Can. J. Math., Vol. XXXV, No. 6, 1983, pp. 961-973

ON LARGE INDUCTIVE DIMENSION OF PROXIMITY SPACES

A. KANDIL

Introduction. The notion of proximity spaces was introduced by Efremovic in [**2**, **3**]. An analysis of proximity spaces was carried out by Smirnov in [**5**].

The study of covering dimension of proximity spaces was originated by Smirnov in [**6**].

In this paper we introduce the concept of δ-large inductive dimension of proximity spaces and study some of its properties.

1. Definitions and basic concepts.

Definition 1. [**5**] A *proximity space* or (δ-*space*) is a pair (X, δ) where X is a set and δ is a mapping from $2^X \times 2^X$ into the set $\{0, 1\}$ satisfying the following axioms:

1. $\delta(A, B) = \delta(B, A) \ \forall \ A, B \in 2^X$.
2. $\delta(A, B \cup C) = \delta(A, B)\, \delta(A, C) \ \forall \ A, B, C \in 2^X$.
3. $\delta(\{x\}, \{y\}) = 0 \Leftrightarrow x = y$.
4. $\delta(X, \emptyset) = 1$.
5. $\delta(A, B) = 1 \Rightarrow \exists \ C, D \in 2^X \ni C \cup D = X$ and $\delta(A, C) \cdot \delta(B, C) = 1.$

Remarks. 1. 2^X denotes the family of all subsets of the set X.

2. The mapping δ in definition 1 is called a *proximity on* X.

3. If $\delta(A, B) = 0$ then we say that the sets A and B are *near*. And if $\delta(A, B) = 1$ then we say that the sets A and B are *far* or *remote*.

The following properties of δ-spaces were proved in [**5**]:

P_1: Every proximity δ on a set X induces a topology τ_δ on X; the formula

$$[A] = \{x \in X : \delta(\{x\}, A) = 0\} \ \forall \ A \in 2^X$$

defines the closure operator on X.

Received April 28, 1981 and in revised form March 1, 1982 and August 25, 1982.

The topological space (X, τ_δ) is a completely regular T_1-space or $T_{3\frac{1}{2}}$-space.

Remark. All topologies considered will be "Tychonoff" that is, completely regular and T_1, otherwise known as $T_{3\frac{1}{2}}$.

P_2: For every completely regular T_1-space (X, τ) there exists at least one proximity δ on the set X such that $\tau_\delta = \tau$.

In this paper we shall consider only the proximities δ on the topological space (X, τ) for which $\tau_\delta = \tau$.

P_3: For every compact T_2-space (X, τ) the proximity δ on X defined by

$$\delta(A, B) = 1 \Leftrightarrow [A] \cap [B] = \emptyset$$

is the unique proximity on X for which $\tau_\delta = \tau$.

Remark. Here $[A]$ denotes the closure of A.

Definition 2. [**4**] Let (X, δ) be a δ-space; then we say that a set B is a *δ-neighbourhood of the set A* and we write $B \Supset A$ if $\delta(A, X \setminus B) = 1$. The family $\alpha \subseteq 2^X$ is called a *δ-family* if

$$\forall\, A \in \alpha\ \exists\, B \in \alpha \ni A \Supset B.$$

A maximal δ-family satisfying the finite intersection property is called a δ-end on X [**4**]. The set of all δ-ends on X is denoted by CX.

P_4: For every δ-space (X, δ) the family

$$\beta_X = \{O_H : H \in \tau_\delta\},$$

is a base for some topology τ_{CX} on CX where

$$O_A = \{\zeta \in CX : A \in \zeta\}.$$

Moreover the topological space (CX, τ_{CX}) is a compactification for (X, τ_δ) generating the proximity δ on X as follows:

$$\delta(A, B) = 1 \Leftrightarrow [A]_{CX} \cap [B]_{CX} = \emptyset.$$

The compact space (CX, τ_{CX}) is called the Smirnov compactification of the space (X, τ_δ).

