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The Weak Ideal Property and Topological Dimension Zero

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Abstract. Following up on previous work, we prove a number of results for C*-algebras with the weak ideal property or topological dimension zero, and some results for C*-algebras with related properties. Some of the more important results include the following:

- The weak ideal property implies topological dimension zero.
- For a separable C*-algebra A, topological dimension zero is equivalent to $\mathrm{RR}(\mathcal{O}_2 \otimes A) = 0$, to $D \otimes A$ having the ideal property for some (or any) Kirchberg algebra D, and to A being residually hereditarily in the class of all C*-algebras B such that $\mathcal{O}_\infty \otimes B$ contains a nonzero projection.
- Extending the known result for $\mathbb{Z}_2$, the classes of C*-algebras with residual (SP), which are residually hereditarily (properly) infinite, or which are purely infinite and have the ideal property, are closed under crossed products by arbitrary actions of abelian 2-groups.
- If A and B are separable, one of them is exact, A has the ideal property, and B has the weak ideal property, then $A \otimes_{\min} B$ has the weak ideal property.
- If X is a totally disconnected locally compact Hausdorff space and A is a $C_0(X)$-algebra all of whose fibers have one of the weak ideal property, topological dimension zero, residual (SP), or the combination of pure infiniteness and the ideal property, then A also has the corresponding property (for topological dimension zero, provided A is separable).
- Topological dimension zero, the weak ideal property, and the ideal property are all equivalent for a substantial class of separable C*-algebras, including all separable locally AH algebras.
- The weak ideal property does not imply the ideal property for separable $\mathcal{Z}$-stable C*-algebras.

We give other related results, as well as counterexamples to several other statements one might conjecture.

1 Introduction

The weak ideal property, introduced by the authors in [33], see Definition 2.3, is the property for which there are good permanence results that seem to be closest to the ideal property; see [33, §8]. (The ideal property fails to pass to extensions [26, Theorem 5.1], to corners [32, Example 2.8], and, by [32, Example 2.7], to fixed point algebras under actions of $\mathbb{Z}_2$. The weak ideal property does all of these.) Topological dimension zero [6] is a non-Hausdorff version of total disconnectedness of the primitive ideal space of a C*-algebra. These two properties are related, although not identical, and the purpose of this paper is to study them and their connections further. Some

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of our results also involve the ideal property, real rank zero, and several forms of pure infiniteness (pure infiniteness [20, Definition 4.1], strong pure infiniteness [21, Definition 5.1], and residual hereditary (proper) infiniteness [33, Definitions 6.1, 6.2]).

For simple C*-algebras, there is a fairly short list of regularity properties generally considered to be important and which often appear as hypotheses or conclusions in important theorems. They include real rank zero, Property (SP), Z-stability, strict comparison, and pure infiniteness. (By contrast, the property of being the linear span of its projections, while considered in early work, seems much less important.) These properties are by now fairly well understood. For nonsimple C*-algebras, there are more regularity properties. They are less well understood, there are fewer theorems, and we do not yet know which regularity properties will turn out to be important. This paper is a contribution towards a better understanding of some of these properties, and, we hope, towards eventually identifying which ones are important.

Even though it is not yet clear which regularity properties will be important in the nonsimple case, the properties we consider (topological dimension zero, the weak ideal property, and the ideal property) have at least proved to be valuable. We illustrate this by giving some already known results in which these properties are used.

For topological dimension zero, if X is the primitive ideal space of a separable C*-algebra, then X has topological dimension zero if and only if X is the primitive ideal space of an AF algebra. (See [4, §3] and the theorem in [4, §5].) If A is a separable purely infinite C*-algebra, then A has real rank zero if and only if A has topological dimension zero and is K_0-liftable [35, Theorem 4.2].

Turning to the ideal property (every ideal is generated, as an ideal, by its projections), we consider AH algebras (in the sense of [25]: the spaces used are connected finite complexes) with the ideal property and with slow dimension growth. Such algebras have stable rank one [25, Theorem 4.1], and can be classified up to shape equivalence by a K-theoretic invariant [25, Theorem 2.15]. If such an algebra A has very slow dimension growth and $K_*(A)$ is torsion free, then A is an AT algebra, that is, a direct limit of finite direct sums of matrix algebras over $C(S^1)$ [14, Theorem 3.6]. The stable rank one and AT algebra results fail without the ideal property. (Counterexamples are easy, but too long for this introduction; we present them at the beginning of Section 7.) Also, a separable purely infinite C*-algebra has the ideal property if and only if it has topological dimension zero [35, Proposition 2.11].

The weak ideal property is much more recent. As noted above, it has better permanence properties than the ideal property. Moreover, under the hypotheses of the theorems above for the ideal property, the weak ideal property actually implies the ideal property. (See Theorem 7.15 and Theorem 2.10.)

We now describe our results. We prove in Section 2 that the weak ideal property implies topological dimension zero in complete generality. For separable C*-algebras which are purely infinite in the sense of [20], it is equivalent to the ideal property and to topological dimension zero. A general separable C*-algebra A has topological dimension zero if and only if $\mathcal{O}_2 \otimes A$ has real rank zero; this is also equivalent to $D \otimes A$ having the ideal property for some (or any) Kirchberg algebra D. We rule out by example other results in this direction which one might conjecture. Topological dimension zero, at least for separable C*-algebras, is also equivalent to a property of the sort considered in [33]. That is, there is an upwards directed class $\mathcal{C}$ such that a separable C*-algebra A has topological dimension zero if and only if A is residually hereditarily in $\mathcal{C}$. (See the end of the introduction for other examples of this kind of property.)

In Section 3, we improve the closure properties under crossed products of the class of C*-algebras residually hereditarily in a class $\mathcal{C}$ by replacing an arbitrary action of $\mathbb{Z}_2$ with an arbitrary action of a finite abelian 2-group (Corollary 3.3). This refinement was overlooked in [33]. It applies to residual hereditary (proper) infiniteness as well as to residual (SP) and to the combination of pure infiniteness and the ideal property. For the weak ideal property and for topological dimension zero, better results are already known [33, Corollary 8.10], [32, Theorem 3.17]. However, for topological dimension zero, in the separable case we remove the technical hypothesis in [32, Theorem 3.14], and show that if a finite group acts on a separable C*-algebra A and the fixed point algebra has topological dimension zero, then A has topological dimension zero.

Section 4 considers minimal tensor products. For a tensor product to have the weak ideal property or topological dimension zero, it is usually necessary that both tensor factors have the corresponding property. In the separable case and with one factor exact, this is sufficient for topological dimension zero. We show by example that this result fails without the exactness hypothesis. For the weak ideal property, we get only partial results: if both factors are separable, one is exact, and one actually has the ideal property, or if one factor is exact and one factor has finite or Hausdorff primitive ideal space, then the tensor product has the weak ideal property.

Proceeding to a $C_0(X)$-algebra A, we show that if X is totally disconnected and the fibers all have the weak ideal property, topological dimension zero, residual (SP), or the combination of pure infiniteness and the ideal property, then A also has the corresponding property (for topological dimension zero, provided A is separable). This result is the analog for these properties of [29, Theorem 2.1] (for real rank zero) and [30, Theorem 2.1] (for the ideal property), but we do not assume that the $C_0(X)$-algebra is continuous. If A is a separable continuous $C_0(X)$-algebra with nonzero fibers and X is second countable, then total disconnectedness of X is also necessary. This is in Section 5. In the short Section 6, we consider locally trivial $C_0(X)$-algebras with fibers that are strongly purely infinite in the sense of [21, Definition 5.1], and show (slightly generalizing the known result for $C_0(X, B)$) that A is again strongly purely infinite. In particular, this applies if the fibers are separable, purely infinite, and have topological dimension zero.

Section 7 gives a substantial class of C*-algebras for which the ideal property, the weak ideal property, and topological dimension zero are all equivalent. This class includes all separable locally AH algebras, as well as a further generalization, the separable LS algebras. We also prove that the weak ideal property implies the ideal property for stable C*-algebras A such that $\mathrm{Prim}(A)$ is Hausdorff. However, we show by example that there is a Z-stable C*-algebra with just one nontrivial ideal which has the weak ideal property but not the ideal property.

Ideals in C*-algebras are assumed to be closed and two sided. We write $\mathbb{Z}_n$ for $\mathbb{Z}/n\mathbb{Z}$, since the p-adic integers will not appear. If $\alpha: G \to \mathrm{Aut}(A)$ is an action of a group G on a C*-algebra A, then A^α denotes the fixed point algebra.

Because of the role they play in this paper, we recall the following definitions [33].

Definition 1.1 ([33, Definition 5.1]) Let $\mathcal{C}$ be a class of C*-algebras. We say that $\mathcal{C}$ is *upwards directed* if, whenever A is a C*-algebra that contains a subalgebra isomorphic to an algebra in $\mathcal{C}$, we have $A \in \mathcal{C}$.

Definition 1.2 ([33, Definition 5.2]) Let $\mathcal{C}$ be an upwards directed class of C*-algebras, and let A be a C*-algebra. We say that A is *hereditarily in* $\mathcal{C}$ if every nonzero hereditary subalgebra of A is in $\mathcal{C}$. We say that A is *residually hereditarily in* $\mathcal{C}$ if A/I is hereditarily in $\mathcal{C}$ for every ideal $I \subset A$ with $I \neq A$.

We gave permanence properties for a general condition defined this way [33, §5]. We recall the conditions of this type considered in [33], and add one more to be proved here.

