with three lemmas which form the basis of our theorems.

LEMMA 1. (Meyer-König [1]: Satz 8, Satz 10).
(a) T_α *is right-translative for* $0 < \alpha < 1$.
(b) T_α *is left-translative if and only if* $1/2 < \alpha < 1$.
(c) S_α *is translative for* $0 < \alpha < 1$.

LEMMA 2. (Meyer-König [2]). *Let* $k \geqq 0$ *be an integer.*
(a) *If* $T_\alpha s$ *exists, then*

$$C_k(T_\alpha s) = (C_kT_\alpha)s = L_k[T_\alpha R_k(C_ks)]. \tag{5}$$

(b) *If* $S_\alpha s$ *exists, then*

$$C_k(S_\alpha s) = (C_kS_\alpha)s = L_k[S_\alpha(C_ks)]. \tag{6}$$

Thus $C_k \cdot T_\alpha \approx C_kT_\alpha$ *and* $C_k \cdot S_\alpha \approx C_kS_\alpha$ *for all sequences to which* T_α, S_α *applies, respectively.*

LEMMA 3. (Parameswaran [5]). *If* k *is a positive integer and* H^*, S^* *are, respectively, a conservative quasi-Hausdorff matrix and a conservative Meyer-König-Ramanujan matrix, then* $C_kH^* = H^{*(k)}C_k$ *and* $C_kS^* = S^{*[k]}C_k$ *where* $[H^{*(k)}]_{n,m} = [H^*]_{n+k,m+k}$ *and* $[S^{*[k]}]_{n,m} = [S^*]_{n+k,m}$ *for* $n, m = 0, 1, \ldots$

THEOREM 1. *For each integer* $k \geqq 0$ *and* $0 < \alpha < 1$,

$$C_k \subset C_k \cdot T_\alpha \approx C_kT_\alpha.$$

PROOF. The assertions follow from (5), the lefthand one by observing that $T_\alpha s$ exists whenever $C_ks \in (c)$ (see [1], p. 263).

REMARK: If A, B are abritrary regular matrices, it is not easy to describe the sequences s for which Bs and $A(Bs)$ will even exist; however if one considers only bounded sequences, then Bs and $A(Bs)$ will not only exist (even if A, B are assumed to merely satisfy the row-norm condition) but they will satisfy also the relations $A \cdot Bs = A(Bs) = (AB)s$; i.e. the product method $A \cdot B$ and the method AB defined by the matrix product of A and B are identical, if we consider only bounded sequences.

THEOREM 2. *Let* H^* *be a conservative quasi-Hausdorff matrix and* S^* *a conservative Meyer-König-Ramanujan matrix. Let* $\kappa, \lambda > 0$ *and let* k *be a positive integer. Then*

(a) $C_\kappa, H^* \subset C_\lambda H^* \approx H^*C_k$ *for bounded sequences, and*

(b) $C_\kappa, S^* \subset C_\lambda S^* \approx S^*C_k$ *for bounded sequences.*

PROOF OF (a): For bounded sequences we have $C_\kappa \approx C_k \subset H^*C_k$ and $H^* \subset C_\lambda H^* \approx C_kH^*$. Hence it is enough to prove that $C_kH^* \approx H^*C_k$ for bounded sequences. For those sequences s we have, by use of Lemma 3, that

$$(7) \qquad (C_kH^*)s = (H^{*(k)}C_k)s = H^{*(k)}(C_ks) = L_k[H^*R_k(C_ks)].$$

Now H^* is translative for bounded sequences ([4], Theorem 7.2). Hence, for bounded sequences s, $H^*R_k(C_ks) \in (c)$ holds if and only if $H^*(C_ks) = (H^*C_k)s \in (c)$, and thus, by (7), $(C_kH^*)s \in (c)$ holds if and only if $(H^*C_k)s \in (c)$.

PROOF OF (b): In the above proof of part (a), if we write S^* instead of H^* and omit the symbol R_k whenever it occurs then part (b) will stand proved.

Note from its proof that the essence of Theorem 2 in fact is that

$$(E): \qquad \left.\begin{aligned} C_kH^* &\approx H^*C_k \\ C_kS^* &\approx S^*C_k \end{aligned}\right\} \begin{aligned} &\text{for bounded sequences} \\ &\text{and positive integer } k. \end{aligned}$$

The theorems below show that in the special cases $H^* = T_\alpha$, $S = S_\alpha$ we can improve on (E) by (i) proving it for a wider class of sequences and (ii) proving a sharper result for bounded sequences.