P_5: For every $T_{3\frac{1}{2}}$-space (X, τ) there exists a one-to-one correspondence between all compactifications of (X, τ) and all proximities δ on X for which $\tau = \tau_\delta$.

P_6: The operator O_H defined in P_4 above satisfies the following properties:

I. $O_{A \cap B} = O_A \cap O_B \forall A, B \in 2^X$.
II. $O_{\cup A_\lambda} \supseteq \cup_\lambda O_{A_\lambda} \forall \{A_\lambda\} \subseteq 2^X$.
III. $\delta(X \setminus A, X \setminus B) = 1 \Rightarrow O_A \cup O_B = CX$.
IV. $X \cap O_A = A^\circ \forall A \in 2^X$.

(A° denotes the interior of the set A.)

V. $O_A \in \tau_{CX} \forall A \in 2^X$.
VI. $O_{A^\circ} = O_A \forall A \in 2^X$.
VII. $[B]_{CX} = CX \setminus O_{(X \setminus B)} \forall B \in 2^X$.
VIII. $[O_H]_{CX} = [H]_{CX} \forall H \in \tau_\delta$.
IX. $H \Supset F$ if and only if $O_H \supseteq [F]_{CX} \forall H, F \in 2^X$.

P_7: Let (X, τ) be a $T_{3\frac{1}{2}}$-space, then the proximity δ_β on X is defined by:

$$\delta_\beta(A, B) = 1 \text{ if and only if } A \text{ and } B \text{ are functionally separated.}$$

It is the finest proximity δ on X for which $\tau = \tau_\delta$. The Smirnov compactification CX in this case coincides with the greatest compactification βX of X. (X, δ_β) will be called a fine δ-space.

P_8: If a subspace F of a fine δ-space (X, δ) has the property that every continuous function $f:F \to I$ is extendable over X, then $(F, \delta/F)$ is a fine δ-space.

Definition 3. We say that the proximity space (X, δ) is *perfect* if and only if

$$[Fr_X H]_{CX} = Fr_{CX} O_H \forall H \in \tau_\delta.$$

Here $Fr_X H$ denotes the boundary of H in X.

The following example shows that not every proximity space is perfect.

Example. Let $\mathbf{R}$ be the real line with the usual topology and let bR be its Alexandrov Compactification. Then the pair $(bR, \mathbf{R})$ defines on $\mathbf{R}$ the following proximity δ; For $A, B \subseteq R$

$$\delta(A, B) = 1 \quad \text{if and only if} \quad [A]_{bR} \cap [B]_{bR} = \emptyset.$$

The proximity space (R, δ) is not perfect. Indeed, if $H = (0, \infty)$ then it is easy to see that

$$[Fr_R\ H]_{bR} \neq Fr_{bR} O_H.$$

LEMMA 1. *The proximity space* (X, δ) *is perfect if and only if for every two disjoint open sets* $H_1, H_2 \in \tau_\delta$ *we have*;

$$O_{H_1 \cup H_2} = O_{H_1} \cup O_{H_2}.$$

Proof. Let (X, δ) be a perfect δ-space; and let $H_1, H_2 \in \tau_\delta$, $H_1 \cap H_2 = \emptyset$. To prove that $O_{H_1 \cup H_2} = O_{H_1} \cup O_{H_2}$, it suffices by P_6 (II), P_6 (VI) and complementation to prove that if $F_1, F_2 \in \tau_\delta^c$ and $F_1 \cup F_2 = X$ then

$$[F_1 \cap F_2]_{CX} \supseteq [F_1]_{CX} \cap [F_2]_{CX}.$$

Suppose $\zeta \in [F_1]_{CX} \cap [F_2]_{CX}$ but $\zeta \notin [F_1 \cap F_2]_{CX}$. Then

$$\zeta \in Fr_{CX}([F_1]_{CX}),$$

otherwise ζ has a neighbourhood V (in CX) such that

$$V \subseteq [F_1]_{CX} \quad \text{and} \quad V \cap F_1 \cap F_2 = \emptyset.$$

But $V \cap F_2$ is a non-empty subset of X contained in $[F_1]_{CX} \cap X = F_1$, contradicting $V \cap F_1 \cap F_2 = \emptyset$. Thus, applying the perfectness of (X, δ) to $H = X \setminus F_1$, we have

$$\zeta \in [Fr_X \cap F_1]_{CX}.$$

But $Fr_X\, F_1 \subseteq F_1 \cap F_2$, proving $\zeta \in [F_1 \cap F_2]_{CX}$, a contradiction.