(1) Let $\mathcal{C}$ be the class of all C*-algebras which contain an infinite projection. Then $\mathcal{C}$ is upwards directed (clear) and a C*-algebra A is purely infinite and has the ideal property if and only if A is residually hereditarily in $\mathcal{C}$. See the equivalence of conditions (ii) and (iv) of [35, Proposition 2.11] (valid, as shown there, even when A is not separable).

(2) Let $\mathcal{C}$ be the class of all C*-algebras that contain an infinite element. Then $\mathcal{C}$ is upwards directed (clear) and a C*-algebra A is (residually) hereditarily infinite [33, Definition 6.1] if and only if A is (residually) hereditarily in $\mathcal{C}$. (See [33, Corollary 6.5]. We should point out that, by [20, Lemma 2.2 (iii)], if D is a C*-algebra, $B \subset D$ is a hereditary subalgebra, and a and b are positive elements of B such that a is Cuntz subequivalent to b relative to D, then a is Cuntz subequivalent to b relative to B.)

(3) Let $\mathcal{C}$ be the class of all C*-algebras that contain a properly infinite element. Then $\mathcal{C}$ is upwards directed (clear) and a C*-algebra A is (residually) hereditarily properly infinite [33, Definition 6.2]) if and only if A is (residually) hereditarily in $\mathcal{C}$. (Lemma 2.2(iii) of [20] plays the same role here as in (2).)

(4) Let $\mathcal{C}$ be the class of all C*-algebras that contain a nonzero projection. Then $\mathcal{C}$ is upwards directed (clear). A C*-algebra A has Property (SP) if and only if A is hereditarily in $\mathcal{C}$, and has residual (SP) [33, Definition 7.1] if and only if A is residually hereditarily in $\mathcal{C}$. (Both statements are clear. Residual (SP) appears, without the name, as a hypothesis in the discussion after [21, Proposition 4.18].)

(5) Let $\mathcal{C}$ be the class of all C*-algebras B such that $K \otimes B$ contains a nonzero projection. Then $\mathcal{C}$ is upwards directed (clear) and a C*-algebra A has the weak ideal property [33, Definition 8.1] if and only if A is residually hereditarily in $\mathcal{C}$. (This is shown at the beginning of the proof of [33, Theorem 8.5].)

(6) Let $\mathcal{C}$ be the class of all C*-algebras B such that $\mathcal{O}_2 \otimes B$ contains a nonzero projection. Then $\mathcal{C}$ is upwards directed. (This is clear.) A separable C*-algebra A has topological dimension zero if and only if A is residually hereditarily in $\mathcal{C}$. (This will be proved in Theorem 2.10.)

2 Topological Dimension Zero

In this section, we prove that the weak ideal property implies topological dimension zero for general C*-algebras (Theorem 2.8). We then give characterizations of topological dimension zero for separable C*-algebras (Theorem 2.10) and purely infinite separable C*-algebras (Theorem 2.9), in terms of other properties of the algebra, in terms of properties of its tensor products with suitable Kirchberg algebras, and (for general separable C*-algebras) of the form of being residually hereditarily in suitable upwards directed classes. We also give two related counterexamples. In particular, there is a separable purely infinite unital nuclear C*-algebra A with one nontrivial ideal such that $\mathcal{O}_2 \otimes A \cong A$ and $\mathrm{RR}(A) = 0$, and an action $\alpha\colon \mathbb{Z}_2 \to \mathrm{Aut}(A)$, such that $\mathrm{RR}(C^*(\mathbb{Z}_2, A, \alpha)) \neq 0$.

We recall two definitions [32]. We call a not necessarily Hausdorff space *locally compact* if the compact (but not necessarily closed) neighborhoods of every point $x \in X$ form a neighborhood base at x.

Definition 2.1 ([6, Remark 2.5 (vi)], [32, Definition 3.2]) Let X be a locally compact but not necessarily Hausdorff topological space. We say that X has *topological dimension zero* if for every $x \in X$ and every open set $U \subset X$ such that $x \in U$, there exists a compact open (but not necessarily closed) subset $Y \subset X$ such that $x \in Y \subset U$. (Equivalently, X has a base for its topology consisting of subsets which are compact and open, but not necessarily closed.) We further say that a C*-algebra A has *topological dimension zero* if $\mathrm{Prim}(A)$ has topological dimension zero.

Definition 2.2 ([32, Definition 3.4]) Let X be a not necessarily Hausdorff topological space. A *compact open exhaustion* of X is an increasing net $(Y_\lambda)_{\lambda\in\Lambda}$ of compact open subsets $Y_\lambda \subset X$ such that $X = \bigcup_{\lambda\in\Lambda} Y_\lambda$.

We further recall ([32, Lemma 3.10]; see [32, Definition 3.9] or [35, p. 53] for the original definition) that if A is a C*-algebra and $I \subset A$ is an ideal, then I is compact if and only if $\mathrm{Prim}(I)$ is a compact open (but not necessarily closed) subset of $\mathrm{Prim}(A)$.

Finally, we recall the definition of the weak ideal property.

Definition 2.3 ([33, Definition 8.1]) Let A be a C*-algebra. We say that A has the *weak ideal property* if, whenever $I \subset J \subset K \otimes A$ are ideals in $K \otimes A$ such that $I \neq J$, it follows that J/I contains a nonzero projection.

Lemma 2.4 *Let A be a C*-algebra with the weak ideal property. Let $J \subset A$ be an ideal with $J \neq A$. Then there exists an ideal $N \subset A$ with $J \subsetneq N$ and such that $K \otimes (N/J)$ is generated as an ideal in $K \otimes (A/J)$ by a single nonzero projection.*

Proof Since A has the weak ideal property and $K \otimes (A/J) \neq 0$, there is a nonzero projection $e \in K \otimes (A/J)$. Let $M \subset K \otimes (A/J)$ be the ideal generated by e. Then there is an ideal $N \subset A$ with $J \subset N \subset A$ such that $M = K \otimes (N/J)$. Since $N/J \neq 0$, it follows that $N \neq J$. ■

Lemma 2.5 *Let A be a C*-algebra, let $F \subset A$ be a finite set of projections, and let $I \subset A$ be the ideal generated by F. Then* $\mathrm{Prim}(I)$ *is a compact open subset of* $\mathrm{Prim}(A)$.

Proof This can be shown by using the same argument as in (iii) implies (i) in the proof of Proposition 2.7 of [35]. However, we can give a more direct proof (not involving the Pedersen ideal). As there, we prove that I is compact (as recalled after

Definition 2.2). So let $(I_\lambda)_{\lambda\in\Lambda}$ be an increasing net of ideals in A such that $\overline{\bigcup_{\lambda\in\Lambda} I_\lambda} = I$. Standard functional calculus arguments produce $\varepsilon > 0$ such that if B is a C*-algebra, $C \subset B$ is a subalgebra, and $p \in B$ is a projection such that $\operatorname{dist}(p, C) < \varepsilon$, then there is a projection $q \in C$ such that $\|q - p\| < 1$, and in particular q is Murray–von Neumann equivalent to p. Write $F = \{p_1, p_2, \ldots, p_n\}$. Choose $\lambda \in \Lambda$ such that $\operatorname{dist}(p_j, I_\lambda) < \varepsilon$ for $j = 1, 2, \ldots, n$. Let $q_1, q_2, \ldots, q_n \in I_\lambda$ be projections obtained from the choice of ε. Then there are partial isometries $s_1, s_2, \ldots, s_n \in A$ such that $p_j = s_j q_j s_j^*$ for $j = 1, 2, \ldots, n$. So $p_1, p_2, \ldots, p_n \in I_\lambda$, whence $I_\lambda = I$. This completes the proof. ∎

Lemma 2.6 *Let A be a C*-algebra, and let $I \subset A$ be an ideal. Suppose that there is a collection $(I_\lambda)_{\lambda\in\Lambda}$ (not necessarily a net) of ideals in A such that I is the ideal generated by $\bigcup_{\lambda\in\Lambda} I_\lambda$ and such that* $\operatorname{Prim}(I_\lambda)$ *has a compact open exhaustion (Definition 2.2) for every $\lambda \in \Lambda$. Then* $\operatorname{Prim}(I)$ *has a compact open exhaustion.*

Proof It is easily checked that a union of open sets with compact open exhaustions also has a compact open exhaustion. ∎

Proposition 2.7 *Let A be a C*-algebra. Then there is a largest ideal $I \subset A$ such that* $\operatorname{Prim}(I)$ *has a compact open exhaustion.*

Proof Let I be the closure of the union of all ideals $J \subset A$ such that $\operatorname{Prim}(J)$ has a compact open exhaustion. Then $\operatorname{Prim}(I)$ has a compact open exhaustion by Lemma 2.6. ∎

Theorem 2.8 *Let A be a C*-algebra with the weak ideal property. Then A has topological dimension zero.*

Proof We will show that for every ideal $I \subset A$, the subset $\operatorname{Prim}(I)$ has a compact open exhaustion. The desired conclusion will then follow [32, Lemma 3.6].