THEOREM 3. *Let* k *be a positive integer and* $0 < \alpha < 1$. *Then*

(i) (a) $C_k \cdot T_\alpha \supset T_\alpha \cdot C_k$

(b) $C_k \cdot T_\alpha \approx T_\alpha \cdot C_k$ if $1/2 < \alpha < 1$

for all sequences to which T_α *is applicable*;

(c) $T_\alpha \not\subset T_\alpha \cdot C_k$ if $0 < \alpha \leqq 1/2$.

(ii) $C_k \cdot S_\alpha \approx S_\alpha \cdot C_k$ *for all sequences to which* S_α *is applicable.*

PROOF. (i) (a): Suppose that $T_\alpha s$ exists and that $T_\alpha(C_ks) \in (c)$. Then $T_\alpha R_k(C_ks) \in (c)$ by Lemma 1 (a) and $C_k(T_\alpha s) \in (c)$ by (5).

(i) (b): Let $C_k(T_\alpha s) \in (c)$. Then (5) yields $T_\alpha R_k(C_ks) \in (c)$. By Lemma 1 (b) then $T_\alpha(C_ks) \in (c)$.

(i) (c): For the case $k = 1$, a statement equivalent to part (c) of the theorem was proved by Meyer-König and Zeller [3]; the following is based on the ideas used by them there. We choose a sequence t such that $T_\alpha t = u$, where

$$u = \left\{ \left(\frac{1-\alpha}{-\alpha} \right)^n \right\}$$

and then define the sequence s by the relation $t = C_k s$. Then

$$C_k(T_\alpha s) = L_k[T_\alpha R_k(C_k s)] = \underline{0} = (0, 0, \dots) \tag{8}$$

since the sequence $w = T_\alpha R_k t = (Z_\alpha)^k (T_\alpha t)$ where Z_α is a "Zweierverfahren" (see [3], p. 301; [6], Section 62) and $w_{n+k} = 0$ for $n = 0, 1, \dots$. From (8) we see that $T_\alpha s = \underline{0} \in (c)$. But $T_\alpha(C_k s) = T_\alpha t = u \notin (c)$. This proves (i) (c).

(ii) Let s be such that $S_\alpha s$ exists. Then $C_k(S_\alpha s) = L_k S_\alpha(C_k s)$ by (6) and hence $C_k(S_\alpha s) \in (c)$ if and only if $S_\alpha(C_k s) \in (c)$.

The following theorem supplements the equivalence (i) (b) in Theorem 3 for the range $0 < \alpha \leqq 1/2$, necessarily for a restricted class of sequences. A sequence (s_n) is called of finite order if $s_n = 0(n^r)$ for some r.

THEOREM 4. *Let $k \geqq 0$ be an integer and* $0 < \alpha \leqq 1/2$. *Then* $C_k \cdot T_\alpha \approx T_\alpha \cdot C_k$ *for all sequences of finite order.*

PROOF. In view of Theorem 3 (i) (a) we need only prove that $C_k \cdot T_\alpha \subset T_\alpha \cdot C_k$ for sequences of finite order. Now, for these sequences $T_\alpha s$ exists and thus, by Lemma 2 (a), $C_k(T_\alpha s) \in (c)$ implies $T_\alpha R_k(C_k s) \in (c)$. As $C_k s$ is of finite order, too, the series $\sum_{n=0}^{\infty} b_n z^n$, where $b_n = t_n - t_{n-1}$, $t = R_k(C_k s)$ has at least 1 as radius of convergence and hence is regular at $z = \alpha$. Hence (by [1], Satz 8) T_α is translative for the sequence t and therefore $T_\alpha(C_k s) \in (c)$.

THEOREM 5. *Let* $0 < \alpha, \beta, \gamma, \delta < 1$ *and* $\kappa, \lambda, \mu > 0$. *Then*

$$T_\alpha \approx S_\beta \subset C_\kappa \approx C_\lambda \cdot T_\gamma \approx T_\delta \cdot C_\mu \approx C_\lambda \cdot S_\gamma \approx S_\delta \cdot C_\mu$$

for bounded sequences.

PROOF. Observe that matrix products may stand for the dot products throughout. It is well known ([1], Satz 25) that $T_\alpha \approx S_\beta \approx B \subset C_\kappa$ (B = Borel's method) for bounded sequences. Now, $C_\kappa \approx C_1 \subset T_\gamma C_1 \approx C_1 T_\gamma$ (by Theorem 2 (a)) and $C_1 T_\gamma \approx C_\lambda T_\gamma \subset C_\lambda C_\kappa \approx C_\kappa$ for bounded sequences. Also $C_\kappa \approx C_\lambda \subset T_\gamma C_\lambda \subset C_\kappa C_\lambda \approx C_\kappa$ for bounded sequences. These relations prove the theorem for Taylor methods. The proof for the Meyer-König methods is similar.

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