Conversely, assume the condition of the lemma. Let $H \in \tau_\delta$ and let $H^* = X \setminus [H]$. Then it is clear that

$$FrH = X \setminus (H \cup H^*).$$

Consequently,

$$(1) \qquad [Fr\ H]_{CX} = [X \setminus (H \cup H^*)]_{CX} = CX \setminus O_{H \cup H^*}$$
$$= CX \setminus (O_{H^*} \cup O_H).$$

Moreover,

$$CX \setminus [O_H]_{CX} = CX \setminus [[H]]_{CX} = O_{H^*}$$

i.e.,

$$(2) \qquad Fr_{CX} O_H = CX \setminus (O_H U O_{H^*}).$$

From 1, 2 we have;

$$[Fr\ H]_{CX} = Fr_{CX} O_H.$$

https://doi.org/10.4153/CJM-1983-052-7 Published online by Cambridge University Press

COROLLARY 1. *Every compact proximity space (X, δ) is perfect.*

Proof. The proof is immediate if we note that

$$O_H = H^\circ \ \forall\ H \in 2^X.$$

LEMMA 2. *A δ-space (X, δ) is perfect if, and only if, $F \subseteq H$ and $\delta(F, FrH) = 1$ imply $H \supset F\ \forall\ H \in \tau_\delta$, $F \in \tau_\delta^c$. (τ_δ^c denotes the family of all closed subsets of the δ-space (X, δ).)*

Proof. Assume that (X, δ) is perfect. Let $H \in \tau_\delta$ and $F \in \tau_\delta^c$ be such that $\delta(F, FrH) = 1$. Then, $X \setminus FrH \supset F$ i.e.,

$$O_{(X \setminus FrH)} \supseteq [F]_{CX}$$

(see P_6 IX). Now, using P_6 IX, it is sufficient to show that $O_H \supseteq [F]_{CX}$. Assume the contrary: i.e.,

$$(1) \quad O_H \not\supseteq [F]_{CX}.$$

From the condition $F \subseteq H$ we have

$$(2) \quad [F]_{CX} \subseteq [O_H]_{CX}.$$

From 1, 2 we have;

$$(3) \qquad [F]_{CX} \cap Fr_{CX}O_H \neq \emptyset.$$

Since (X, δ) is perfect,

$$Fr_{CX}O_H = [Fr_X\, H]_{CX}.$$

And since

$$O_{(X \setminus FrH)} = CX \setminus [FrH]_{CX} \quad [\text{See } P_6 \text{ VII}],$$

hence

$$(4) \qquad O_{(X \setminus FrH)} = CX \setminus Fr_{CX}O_H.$$

From (3), (4) we have

$$[F]_{CX} \not\subseteq O_{(X \setminus FrH)}$$

which contradicts the assumption;

$$[F]_{CX} \subseteq O_{(X \setminus FrH)}.$$

ii) Conversely, assume that the condition of the lemma is satisfied, i.e., $\forall\ H \in \tau_\delta$ and $F \in \tau_\delta^c$ such that $F \subseteq H$ and $\delta(F, FrH) = 1$ we have $H \supset F$.