So let $I \subset A$ be an ideal. By Proposition 2.7, there is a largest ideal $J \subset I$ such that $\operatorname{Prim}(J)$ has a compact open exhaustion. We prove that $J = I$. Suppose not. Use Lemma 2.4 with I in place of A to find an ideal $N \subset I$ with $J \subsetneqq N$ and such that $K \otimes (N/J)$ is generated by one nonzero projection. Then $\operatorname{Prim}(K \otimes (N/J))$ is a compact open subset of $\operatorname{Prim}(K \otimes (I/J))$ by Lemma 2.5. So $\operatorname{Prim}(N/J)$ is a compact open subset of $\operatorname{Prim}(I/J)$. Since $\operatorname{Prim}(J)$ has a compact open exhaustion, we can apply [32, Lemma 3.7] (taking $U = \operatorname{Prim}(J)$) to deduce that $\operatorname{Prim}(N)$ has a compact open exhaustion. Since $J \subsetneqq N$, we have a contradiction. Thus $J = I$, and $\operatorname{Prim}(I)$ has a compact open exhaustion. ∎

The list of equivalent conditions in the next theorem extends the list in [35, Corollary 4.3], by adding condition (v). As discussed in the introduction, this condition is better behaved than the related condition (iv).

Theorem 2.9 *Let A be a separable C*-algebra that is purely infinite in the sense of [20, Definition 4.1]. Then the following are equivalent.*

(i) *$\mathcal{O}_2 \otimes A$ has real rank zero.*
(ii) *$\mathcal{O}_2 \otimes A$ has the ideal property.*

(iii) *A has topological dimension zero.*
(iv) *A has the ideal property.*
(v) *A has the weak ideal property.*

Proof The equivalence of conditions (i)–(iv) is [35, Corollary 4.3]. That (iv) implies (v) is trivial. That (v) implies (iii) is Theorem 2.8. ■

We presume that Theorem 2.9 holds without separability. However, some of the results used in the proof of [35, Corollary 4.3] are only known in the separable case, and it seems likely to require some work to generalize them.

Recall that a Kirchberg algebra is a simple separable nuclear purely infinite C*-algebra.

Theorem 2.10 *Let A be a separable C*-algebra. Then the following are equivalent.*

(i) *A has topological dimension zero.*
(ii) *$\mathcal{O}_2 \otimes A$ has real rank zero.*
(iii) *$\mathcal{O}_2 \otimes A$ has the ideal property.*
(iv) *$\mathcal{O}_2 \otimes A$ has the weak ideal property.*
(v) *$\mathcal{O}_\infty \otimes A$ has the ideal property.*
(vi) *$\mathcal{O}_\infty \otimes A$ has the weak ideal property.*
(vii) *There exists a Kirchberg algebra D such that $D \otimes A$ has the weak ideal property.*
(viii) *For every Kirchberg algebra D, the algebra $D \otimes A$ has the ideal property.*
(ix) *A is residually hereditarily in the class of all C*-algebras B such that $\mathcal{O}_2 \otimes B$ contains a nonzero projection.*
(x) *A is residually hereditarily in the class of all C*-algebras B such that $K \otimes \mathcal{O}_2 \otimes B$ contains a nonzero projection.*
(xi) *A is residually hereditarily in the class of all C*-algebras B such that $\mathcal{O}_\infty \otimes B$ contains a nonzero projection.*

We presume that Theorem 2.10 also holds without separability.

To put conditions (ix)–(xi) in context, we point out that it is clear that the classes used in them are upwards directed in the sense of Definition 1.1. Applying the results of [33, §5] does not give any closure properties for the collection of C*-algebras with topological dimension zero which are not already known. We do get something new, which is at least implicitly related to this characterization; see Theorem 3.6.

The conditions in Theorem 2.10 are not equivalent to A having the weak ideal property, since there are nonzero simple separable C*-algebras A, such as those classified in [40], for which $K \otimes A$ has no nonzero projections. They are also not equivalent to $\mathrm{RR}(\mathcal{O}_\infty \otimes A) = 0$. See Example 2.13.

Proof of Theorem 2.10 Since A has topological dimension zero if and only if $\mathcal{O}_2 \otimes A$ has topological dimension zero, and since $\mathcal{O}_2 \otimes A$ is purely infinite [20, Proposition 4.5], the equivalence of (i)–(iv) follows by applying Theorem 2.9 to $\mathcal{O}_2 \otimes A$. Since $\mathcal{O}_\infty \otimes A$ is purely infinite [20, Proposition 4.5] and $\mathcal{O}_2 \otimes \mathcal{O}_\infty \cong \mathcal{O}_2$, the equivalence of (iii), (v), and (vi) follows by applying Theorem 2.9 to $\mathcal{O}_\infty \otimes A$.

We prove the equivalence of (i) and (ix). Let $\mathcal{C}$ be the class of all C*-algebras B such that $\mathcal{O}_2 \otimes B$ contains a nonzero projection.

Assume that A has topological dimension zero; we prove that A is residually hereditarily in $\mathcal{C}$. Let $I \subset A$ be an ideal, and let $B \subset A/I$ be a nonzero hereditary subalgebra. Then A/I has topological dimension zero by [6, Proposition 2.6] and [32, Lemma 3.6]. It follows from [32, Lemma 3.3] that B has topological dimension zero. Use (iii) implies (i) in Theorem 2.9 to conclude that $\mathcal{O}_2 \otimes B$ contains a nonzero projection.

Conversely, assume that A is residually hereditarily in $\mathcal{C}$. We actually prove that $\mathcal{O}_2 \otimes A$ has the weak ideal property. By (v) implies (iii) in Theorem 2.9, and since $\mathcal{O}_2 \otimes A$ is purely infinite [20, Proposition 4.5], it will follow that $\mathcal{O}_2 \otimes A$ has topological dimension zero. Since $\operatorname{Prim}(\mathcal{O}_2 \otimes A) \cong \operatorname{Prim}(A)$, it will follow that A has topological dimension zero.

Thus, let $I \subset J \subset \mathcal{O}_2 \otimes A$ be ideals such that $J \neq I$; we must show that $K \otimes (J/I)$ contains a nonzero projection. Since $\mathcal{O}_2$ is simple and nuclear, there are ideals $I_0 \subset J_0 \subset A$ such that $I = \mathcal{O}_2 \otimes I_0$ and $J = \mathcal{O}_2 \otimes J_0$; moreover, $J/I \cong \mathcal{O}_2 \otimes (J_0/I_0)$. Since J_0/I_0 is a nonzero hereditary subalgebra of A/I_0, the definition of being hereditarily in $\mathcal{C}$ implies that $\mathcal{O}_2 \otimes (J_0/I_0)$ contains a nonzero projection, so $K \otimes (J/I) \cong K \otimes \mathcal{O}_2 \otimes (J_0/I_0)$ does also. This completes the proof of the equivalence of (i) and (ix).

We prove the equivalence of (ix) and (xi) by showing that the two classes involved are equal, that is, by showing that if B is any C*-algebra, then $\mathcal{O}_2 \otimes B$ contains a nonzero projection if and only if $\mathcal{O}_\infty \otimes B$ contains a nonzero projection. If $\mathcal{O}_2 \otimes B$ contains a nonzero projection, use an injective (nonunital) homomorphism $\mathcal{O}_2 \to \mathcal{O}_\infty$ to produce an injective homomorphism of the minimal tensor products $\mathcal{O}_2 \otimes_{\min} B \to \mathcal{O}_\infty \otimes_{\min} B$. Since $\mathcal{O}_2$ and $\mathcal{O}_\infty$ are nuclear, we have an injective homomorphism $\mathcal{O}_2 \otimes B \to \mathcal{O}_\infty \otimes B$, and hence a nonzero projection in $\mathcal{O}_\infty \otimes B$. Using an injective (unital) homomorphism from $\mathcal{O}_\infty$ to $\mathcal{O}_2$, the same argument also shows that if $\mathcal{O}_\infty \otimes B$ contains a nonzero projection, then so does $\mathcal{O}_2 \otimes B$.

The proof of the equivalence of (ix) and (x) is essentially the same as in the previous paragraph, using injective homomorphisms

$$\mathcal{O}_2 \longrightarrow K \otimes \mathcal{O}_2 \quad \text{and} \quad K \otimes \mathcal{O}_2 \longrightarrow \mathcal{O}_2 \otimes \mathcal{O}_2 \xrightarrow{\cong} \mathcal{O}_2.$$

We have now proved the equivalence of all the conditions except (vii) and (viii).

It is trivial that (vi) implies (vii) and that (viii) implies (v).

Assume (vii), so that there is a Kirchberg algebra D_0 such that $D_0 \otimes A$ has the weak ideal property. We prove (viii). Let D be any Kirchberg algebra. By Theorem 2.8, the algebra $D_0 \otimes A$ has topological dimension zero. Since

$$\operatorname{Prim}(D_0 \otimes A) \cong \operatorname{Prim}(A) \cong \operatorname{Prim}(D \otimes A),$$

$D \otimes A$ has topological dimension zero. Apply the already proved implication from (i) to (v) with $D \otimes A$ in place of A, concluding that $\mathcal{O}_\infty \otimes D \otimes A$ has the ideal property. Since $\mathcal{O}_\infty \otimes D \cong D$ (by [19, Theorem 3.15]), we see that $D \otimes A$ has the ideal property. ■

A naive look at condition (i) of Theorem 2.9 and the permanence properties for C*-algebras which are residually hereditarily in some class $\mathcal{C}$ (see [33, Corollary 5.6 and Theorem 5.3]) might suggest that if $\mathcal{O}_\infty \otimes A$ has real rank zero and one has an arbitrary action of $\mathbb{Z}_2$ on $\mathcal{O}_\infty \otimes A$ or a spectrally free [33, Definition 1.3] action of any discrete group on $\mathcal{O}_\infty \otimes A$, then the crossed product should also have real rank zero. This is false. We give an example of a nonsimple purely infinite unital nuclear

C*-algebra A satisfying the Universal Coefficient Theorem (in fact, with $\mathcal{O}_2 \otimes A \cong A$), with exactly one nontrivial ideal, and such that $\mathrm{RR}(A) = 0$, and a spectrally free action $\alpha: \mathbb{Z}_2 \to \mathrm{Aut}(A)$, such that $C^*(\mathbb{Z}_2, A, \alpha)$ does not have real rank zero.