Let $H \in \tau_\delta$, then it is clear that

$$[FrH]_{CX} \subseteq Fr_{CX}\, O_H.$$

Now, let $\zeta \in Fr_{CX}\, O_H$. Then, $\zeta \in [O_H]_{CX} \backslash O_H$. And hence

$$O_H \cap O_{H^*} = O_{H \cap H^*} \neq \emptyset \ \forall\, H^* \in \zeta.$$

Consequently

(5) $\quad [O_H \cap O_{H^*}]_{CX} = [\,[H \cap H^*]_X]_{CX} \ \forall\, H^* \in \zeta$

and

(6) $\quad \zeta \notin O_H.$

We need to prove that $\zeta \in [FrH]_{CX}$. In fact if $\zeta \notin [FrH]_{CX}$ then there exists $H_\circ \in \zeta$ such that

(7) $\quad X \setminus FrH \supseteq [H_\circ]_X.$

From 5 we have

(8) $\quad \zeta \in [\,[H \cap H_\circ]_X]_{CX}.$

And from 7 we have

(9) $\quad [H_\circ]_X \subseteq H \cup (X \setminus [H]_X).$

But (9) implies that

$$[H_\circ] \cap H \in \tau_\delta^c.$$

Thus we have

(10) $\quad [H \cap H_\circ]_X \subseteq [H_\circ]_X \cap H \subseteq H.$

Moreover, we have

$$[H \cap H_\circ]_X \subseteq [H_\circ]_X \subseteq X \setminus FrH.$$

Consequently

$$[H_\circ \cap H]_X \Subset X \setminus FrH.$$

From 10, 11 and the condition of the lemma we have

$$H \supseteq [H \cap H]_X.$$

Thus

(12) $\quad [\,[H_\circ \cap H]\,]_{CX} \subseteq O_H.$

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Therefore from (8) and (12) we have

$$\zeta \in O_H$$

which contradicts condition (6).

This complete the proof of the lemma.

COROLLARY 1. *Every fine* δ*-space* (X, δ_β) *is perfect.*

Proof. Let $H \in \tau_{\delta_\beta}$ and let F be an arbitrary closed subset of X such that $F \subseteq H$ and $\delta_\beta (F, FrH) = 1$.

From the definition of δ_β, there exists a continuous real function, $f:X \to [0, 1]$ such that; $f(x) = 0 \ \forall \ x \in F$ and $f(x) = 1 \ \forall \ x \in FrH$.

Introduce a function $g:X \to [0, 1]$ as follows:

$$g(x) = f(x) \quad \forall \ x \in [H]_X \quad \text{and,}$$
$$g(x) = 1 \quad \forall \ x \in X \setminus H.$$

It is easy to see that g is continuous and separates the two sets F and $X \setminus H$. I.e., $H \supsetneq F$.

Thus from the above lemma it follows that (X, δ_β) is perfect.

Definition 4. A proximity space (X, δ) is called a *semicompact* δ*-space* if and only if the following condition is satisfied:

For $A, B \in \tau_\delta^c$, $\delta(A, B) = 1$ if and only if there exists an open subset H of X with compact boundary such that

$$A \subseteq H \subseteq [H] \subseteq X \setminus B.$$

PROPOSITION 1. [7]. I. *Every semicompact* δ*-space* (X, δ) *has a basis of open sets with compact boundaries.*

II. *Every* $T_{3\frac{1}{2}}$*-space* (X, τ) *having a basis of open sets with compact boundaries induces a semicompact* δ*-space* (X, δ) *by the following proximity relation*;

For $A, B \subseteq X$, $\delta(A, B) = 1$ *if and only if there exists an open subset* H *of* X *with compact boundary such that*

$$A \subseteq H \subseteq [H] \subseteq X \setminus B.$$

LEMMA 3. *If* (X, δ) *is a semicompact* δ*-space and* σ *is the family of all open subsets of* X *with compact boundaries, then for every closed subset* F *of* X *contained in some element* H *from* σ, *we have* $H \supsetneq F$.

Proof. This follows immediately from Definition 4, and Proposition 1 (I).