To put our example in context, we recall the following. First, [10, Example 9] gives an example of a pointwise outer action α of $\mathbb{Z}_2$ on a simple unital AF algebra A such that $C^*(\mathbb{Z}_2, A, \alpha)$ does not have real rank zero. Second, if A is purely infinite and simple, then for any action $\alpha: \mathbb{Z}_2 \to \mathrm{Aut}(A)$ the crossed product is again purely infinite [16, Corollary 4.4]. If α is pointwise outer, then $C^*(\mathbb{Z}_2, A, \alpha)$ is again simple, so automatically has real rank zero. Otherwise, α must be an inner action. (See Lemma 2.11.) Then $C^*(\mathbb{Z}_2, A, \alpha) \cong A \oplus A$, so has real rank zero. Thus, no such example is possible when A is purely infinite and simple. Third, it is possible for A to satisfy $\mathcal{O}_2 \otimes A \cong A$ but to have $\mathcal{O}_2 \otimes C^*(\mathbb{Z}_2, A, \alpha) \ncong C^*(\mathbb{Z}_2, A, \alpha)$. See [15, Lemma 4.7], where this happens with $A = \mathcal{O}_2$.

The following lemma is well known, but we are not aware of a reference.

Lemma 2.11 *Let A be a simple C*-algebra, let G be a finite cyclic group, and let $\alpha: G \to \mathrm{Aut}(A)$ be an action of G on A. Let $g_0 \in G$ be a generator of G. If α_{g_0} is inner, then α is an inner action, that is, there is a homomorphism $g \mapsto u_g$ from G to the unitary group of $M(A)$ such that $\alpha_g(a) = u_g a u_g^*$ for all $g \in G$ and $a \in A$.*

Proof Let n be the order of G. By hypothesis, there is a unitary $v \in M(A)$ such that $\alpha_{g_0}(a) = vav^*$ for all $a \in A$. Then $a = \alpha_{g_0}^n(a) = v^n a v^{-n}$ for all $a \in A$. Simplicity of A implies that the center of $M(A)$ contains only scalars, so there is $\lambda \in S^1$ such that $v^n = \lambda \cdot 1$. Now choose $\omega \in S^1$ such that $\omega^n = \lambda^{-1}$, giving $(\omega v)^n = 1$. Define $u_{g_0^k} = \omega^k v^k$ for $k = 0, 1, \dots, n-1$. ∎

Example 2.12 There are a separable purely infinite unital nuclear C*-algebra A and an action $\alpha: \mathbb{Z}_2 \to \mathrm{Aut}(A)$ with the following properties. The algebra A has exactly one nontrivial ideal I and satisfies the Universal Coefficient Theorem; moreover, $\mathcal{O}_2 \otimes A \cong A$ and $\mathrm{RR}(A) = 0$. The action α is strongly pointwise outer [39, Definition 4.11], [33, Definition 1.1] and spectrally free [33, Definition 1.3], but $\mathrm{RR}(C^*(\mathbb{Z}_2, A, \alpha)) \neq 0$.

To start the construction, let $\nu: \mathbb{Z}_2 \to \mathrm{Aut}(\mathcal{O}_2)$ be the action considered in [15, Lemma 4.7]. Define $B = C^*(\mathbb{Z}_2, \mathcal{O}_2, \nu)$; [15, Lemma 4.7] implies that B is a Kirchberg algebra (simple, separable, nuclear, and purely infinite) that is unital and satisfies the Universal Coefficient Theorem, and moreover that $K_0(B) \cong \mathbb{Z}[\frac{1}{2}]$ and $K_1(B) = 0$.

Let P be the unital Kirchberg algebra satisfying the Universal Coefficient Theorem, $K_0(P) = 0$, and $K_1(P) \cong \mathbb{Z}$. The Künneth formula [43, Theorem 4.1] implies that $K_0(P \otimes \mathcal{O}_4) = 0$ and $K_1(P \otimes \mathcal{O}_4) \cong \mathbb{Z}_3$.

The algebras $\mathcal{O}_4$ and $P \otimes \mathcal{O}_4$ are both in the classifiable class $\mathcal{C}$ of purely infinite simple separable nuclear C*-algebras defined in the introduction to [41, § 3]. It follows from [41, Proposition 5.4] that every possible six term exact sequence

$$
\begin{tikzcd}
K_0(P \otimes \mathcal{O}_4) \arrow[r] & M_0 \arrow[r] & K_0(\mathcal{O}_4) \\
K_1(\mathcal{O}_4) \arrow[u, "\partial"] & M_1 \arrow[l] & K_1(P \otimes \mathcal{O}_4) \arrow[l] \arrow[u, "\exp"']
\end{tikzcd}
\tag{2.1}
$$

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(for any possible choice of abelian groups M_0 and M_1 and homomorphisms exp and ∂) is realized as the K-theory of an exact sequence

$$0 \longrightarrow I \xrightarrow{\mu_0} E_0 \xrightarrow{\pi_0} D \longrightarrow 0, \tag{2.2}$$

in which I and D are stable C*-algebras in $\mathcal{C}$, and

$$K_*(I) \cong K_*(P \otimes \mathcal{O}_4), \quad K_*(D) \cong K_*(\mathcal{O}_4), \quad K_0(E_0) \cong M_0, \quad K_1(E_0) \cong M_1.$$

Moreover, as in the introduction to [41, §4], we may require that the extension be in the standard form described there. In particular, it will then be essential. Choose the exact sequence (2.2) such that the connecting map

$$\exp\colon K_0(\mathcal{O}_4) \to K_1\big(P \otimes \mathcal{O}_4\big) \tag{2.3}$$

is an isomorphism. Classification in the simple case (see [18], [38, Theorem 4.2.4]) gives $I \cong K \otimes P \otimes \mathcal{O}_4$ and $D \cong K \otimes \mathcal{O}_4$. The algebra E_0 has a countable approximate identity consisting of projections [41, Proposition 4.4]. In particular, there exists a projection $p \in E_0$ such that $\pi_0(p) \neq 0$. Since the extension is essential, p is full.

We identify the algebras $\pi_0(p)D\pi_0(p)$ and $p\mu_0(I)p$ in the extension

$$0 \longrightarrow p\mu_0(I)p \xrightarrow{\mu_0} pE_0p \xrightarrow{\pi_0} \pi_0(p)D\pi_0(p) \longrightarrow 0. \tag{2.4}$$

Since exp in (2.1) has been chosen to be an isomorphism, $[\pi_0(p)] = 0$ in $K_0(D)$. Therefore classification in the simple case implies that $\pi_0(p)D\pi_0(p) \cong M_3(\mathcal{O}_4)$ (see [18], [38, Theorem 4.2.4]). Since p is full, $p\mu_0(I)p \neq 0$. The extension (2.4) does not split, because $\exp \neq 0$ in (2.1), so $p\mu_0(I)p$ is not unital. Therefore $p\mu_0(I)p$ is stable. So $p\mu_0(I)p \cong K \otimes P \otimes \mathcal{O}_4$. Setting $E = pE_0p$, the extension (2.4) is therefore isomorphic to an extension

$$0 \longrightarrow K \otimes P \otimes \mathcal{O}_4 \xrightarrow{\mu} E \xrightarrow{\pi} M_3(\mathcal{O}_4) \longrightarrow 0, \tag{2.5}$$

whose K-theory is as in (2.1) with the choice (2.3).

Define $A = \mathcal{O}_2 \otimes E$. Let

$$\iota_0\colon \mathbb{Z}_2 \to \mathrm{Aut}(K \otimes P \otimes \mathcal{O}_4), \quad \iota\colon \mathbb{Z}_2 \to \mathrm{Aut}(E), \quad \text{and} \quad \iota_1\colon \mathbb{Z}_2 \to \mathrm{Aut}(M_3(\mathcal{O}_4))$$

be the trivial actions, and let

$$\alpha_0 = \nu \otimes \iota_0\colon \mathbb{Z}_2 \to \mathrm{Aut}(\mathcal{O}_2 \otimes K \otimes P \otimes \mathcal{O}_4), \quad \alpha = \nu \otimes \iota\colon \mathbb{Z}_2 \to \mathrm{Aut}(A),$$

and

$$\alpha_1 = \nu \otimes \iota_1\colon \mathbb{Z}_2 \to \mathrm{Aut}(\mathcal{O}_2 \otimes M_3(\mathcal{O}_4))$$

be the obvious actions on the tensor products. Tensoring the sequence (2.5) with $\mathcal{O}_2$ and equipping the algebras with these actions gives an equivariant exact sequence

$$0 \longrightarrow \mathcal{O}_2 \otimes K \otimes P \otimes \mathcal{O}_4 \longrightarrow A \longrightarrow \mathcal{O}_2 \otimes M_3(\mathcal{O}_4) \longrightarrow 0. \tag{2.6}$$

Using the isomorphisms $\mathcal{O}_2 \otimes M_3(\mathcal{O}_4) \cong \mathcal{O}_2$ and $\mathcal{O}_2 \otimes P \otimes \mathcal{O}_4 \cong \mathcal{O}_2$, we can rewrite (2.6) as $0 \to K \otimes \mathcal{O}_2 \to A \to \mathcal{O}_2 \to 0$. Therefore, [5, Theorem 3.14, Corollary 3.16] imply that $\mathrm{RR}(A) = 0$. It follows from [37, Lemma 2.8.2] that taking crossed products in (2.6) gives an exact sequence

$$0 \longrightarrow C^*\big(\mathbb{Z}_2, \mathcal{O}_2 \otimes K \otimes P \otimes \mathcal{O}_4, \alpha_0\big) \longrightarrow C^*(\mathbb{Z}_2, A, \alpha) \longrightarrow C^*\big(\mathbb{Z}_2, \mathcal{O}_2 \otimes M_3(\mathcal{O}_4), \alpha_1\big) \longrightarrow 0.$$

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This sequence reduces to

$$0 \longrightarrow B \otimes K \otimes P \otimes \mathcal{O}_4 \longrightarrow B \otimes E \longrightarrow B \otimes M_3(\mathcal{O}_4) \longrightarrow 0, \tag{2.7}$$

in which the maps are obtained from those of (2.5) by tensoring them with id_B. It follows from the Künneth formula [43, Theorem 4.1] that

$$K_0(B \otimes M_3(\mathcal{O}_4)) \cong K_1\big(B \otimes K \otimes P \otimes \mathcal{O}_4\big) \cong \mathbb{Z}[\tfrac{1}{2}] \otimes \mathbb{Z}_3 \cong \mathbb{Z}_3.$$

Consider the connecting map $K_0(B \otimes M_3(\mathcal{O}_4)) \to K_1(B \otimes K \otimes P \otimes \mathcal{O}_4)$ associated with (2.7). By naturality, it is the tensor product of the isomorphism (2.3) with $\mathrm{id}_{\mathbb{Z}[1/2]}$, and is hence nonzero. Since every class in $K_0(B \otimes M_3(\mathcal{O}_4))$ is represented by a projection in $B \otimes M_3(\mathcal{O}_4)$, it follows from the six term exact sequence in K-theory that there are projections in $B \otimes M_3(\mathcal{O}_4)$ which do not lift to projections in $B \otimes E$. Therefore, [5, Theorem 3.14] implies that $\mathrm{RR}(B \otimes E) \neq 0$. Thus $\mathrm{RR}(C^*(\mathbb{Z}_2, A, \alpha)) \neq 0$.

It remains to prove the claim that α is strongly pointwise outer and spectrally free. Since the group $\mathbb{Z}_2$ is finite, these are equivalent by [33, Theorem 1.16], so we prove strong pointwise outerness. Let $g \in \mathbb{Z}_2$ be the nontrivial element. This then reduces to proving that the automorphisms $(\alpha_0)_g = \nu_g \otimes \mathrm{id}_{K \otimes P \otimes \mathcal{O}_4} \in \mathrm{Aut}(\mathcal{O}_2 \otimes K \otimes P \otimes \mathcal{O}_4)$ and $(\alpha_1)_g = \nu_g \otimes \mathrm{id}_{M_3(\mathcal{O}_4)} \in \mathrm{Aut}(\mathcal{O}_2 \otimes M_3(\mathcal{O}_4))$ are outer. The automorphism $\nu_g \in \mathrm{Aut}(\mathcal{O}_2)$ is outer, since otherwise the action ν would be inner by Lemma 2.11, so the crossed product would be $\mathcal{O}_2 \oplus \mathcal{O}_2$. We can now apply [33, Proposition 1.19 (2)] twice, both times using $\nu\colon \mathbb{Z}_2 \to \mathrm{Aut}(\mathcal{O}_2)$ in place of $\alpha\colon G \to \mathrm{Aut}(A)$, and in one case using $K \otimes P \otimes \mathcal{O}_4$ in place of B and in the other case using $M_3(\mathcal{O}_4)$.

We would like to get outerness of $(\alpha_0)_g \in \mathrm{Aut}(\mathcal{O}_2 \otimes K \otimes P \otimes \mathcal{O}_4)$ from [45, Theorem 1], but that theorem is only stated for unital C*-algebras.

Example 2.13 There is a separable purely infinite unital nuclear C*-algebra A with exactly one nontrivial ideal and which has the ideal property but such that $\mathcal{O}_\infty \otimes A$ does not have real rank zero.

Let E be as in (2.5) in Example 2.12, with the property that the connecting map in (2.3) is nonzero. Set $A = \mathcal{O}_\infty \otimes E$. Since $\mathcal{O}_\infty \otimes K \otimes P \otimes \mathcal{O}_4$ and $\mathcal{O}_\infty \otimes M_3(\mathcal{O}_4)$ have the weak ideal property (for trivial reasons), it follows from [33, Theorem 8.5 (5)] that A has the weak ideal property, and then from Theorem 2.9 that A has the ideal property. However, A is by construction not K_0-liftable in the sense of [35, Definition 3.1], so [35, Corollary 4.3(i)] implies that $\mathcal{O}_\infty \otimes A$ (which is of course isomorphic to A) does not have real rank zero.

3 Permanence Properties for Crossed Products

We proved [33] that if $\mathcal{C}$ is an upwards directed class of C*-algebras, α is a completely arbitrary action of $\mathbb{Z}_2$ on a C*-algebra A, and A^α is (residually) hereditarily in $\mathcal{C}$, then A is (residually) hereditarily in $\mathcal{C}$. (See [33, Theorem 5.5].) In particular, by considering dual actions, it follows [33, Corollary 5.6] that crossed products by arbitrary actions of $\mathbb{Z}_2$ preserve the class of C*-algebras that are (residually) hereditarily in $\mathcal{C}$. Here we show how one can easily extend the first result to arbitrary groups of order a power of 2 and the second result to arbitrary abelian groups of order a power of 2. This should have been done in [33], but was overlooked there. We believe these results should be true for any finite group in place of $\mathbb{Z}_2$, or at least any finite abelian group, but we do not know how to prove them in this generality.

The following lemma is surely well known.

Lemma 3.1 *Let G be a topological group, let A be a C^*-algebra, and let $\alpha: G \to \mathrm{Aut}(A)$ be an action of G on A. Let $N \subset G$ be a closed normal subgroup. Then there is an action $\overline{\alpha}: G/N \to \mathrm{Aut}(A^{\alpha|_N})$ such that for $g \in G$ and $a \in A^{\alpha|_N}$ we have $\overline{\alpha}_{gN}(a) = \alpha_g(a)$. Moreover, $(A^{\alpha|_N})^{\overline{\alpha}} = A^{\alpha}$.*

Proof The only thing requiring proof is that if $g \in G$ and $a \in A^{\alpha|_N}$, then $\alpha_g(a) \in A^{\alpha|_N}$. So let $k \in N$. Since $g^{-1}kg \in N$, we get $\alpha_k(\alpha_g(a)) = \alpha_g(\alpha_{g^{-1}kg}(a)) = \alpha_g(a)$. ■

Theorem 3.2 *Let $\mathcal{C}$ be an upwards directed class of C^*-algebras. Let G be a finite 2-group, and let $\alpha: G \to \mathrm{Aut}(A)$ be an arbitrary action of G on a C^*-algebra A.*

(i) *If A^{α} is hereditarily in $\mathcal{C}$, then A is hereditarily in $\mathcal{C}$.*
(ii) *If A^{α} is residually hereditarily in $\mathcal{C}$, then A is residually hereditarily in $\mathcal{C}$.*

Proof We prove both parts at once. We use induction on the number $n \in \mathbb{Z}_{\geq 0}$ such that the order of G is 2^n. When $n = 0$, the statement is trivial. So assume that $n \in \mathbb{Z}_{\geq 0}$, that the statement is known for all groups of order 2^n, that G is a group with $\mathrm{card}(G) = 2^{n+1}$, that A is a C^*-algebra, that $\alpha: G \to \mathrm{Aut}(A)$ is an action, and that A^{α} is (residually) hereditarily in $\mathcal{C}$. The Sylow theorems provide a subgroup $N \subset G$ such that $\mathrm{card}(N) = 2^n$. Since N has index 2, N must be normal. Let $\overline{\alpha}: G/N \to \mathrm{Aut}(A^{\alpha|_N})$ be as in Lemma 3.1. Then $(A^{\alpha|_N})^{\overline{\alpha}} = A^{\alpha}$ is (residually) hereditarily in $\mathcal{C}$. Since $G/N \cong \mathbb{Z}_2$, it follows from [33, Theorem 5.5] that $A^{\alpha|_N}$ is (residually) hereditarily in $\mathcal{C}$. The induction hypothesis now implies that A is (residually) hereditarily in $\mathcal{C}$. ■

Corollary 3.3 *Let $\mathcal{C}$ be an upwards directed class of C^*-algebras. Let G be a finite abelian 2-group, and let $\alpha: G \to \mathrm{Aut}(A)$ be an arbitrary action of G on a C^*-algebra A.*

(i) *If A is hereditarily in $\mathcal{C}$, then $C^*(G, A, \alpha)$ and A^{α} are hereditarily in $\mathcal{C}$.*
(ii) *If A is residually hereditarily in $\mathcal{C}$, then $C^*(G, A, \alpha)$ and A^{α} are residually hereditarily in $\mathcal{C}$.*

Proof For $C^*(G, A, \alpha)$, apply Theorem 3.2 with $C^*(G, A, \alpha)$ in place of A and the dual action $\widehat{\alpha}$ in place of α.