COROLLARY 1. *Every semicompact δ-space (X, δ) is perfect.*

Proof. Let σ be the family of all open subsets of X with compact boundaries, and H be an arbitrary open subset of X. Assume that F is a closed subset of X for which $F \subseteq H$ and $\delta(F, FrH) = 1$. Then there exists $H^* \in \sigma$ such that

$$FrH \subseteq H^* \subseteq [H^*] \subseteq X \setminus F.$$

From above we have $Fr(H \setminus [H^*])$ is compact, indeed;

$$\begin{aligned} Fr(H \setminus [H^*]) &= Fr(X \setminus [H^*]) \cup (X \setminus H)) \\ &= Fr[H^* \cup (X \setminus H)] \\ &\subseteq Fr(H^* \cup (X \setminus H)) \\ &\subseteq Fr\, H^* \cup Fr\, H \\ &\subseteq [H^*]. \end{aligned}$$

It is easy to see that

$$Fr(H \setminus [H^*]) \cap H^* = \emptyset.$$

Thus

$$Fr(H \setminus [H^*]) \subseteq Fr\, H^*.$$

Consequently, $H \setminus [H^*] \in \sigma$. Since $F \subseteq H \setminus [H^*] \subseteq H$, from Lemma 3 we have

$$H \supseteq H \setminus [H^*] \Supset F.$$

Therefore $H \Supset F$. It follows from Lemma 2 that (X, δ) is perfect.

Definition 5. A perfect δ-space (X, δ) is called a *strongly perfect δ-space* (or *S-perfect δ-space*) if every closed subspace of (X, δ) is perfect.

Using the following properties:

1) Every closed subspace of a compact space is compact.
2) Every closed subspace F of a normal fine δ-space is a normal fine δ-space (see P_8).

The following statement may be easily proved.

LEMMA 4. *Every compact δ-space, and every normal fine δ-space are S-perfect δ-spaces.*

PROPOSITION 2. **[6]**. *Every δ-space is homeomorphic with a closed subset of a fine δ-space.*

From Proposition 2, Example 1 and Corollary 1 of Lemma 2 we deduce that not every perfect δ-space is S-perfect.

Definition 6. Let (X, δ) be a δ-space. Then we say that the set $L \subseteq X$ is a *δ-partition between A and B* if there exist open sets $U, W \subseteq X$ such that;

$$U \supset A,\ W \supset B,\ U \cap W = \emptyset \text{ and } U \cup W = X \setminus L.$$

It is clear that if L is a δ-partition between A and B then

$$\delta(A \cup B, L) = 1 = \delta(A, L) = \delta(B, L).$$

LEMMA 5. *Let (X, δ) be a δ-space and let F_1 and F_2 be two of its closed subsets. If $\delta(F_1, F_2) = 1$ and ψ^* is a partition between $[F_1]_{CX}$, $[F_2]_{CX}$ in CX (in the topological sense* [**1**]) *then;*

$$\psi = \psi^* \cap X \text{ is a } \delta\text{-partition between } F_1, F_2 \text{ in } X.$$

Proof. This is clear.

LEMMA 5. *If the closed subset ψ of a perfect proximity space (X, δ) is a δ-partition between two closed far sets $F_1, F_2 \subseteq X$ then $[\psi]_{CX}$ is a partition between $[F_1]_{CX}$ and $[F_2]_{CX}$ in CX.*

Proof. Let ψ be a δ-partition between F_1 and F_2; then by definition there exist U_1 and $U_2 \in \tau_\delta$ such that:

$$X \setminus \psi = U_1 \cup U_2,\ U_1 \cap U_2 = \emptyset \text{ and } U_i \supset F_i,\ i = 1, 2.$$

Let

$$\psi_i = \psi \cup U_i, \quad i = 1, 2.$$

Then

$$\psi_i \in \tau_\delta^c \quad \text{and}$$

$$\delta(\psi_1, F_2) = \delta(\psi, F_1) = 1.$$

Consequently

$$[F_1]_{CX} \subseteq CX \setminus [\psi_2]_{CX} = O_{U_1},$$

$$[F_2]_{CX} \subseteq CX \setminus [\psi_1]_{CX} = O_{U_2}.$$

Since $U_2 \cap U_i = \emptyset$, then

$$O_{U_1} \cap O_{U_2} = O_{U_1 \cap U_2} = \emptyset.$$

Now from $X \setminus \psi = U_1 \cup U_2$, (X, δ) is perfect and P_6 VIII we have

$$CX \setminus [\psi]_{CX} = O_{(X \setminus \psi)} = O_{(U_1 \cup U_2)} = O_{U_1} \cup O_{U_2},$$

i.e., $C\Psi$ is a partition between CF_1 and CF_2 in CX, where $C\Psi = [\Psi]_{CX}$, [**5**].