For A^{α}, use the proposition in [42] to see that A^{α} is isomorphic to a corner of $C^*(G, A, \alpha)$, and apply [33, Proposition 5.10]. ■

Presumably Corollary 3.3 is valid for crossed products by coactions of not necessarily abelian 2-groups. Indeed, possibly the appropriate context is that of actions of finite dimensional Hopf C^*-algebras. We will not pursue this direction here.

Corollary 3.4 *Let G be a finite 2-group, and let $\alpha: G \to \mathrm{Aut}(A)$ be an arbitrary action of G on a C^*-algebra A. Suppose A^{α} has one of the following properties: residual hereditary infiniteness, residual hereditary proper infiniteness, residual (SP), or the combination of the ideal property and pure infiniteness. Then A has the same property.*

Proof As discussed in the introduction, for each of these properties there is an upwards directed class $\mathcal{C}$ such that a C*-algebra has the property if and only if it is residually hereditarily in the class $\mathcal{C}$. Apply Theorem 3.2. ∎

Corollary 3.5 *Let G be a finite abelian 2-group, and let $\alpha: G \to \mathrm{Aut}(A)$ be an arbitrary action of G on a C*-algebra A. Suppose A has one of the following properties: residual hereditary infiniteness, residual hereditary proper infiniteness, residual (SP), or the combination of the ideal property and pure infiniteness. Then $C^*(G, A, \alpha)$ and A^α have the same property.*

Proof The proof is the same as that of Corollary 3.4, using Corollary 3.3 instead of Theorem 3.2. ∎

We omit the weak ideal property in Corollary 3.4 and Corollary 3.5, because better results are already known [33, Theorem 8.9, Corollary 8.10]. We also already know [32, Theorem 3.17] that topological dimension zero is preserved by crossed products by actions of arbitrary finite abelian groups, not just abelian 2-groups. The result analogous to Corollary 3.4 is [32, Theorem 3.14], but it has an extra technical hypothesis. In the separable case, we remove this hypothesis.

Theorem 3.6 *Let $\alpha: G \to \mathrm{Aut}(A)$ be an action of a finite group G on a separable C*-algebra A. Suppose that A^α has topological dimension zero. Then A has topological dimension zero.*

Proof Define an action $\beta: G \to \mathrm{Aut}(\mathcal{O}_2 \otimes A)$ by $\beta_g = \mathrm{id}_{\mathcal{O}_2} \otimes \alpha_g$ for $g \in G$. The implication from (i) to (iv) in Theorem 2.10 shows that $(\mathcal{O}_2 \otimes A)^\beta = \mathcal{O}_2 \otimes A^\alpha$ has the weak ideal property. Theorem 8.9 of [33] now implies that $\mathcal{O}_2 \otimes A$ has the weak ideal property. So A has topological dimension zero by the implication from (iv) to (i) in Theorem 2.10. ∎

4 Permanence Properties for Tensor Products

In this section, we consider permanence properties for tensor products. One of its purposes is to serve as motivation for the results on $C_0(X)$-algebras in Section 5. The new positive result is Theorem 4.4: if A and B are nonzero separable C*-algebras and A is exact, then $A \otimes_{\min} B$ has topological dimension zero if and only if A and B have topological dimension zero. The exactness hypothesis is necessary (Example 4.1). Still assuming this exactness hypothesis, we also give partial results for the weak ideal property, when one of the tensor factors actually has the ideal property and both are separable (Theorem 4.8), and when one of them has finite or Hausdorff primitive ideal space (Proposition 4.10 and Proposition 4.11).

The properties we are considering are certainly not preserved by taking tensor products with arbitrary C*-algebras. For example, the algebra $\mathbb{C}$ has all of topological dimension zero, the ideal property, the weak ideal property, and residual (SP), but $C([0,1]) \otimes \mathbb{C}$ has none of these. The algebra $\mathcal{O}_2$ is purely infinite and has the ideal property, but $C([0,1]) \otimes \mathcal{O}_2$ does not have the ideal property.

There is thus no hope of any general theorem about tensor products for properties of the form "residually hereditarily in $\mathcal{C}$" when only one tensor factor has the property. Permanence theorems will therefore have to assume that both factors have the property in question. The following example shows that we will also need to assume that at least one tensor factor is exact.

Example 4.1 We show that there are separable unital C*-algebras A and C (neither of which is exact) that have topological dimension zero and such that $A \otimes_{\min} C$ does not have topological dimension zero. In fact, A and C even have real rank zero, and C is simple. We also show that there are separable unital C*-algebras B and D that are purely infinite and have the ideal property, but such that $B \otimes_{\min} D$ does not have the ideal property. In fact, B and D even tensorially absorb $\mathcal{O}_2$, and D is simple.

Since topological dimension zero and the weak ideal property are preserved by passing to quotients, it follows that no other tensor product of A and C has topological dimension zero. Also using the implication from (iv) to (iii) in Theorem 2.10, it follows that no other tensor product of B and D even has the weak ideal property.

Let A and C be as in [34, Theorem 2.6]. As there, A and C are separable unital C*-algebras with real rank zero. Real rank zero passes to ideals and quotients, and therefore clearly implies the weak ideal property. So A and C have topological dimension zero by Theorem 2.8. Also, C is simple and $A \otimes_{\min} C$ does not have the ideal property [34, Theorem 2.6]. These are the same algebras A and C as used in the proof of [35, Proposition 4.5]. Thus, $A \otimes_{\min} C$ does not have topological dimension zero [35, Proposition 4.5 (1)]. This shows that A and C have the required properties. Also, $\mathcal{O}_2 \otimes A \otimes_{\min} C$ does not have the ideal property [35, Proposition 4.5 (2)]. Thus taking $B = \mathcal{O}_2 \otimes A$ and $D = \mathcal{O}_2 \otimes C$ gives algebras B and D with the required properties.

We have several positive results, but no answers for several obvious questions. We recall known results, then give the new result we can prove (on topological dimension zero) and our partial results for the weak ideal property. We conclude with open questions.

In order to get

$$\operatorname{Prim}(A \otimes_{\min} B) \cong \operatorname{Prim}(A) \times \operatorname{Prim}(B), \tag{4.1}$$

we will assume one of the algebras is exact and both are separable. In Theorem 4.4, Theorem 4.8, and Corollary 4.9, these assumptions can be replaced by any other hypotheses which imply a natural homeomorphism as in (4.1). Proposition 2.17 of [3] gives a number of conditions that imply this for the spaces of prime ideals in place of the primitive ideal spaces, but for separable C*-algebras this is the same thing.

Theorem 4.2 ([34, Corollary 1.3]) *Let A and B be C*-algebras with the ideal property. Assume that A is exact. Then $A \otimes_{\min} B$ has the ideal property.*

Theorem 4.3 ([35, Proposition 4.6]) *Let A and B be C*-algebras with the ideal property. Assume that B is purely infinite and A is exact. Then $A \otimes_{\min} B$ is purely infinite and has the ideal property.*

Theorem 4.4 *Let A and B be nonzero separable C*-algebras. Assume that A is exact. Then $A \otimes_{\min} B$ has topological dimension zero if and only if both A and B have topological dimension zero.*

Proof By [3, Proposition 2.17] (see [3, Remark 2.11] for the notation in [3, Proposition 2.16], to which it refers), the spaces of closed prime ideals satisfy

$$\operatorname{prime}(A \otimes_{\min} B) \cong \operatorname{prime}(A) \times \operatorname{prime}(B),$$

with the homeomorphism being implemented in the obvious way ([3, Proposition 2.16 (iii)]). Since A, B, and $A \otimes_{\min} B$ are all separable, [36, Proposition 4.3.6] implies that prime ideals are primitive; the reverse is well known. So

$$\operatorname{Prim}(A \otimes_{\min} B) \cong \operatorname{Prim}(A) \times \operatorname{Prim}(B). \tag{4.2}$$

Assume A and B have topological dimension zero. Then (see Definition 2.1) we need to prove that if X and Y are locally compact, but not necessarily Hausdorff spaces that have topological dimension zero, then $X \times Y$ has topological dimension zero. So let $(x, y) \in X \times Y$, and let $W \subset X \times Y$ be an open set with $(x, y) \in W$. By the definition of the product topology, there are open subsets $U_0 \subset X$ and $V_0 \subset Y$ such that $x \in U_0$, $y \in V_0$, and $U_0 \times V_0 \subset W$. By the definition of topological dimension zero, there are compact (but not necessarily closed) open subsets $U \subset X$ and $V \subset Y$ such that $x \in U \subset U_0$ and $y \in V \subset V_0$. Then $U \times V$ is a compact open subset of $X \times Y$ such that $(x, y) \in U \times V \subset W$.

Now assume $A \otimes_{\min} B$ has topological dimension zero. We prove that B has topological dimension zero; the proof that A has topological dimension zero is the same. By (4.2), it is enough to prove that if X and Y are nonempty locally compact, but not necessarily Hausdorff spaces, and $X \times Y$ has topological dimension zero, then X has topological dimension zero. So let $x \in X$ and let $U \subset X$ be an open set that contains x. Fix any point $y_0 \in Y$. Then $U \times Y$ is an open subset of $X \times Y$ that contains (x, y_0). Therefore there is a compact (but not necessarily closed) open subset $W \subset U \times Y$ such that $(x, y_0) \in W$. Let $p\colon X \times Y \to X$ be the projection to the first coordinate. Then p is a continuous open map. Therefore the set $V = p(W)$ is a compact (but not necessarily closed) open subset of X, and clearly $x \in V \subset U$. ∎

The first result for the weak ideal property requires some preparation.

Notation 4.5 Let A be a C*-algebra. For an open set $U \subset \operatorname{Prim}(A)$, we let $I_A(U) \subset A$ be the corresponding ideal. Thus

$$\operatorname{Prim}(I_A(U)) \cong U \quad \text{and} \quad \operatorname{Prim}(A/I_A(U)) \cong \operatorname{Prim}(A) \setminus U.$$

Lemma 4.6 *Let A be a C*-algebra, let $U \subset \operatorname{Prim}(A)$ be open, and let $p \in A/I_A(U)$ be a projection. Then there exist an open subset $V \subset \operatorname{Prim}(A)$, a compact (but not necessarily closed) subset $L \subset \operatorname{Prim}(A)$, and a projection $q \in A/I_A(V)$, such that $V \subset L \subset U$ and the image of q in $A/I_A(U)$ is equal to p.*

Proof For $P \in \operatorname{Prim}(A)$, let $\pi_P\colon A \to A/P$ be the quotient map, and for an open subset $W \subset \operatorname{Prim}(A)$, let $\kappa_W\colon A \to A/I_A(W)$ be the quotient map. Choose $a \in A_{\mathrm{sa}}$

such that $\kappa_U(a) = p$. Define

$$V = \left\{ P \in \mathrm{Prim}(A) : \|\pi_P(a^2 - a)\| > \tfrac{1}{8} \right\}$$

and

$$L = \left\{ P \in \mathrm{Prim}(A) : \|\pi_P(a^2 - a)\| \geq \tfrac{1}{8} \right\}.$$

We apply results in [9] to these sets. These results are actually stated in terms of functions on the space $\widehat{A}$ of unitary equivalence classes of irreducible representations of A, with the topology being the inverse image of the topology on $\mathrm{Prim}(A)$ under the standard surjection $\widehat{A} \to \mathrm{Prim}(A)$, but they clearly apply to $\mathrm{Prim}(A)$. It follows that V is open [9, Proposition 3.3.2], and that L is compact [9, Proposition 3.3.7]. Obviously $V \subset L$. Clearly $\pi_P(a^2 - a) = 0$ for all $P \in \mathrm{Prim}(A) \setminus U$, so $L \subset U$.

Lemma 3.3.6 of [9] implies that $\|\kappa_V(a^2 - a)\| \leq \frac{1}{8}$. Therefore $\frac{1}{2} \notin \mathrm{sp}(\kappa_V(a))$. Thus we can define a projection $q \in A/I_A(V)$ by $q = \chi_{(\frac{1}{2},\infty)}(\kappa_V(a))$. The image of q in $A/I_A(U)$ is clearly equal to p. ■

Lemma 4.7 *Let X_1 and X_2 be topological spaces, let $W \subset X_1 \times X_2$ be an open subset, let $x \in X_1$, let $L \subset X_2$ be compact, and suppose that $\{x\} \times L \subset W$. Then there exists an open set $U \subset X_1$ such that $x \in U$ and $U \times L \subset W$.*

We do not assume that X_1 and X_2 are Hausdorff. In particular, L need not be closed.

Proof of Lemma 4.7 For each $y \in L$, choose open sets $V_1(y) \subset X_1$ and $V_2(y) \subset X_2$ such that $(x, y) \in V_1(y) \times V_2(y) \subset W$. Use compactness of L to choose $n \in \mathbb{Z}_{\geq 0}$ and $y_1, y_2, \ldots, y_n \in L$ such that $V_2(y_1), V_2(y_2), \ldots, V_2(y_n)$ cover L. Take

$$U = \bigcap_{j=1}^{n} V_1(y_j). \qquad ■$$

Theorem 4.8 *Let A_1 and A_2 be separable C^*-algebras. Assume that A_1 or A_2 is exact, that A_1 has the ideal property, and that A_2 has the weak ideal property. Then $A_1 \otimes_{\min} A_2$ has the weak ideal property.*

In the diagram (4.5) in the proof below, one should think of the subquotients as corresponding to locally closed subsets of $\mathrm{Prim}(A_1) \times \mathrm{Prim}(A_2)$. Thus, the algebra in the middle of the top row corresponds to $V_1 \times (V_2 \setminus T)$. It contains a useful nonzero projection, obtained as the tensor product of suitable projections in $I_{A_1}(V_1)$ and $I_{A_2}(V_2)/I_{A_2}(T)$. This subset is not open, so the algebra is not a subalgebra of $A_1 \otimes_{\min} A_2$. A main point in the proof is that, given V_2 and a nonzero projection

$$e_2 \in I_{A_2}(V_2)/I_{A_2}(S_2)$$

(see (4.3) for the definition of S_2), the sets V_1 and T have been chosen so that there is a projection $p_2 \in I_{A_2}(V_2)/I_{A_2}(T)$ whose image is e_2, and so that the set

$$Y \cup [V_1 \times (V_2 \setminus T)]$$

is open.

We do not get a proof that the tensor product of two algebras with the weak ideal property again has the weak ideal property, because we do not know how to reduce the size of V_2 (to go with an analogous subset $T_1 \subset V_1$) without changing the projection e_2.

Proof of Theorem 4.8 Replacing A_2 by $K \otimes A_2$, we may assume that if $I, J \subset A_2$ are ideals such that $I \subsetneqq J$, then J/I contains a nonzero projection.

Define $X_j = \mathrm{Prim}(A_j)$ for $j = 1, 2$. Using [3] in the same way as in the proof of Theorem 4.4, we identify $\mathrm{Prim}(A_1 \otimes_{\min} A_2) = X_1 \times X_2$. The identification is given by the map from $X_1 \times X_2$ to $\mathrm{Prim}(A_1 \otimes_{\min} A_2)$ sending $(P_1, P_2) \in X_1 \times X_2$ to the primitive ideal obtained as the kernel of $A_1 \otimes_{\min} A_2 \to (A_1/P_1) \otimes_{\min} (A_2/P_2)$. The lattice of ideals of $A_1 \otimes_{\min} A_2$ can thus be canonically identified with the lattice of open subsets of $X_1 \times X_2$ when this space is equipped with the product topology. We simplify Notation 4.5 by writing $I_j(U)$ for $I_{A_j}(U)$ when $U \subset X_j$ is open, and $I(W)$ for $I_{A_1 \otimes_{\min} A_2}(W)$ when $W \subset X_1 \times X_2$ is open. We then get canonical isomorphisms $I_1(U_1) \otimes_{\min} I_2(U_2) \cong I(U_1 \times U_2)$ for open subsets $U_1 \subset X_1$ and $U_2 \subset X_2$.

We need to show that if $Y, Z \subset X_1 \times X_2$ are open subsets such that $Y \subsetneqq Z$, then $I(Z)/I(Y)$ contains a nonzero projection.

Choose $x_1 \in X_1$ and $x_2 \in X_2$ such that $(x_1, x_2) \in Z \setminus Y$. Choose open sets $U \subset X_1$ and $V_2 \subset X_2$ such that $x_1 \in U$, $x_2 \in V_2$, and $U \times V_2 \subset Z$ Define

$$S_2 = \{ y \in V_2 : (x_1, y) \in Y \}, \tag{4.3}$$

which is an open proper subset of V_2. By the reduction at the beginning of the proof, there is a nonzero projection $e_2 \in I_2(V_2)/I_2(S_2)$. Use Lemma 4.6 with $A = I_2(V_2)$ and with S_2 in place of U to choose subsets $T \subset L \subset S_2$ such that L is compact, T is open, and there is a projection $p_2 \in I_2(V_2)/I_2(T)$ whose image in $I_2(V_2)/I_2(S_2)$ is equal to e_2. Use Lemma 4.7 to choose an open set $V_1 \subset U$ such that $x_1 \in V_1$ and $V_1 \times L \subset Y$. Define $S_1 = V_1 \cap (X_1 \setminus \overline{\{x_1\}})$, which is an open proper subset of V_1. Since A_1 has the ideal property, there is a projection $p_1 \in I_1(V_1)$ whose image $e_1 \in I_1(V_1)/I_1(S_1)$ is nonzero.

We claim that

$$Y \cap [(V_1 \setminus S_1) \times (V_2 \setminus S_2)] = \varnothing. \tag{4.4}$$

The definitions of the sets involved imply that

$$V_1 \setminus S_1 \subset \overline{\{x_1\}} \quad \text{and} \quad V_2 \setminus S_2 = \{ y \in V_2 : (x_1, y) \notin Y \}.$$

Therefore

$$(V_1 \setminus S_1) \times (V_2 \setminus S_2) \subset \overline{\{x_1\} \times (V_2 \setminus S_2)} \quad \text{and} \quad [\{x_1\} \times (V_2 \setminus S_2)] \cap Y = \varnothing.$$

Since Y is open, the claim follows.