2. Definition and basic properties of the dimension δ Ind in δ-spaces.

Definition 7. To every δ-space X one assigns the δ-large inductive dimension of X denoted by δ-Ind X, which is a natural number or -1 or ∞. The definition of δ Ind X consists of the following conditions:

1_1. δ Ind $X = -1$ if and only if $X = \emptyset$.

1_2. δ Ind $X \leqq n$ where $n = 0, 1, 2, \ldots.$ if for every two closed far sets A, $B \subseteq X$ there exists a δ-partition L between A and B such that;

$$\delta \text{ Ind } L \leqq n - 1.$$

1_3. δ Ind $X = n$ if and only if δ Ind $X \leqq n$ and δ Ind $X > n - 1$.

1_4. δ Ind $X = \infty$ if and only if δ Ind $X > n$ for $n = -1, 0, 1, 2, \ldots.$.

THEOREM 1. *For every δ-space X we have*

$$\delta \text{ Ind } X \leqq \text{Ind } CX.$$

Proof. We shall apply induction with respect to Ind CX. If Ind $CX = -1$ then $CX = \emptyset = X$ and our inequality holds. Assume that the inequality holds for all δ-spaces X with Ind $CX < n$ for some $n \geqq 0$, and consider a δ-space X such that Ind $CX = n$.

Let F_1 and F_2 be far closed sets in X.

Then the sets CF_1 and CF_2 are disjoint in CX so that there exists a partition $\tilde{\psi}$ in CX between CF_1 and CF_2, such that Ind $\tilde{\psi} \leqq n - 1$.

From Lemma 5 we have $\psi = \tilde{\psi} \cap X$ is a δ-partition in X between F_1 and F_2. Since $C\psi = [\psi]_{CX}$ it follows from Theorem 2.2.1 [**1**] and the inductive assumption that

$$\delta \text{ Ind } \psi \leqq n - 1,$$

so that

$$\delta \text{ Ind } X \leqq n = \text{Ind } CX.$$

THEOREM 2. *For every S-perfect δ-space X we have*

$$\delta \text{ Ind } X = \text{Ind } CX.$$

Proof. From Theorem 1 it suffices to show that

$$\text{Ind } CX \leqq \delta \text{ Ind } X.$$

As in the proof of Theorem 1 we shall suppose that δ Ind $X < \infty$ and apply induction with respect to δ Ind X.

Our inequality holds if δ Ind $X = -1$.

Assume that the inequality is proved for all S-perfect δ-spaces with dimension δ Ind less than $n \geqq 0$, and consider an S-perfect δ-space X such that δ Ind $X = n$. Let $\widetilde{F}_1$ and $\widetilde{F}_2$ be disjoint closed sets in CX. Then there exist open sets $\widetilde{V}_1, \widetilde{V}_2 \subseteq CX$ such that

$$\widetilde{F}_i \subseteq \widetilde{V}_i, \quad i = 1, 2 \quad \text{and}$$

$$[\widetilde{V}_1]_{CX} \cap [\widetilde{V}_2]_{CX} = \emptyset.$$

The sets $V_i = [\widetilde{V}_i]_{CX} \cap X$ are closed in X and far, so that there exists a δ-partition ψ in X between V_1 and V_2 such that δ Ind $\leqq n - 1$.

From Lemma 6 the set $C\psi$ is a partition between CV_1 and CV_2 in CX. And from the induction assumption we have Ind $C\psi \leqq n - 1$.

Since

$$\widetilde{F}_i \subseteq [\widetilde{V}_i]_{CX}$$

then $C\psi$ is a partition between F_1 and F_2; consequently

$$\text{Ind } CX \leqq \delta \text{ Ind } X.$$

Corollary 1. *For every compact proximity space X the topological* Ind X *coincides with* δ Ind X.

Proof. This is immediate from Theorem 2 and Corollary 1 of Lemma 1.

Corollary 2. *Every normal fine δ-space X has*

$$\delta \text{ Ind } X = \text{Ind } \beta X.$$

Proof. This follows immediately from Theorem 2 and Lemma 4.

Corollary 3. *If X is an S-perfect δ-space and M is a closed subset of X, then*

$$\delta \text{ Ind } M \leqq \delta \text{ Ind } X.$$

Proof. From Definition 4 and the above theorem we have

$$\delta \text{ Ind } M = \text{Ind } CM = \text{Ind } [M]_{CX} \leqq \text{Ind } CX = \delta \text{ Ind } X.$$

Corollary 4. *For every S-perfect δ-space we have*

$$\delta dX \leqq \delta \text{ Ind } X,$$

where δdX is the covering dimension of (X, δ), (*see* [**5**]).

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Proof. From Theorem 1 in [**6**] we have

$$\delta dX = \dim CX.$$

From Theorem 2 we have δ Ind X = Ind CX, and from Theorem 3.1.28 in [**1**] we have dim $CX \leqq$ Ind CX. Thus we have $\delta dX \leqq \delta$ Ind X for every S-perfect space.

COROLLARY 5. *If* (X, δ) *is an* S*-perfect proximity space, and* A, B *are closed subsets of* (X, δ), *then,*

$$\delta \operatorname{Ind} (A \cup B) \leqq \delta \operatorname{Ind} A + \delta \operatorname{Ind} B + 1.$$

Proof.

$$\begin{aligned} \delta \operatorname{Ind} (A \cup B) &= \operatorname{Ind} [A \cup B]_{CX} = \operatorname{Ind} [A]_{CX} \cup [B]_{CX} \\ &\leqq \operatorname{Ind} [A]_{CX} + \operatorname{Ind} [B]_{CX} + 1 \quad (\text{see } [\mathbf{1}]) \\ &= \delta \operatorname{Ind} A + \delta \operatorname{Ind} B + 1. \end{aligned}$$

Definition 8. If (X, δ) is a δ-space, then the set $H \in 2^X$ is called a δ-*singular set* if $\delta(X \setminus H, H) = 1$.

THEOREM 3. *The perfect* δ*-space* X *has* δ Ind $X = O$ *if and only if for every closed set* $F \subseteq X$ *and for every* δ*-neighbourhood* U *of* F *there exists a* δ*-singular set* H *such that*

$$F \subseteq H \subseteq U.$$

Proof. Let δ Ind $X = 0$ and let F be a closed subset of the δ-space (X, δ), and let $U \supset F$; then $\delta(F, X \setminus U) = 1$.

Therefore, the empty set $\emptyset$ is a δ-partition between F and $X \setminus U$. Thus

$$\exists\, U_1, U_2 \in \tau_\delta \quad \text{such that}$$

$$X = U_1 \cup U_2, U_1 \cap U_2 = \emptyset$$

and

$$U_1 \supset F, U_2 \supset X \setminus U.$$

But

$$CX = O_X = O_{(U_1 \cup U_2)} = O_{U_1} \cup O_{U_2},$$

and

$$O_{U_1} \cap O_{U_2} = \emptyset$$

because $U_1 \cap U_2 = \emptyset$. Then O_{U_1} and O_{U_2} are open-closed sets in CX. i.e.,

$$\delta(O_{H_1} \cap X, O_{H_2} \cap X) = 1$$

which implies that $\delta(U_1, U_2) = 1$, i.e.,

$$\delta(U_1, X \setminus U_1) = 1.$$

It is clear that

$$U \Supset U_1 \Supset F.$$

The converse is clear.

COROLLARY. *For every perfect δ-space X the conditions*

$$\delta \operatorname{Ind} X = 0 \quad \text{and} \quad \delta dX = 0$$

are equivalent.

Proof. This is immediate from the above Theorem (3) and Theorem (6) in [**6**].

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Mansoura University,
Mansoura, Egypt