We now want to construct a commutative diagram as follows:

$$\begin{tikzcd} I(V_1 \times V_2) \arrow[r, "\pi"] \arrow[d, "\iota"] & I(V_1 \times V_2)/I(V_1 \times T) \arrow[r, "\kappa"] \arrow[d, "\varphi"] & I(V_1 \times V_2)/I(R) \arrow[d, "\psi"] \\ I(Z) \arrow[r, "\sigma"'] & I(Z)/I(Y) \arrow[r, "\rho"'] & I(Z)/I(Y \cup R). \end{tikzcd} \tag{4.5}$$

The maps π and σ are the obvious quotient maps, and ι is the obvious inclusion, coming from $V_1 \times V_2 \subset U \times V_2 \subset Z$. We define $R = (V_1 \times S_2) \cup (S_1 \times V_2)$, which is an open subset of $V_1 \times V_2$. In particular, $R \subset Z$. The map κ is then the quotient map arising from the inclusion $V_1 \times T \subset V_1 \times S_2 \subset R$, and ρ is the quotient map arising from the inclusion $Y \subset Y \cup R$. Since π is surjective, the map φ is unique if it exists. For existence, we must show that $\operatorname{Ker}(\pi) \subset \operatorname{Ker}(\sigma \circ \iota)$. This inclusion follows from

$$\operatorname{Ker}(\pi) = I(V_1 \times T), \quad \operatorname{Ker}(\sigma \circ \iota) = I\big((V_1 \times V_2) \cap Y \big),$$
$$V_1 \times T \subset V_1 \times L \subset Y, \quad \text{and} \quad T \subset S_2 \subset V_2.$$

It remains to construct ψ. Since κ is surjective, the map ψ is unique if it exists. We claim that $\operatorname{Ker}(\kappa) = \operatorname{Ker}(\rho \circ \varphi)$. Since π is surjective, it suffices to prove that $\operatorname{Ker}(\kappa \circ \pi) = \operatorname{Ker}(\rho \circ \varphi \circ \pi)$. We easily check that

$$\operatorname{Ker}(\kappa \circ \pi) = I(R) \quad \text{and} \quad \operatorname{Ker}(\rho \circ \varphi \circ \pi) = I\big((Y \cup R) \cap (V_1 \times V_2) \big).$$

It follows from (4.4) that $(Y \cup R) \cap (V_1 \times V_2) = R$, proving the claim. The claim implies not only that there is a map ψ making the right hand square commute, but also that ψ is injective.

The identification $\operatorname{Prim}(A_1 \otimes_{\min} A_2) = X_1 \times X_2$ gives identifications

$$I(V_1 \times V_2)/I(V_1 \times T) = I_1(V_1) \otimes_{\min} [I_2(V_2)/I_2(T)]$$

and

$$I(V_1 \times V_2)/I(R) = [I_1(V_1)/I_1(S_1)] \otimes_{\min} [I_2(V_2)/I_2(S_2)],$$

with respect to which κ becomes the tensor product of the quotient maps

$$I_1(V_1) \to I_1(V_1)/I_1(S_1) \quad \text{and} \quad I_2(V_2)/I_2(T) \to I_2(V_2)/I_2(S_2).$$

Define $q \in I(Z)/I(Y)$ by $q = \varphi(p_1 \otimes p_2)$. Then q is a projection. Moreover, $\rho(q) = (\psi \circ \kappa)(p_1 \otimes p_2) = \psi(e_1 \otimes e_2)$. Since $e_1 \neq 0$, $e_2 \neq 0$, and ψ is injective, it follows that $q \neq 0$. Thus $I(Z)/I(Y)$ contains a nonzero projection, as desired. ∎

Using results from Section 7 below, we can now give a case in which the tensor product of C*-algebras with the weak ideal property again has this property.

Corollary 4.9 *Let A and B be separable C*-algebras. Assume that A or B is exact, and that A is in the class $\mathcal{W}$ of Theorem 7.15. If A and B have the weak ideal property, then $A \otimes_{\min} B$ has the weak ideal property.*

The class $\mathcal{W}$ is the smallest class of separable C*-algebras that contains the separable locally AH algebras, the separable LS algebras, the separable type I C*-algebras, and the separable purely infinite C*-algebras, and is closed under finite and countable direct sums and under minimal tensor products when one tensor factor is exact.

Proof of Corollary 4.9 By Theorem 2.8, the algebra A has topological dimension zero. Combine Lemma 7.5, Lemma 7.6, Lemma 7.13, Lemma 7.12 (ii), Proposition 7.14, and Theorem 2.9, to see that A is in the class $\mathcal{P}$ of Notation 7.3. Thus A has the ideal property. So $A \otimes_{\min} B$ has the weak ideal property by Theorem 4.8. ∎

Combining Proposition 7.16 below with Theorem 4.8 and with [33, Theorem 8.5 (6)], one immediately sees that if A and B are separable C*-algebras with the weak ideal property, one of which is exact, and $\mathrm{Prim}(A)$ is Hausdorff, then $A \otimes_{\min} B$ has the weak ideal property. A different argument allows one to prove this without separability. We give it here, although it is based on material on $C_0(X)$-algebras in the next section. We first consider the case in which $\mathrm{Prim}(A)$ is finite but not necessarily Hausdorff.

Proposition 4.10 *Let A and B be C*-algebras with the weak ideal property such that* $\mathrm{Prim}(A)$ *is finite and A or B is exact. Then* $A \otimes_{\min} B$ *has the weak ideal property.*

Proof First suppose that A is simple. Using [3, Proposition 2.17 (2)] and parts (ii) and (iv) of [3, Proposition 2.16], we see that $J \mapsto A \otimes_{\min} J$ is a one-to-one correspondence from the ideals of B to the ideals of $A \otimes_{\min} B$; moreover, if $J_1 \subset J_2 \subset B$ are ideals, then $(A \otimes_{\min} J_2)/(A \otimes_{\min} J_1) \cong A \otimes_{\min} (J_2/J_1)$. Now let $L_1, L_2 \subset A \otimes_{\min} B$ be ideals with $L_1 \subset L_2$. It follows that there exist ideals $J_1, J_2 \subset B$ with $J_1 \subset J_2$ such that $L_2/L_1 \cong A \otimes_{\min} (J_2/J_1)$. There are nonzero projections $p_1 \in K \otimes A$ and $p_2 \in K \otimes (J_2/J_1)$, so $p_1 \otimes p_2$ is a nonzero projection in

$$[K \otimes A] \otimes_{\min} [K \otimes (J_2/J_1)] \cong K \otimes (L_2/L_1).$$

We prove the general case by induction on $\mathrm{card}(\mathrm{Prim}(A))$. We just did the case $\mathrm{card}(\mathrm{Prim}(A)) = 1$. So let $n \in \mathbb{Z}_{>0}$ and suppose the result is known whenever $\mathrm{card}(\mathrm{Prim}(A)) < n$. Assume that $\mathrm{card}(\mathrm{Prim}(A)) = n$. Choose a nontrivial ideal $I \subset A$. By [3, Proposition 2.17 (2)] and [3, Proposition 2.16 (iv)], the sequence

$$0 \longrightarrow I \otimes_{\min} B \longrightarrow A \otimes_{\min} B \longrightarrow (A/I) \otimes_{\min} B \longrightarrow 0$$

is exact. The algebras $I \otimes_{\min} B$ and $(A/I) \otimes_{\min} B$ have the weak ideal property by the induction hypothesis, so $A \otimes_{\min} B$ has the weak ideal property by [33, Theorem 8.5 (5)]. ∎

Much of the proof of the following proposition will be reused in the proof of Proposition 7.16.

Proposition 4.11 *Let A and B be C*-algebras such that A or B is exact and* $\mathrm{Prim}(A)$ *is Hausdorff. If A and B have the weak ideal property, then* $A \otimes_{\min} B$ *has the weak ideal property.*

Proof Set $X = \mathrm{Prim}(A)$. We first claim that A is a continuous $C_0(X)$-algebra with fiber $A_P = A/P$ for $P \in X$. In the language of continuous fields, this is [13, Theorem 2.3]. To get it in our language, apply [24, Theorem 3.3], taking $\alpha\colon \mathrm{Prim}(A) \to X$ to be the identity map. Identifying continuous $C_0(X)$-algebras and continuous C*-bundles as in Proposition 5.6 (iii), we use [22, Corollary 2.8] to see that $A \otimes_{\min} B$ is a continuous $C_0(X)$-algebra, with fibers $(A \otimes_{\min} B)_P = (A/P) \otimes_{\min} B$ for $P \in X$.

The algebra A has topological dimension zero by Theorem 2.8. Since X is Hausdorff, it follows that X is totally disconnected. For every $P \in X$, the quotient A/P is simple because $\{P\}$ is closed, and has the weak ideal property by [33, Theorem 8.5 (5)]. So the fiber $(A \otimes_{\min} B)_P = (A/P) \otimes_{\min} B$ has the weak ideal property by

Proposition 4.10. Theorem 5.14 (iii) now implies that $A \otimes_{\min} B$ has the weak ideal property. ∎

Question 4.12 Let A and B be C*-algebras with A exact. If A and B have the weak ideal property, does $A \otimes_{\min} B$ have the weak ideal property?

Question 4.13 Let A and B be C*-algebras with A exact. If A and B have residual (SP), does $A \otimes_{\min} B$ have residual (SP)?

5 Permanence Properties for Bundles Over Totally Disconnected Spaces

We now turn to section algebras of continuous fields over totally disconnected base spaces. We prove that if A is the section algebra of a bundle over a totally disconnected space, and the fibers all have one of the properties residual (SP), topological dimension zero, the weak ideal property, or the combination of the ideal property and pure infiniteness, then A also has the same property. Moreover, if A has one of these properties, so do all the fibers.

The section algebra of a continuous field over a space that is not totally disconnected will not have the weak ideal property except in trivial cases, and the same is true of the other properties involving the existence of projections in ideals. Indeed, we prove that for a continuous field over a second countable locally compact Hausdorff space with nonzero fibers, if the section algebra is separable and has one of the four properties above, then the base space must be totally disconnected.

The fact that the properties we consider are equivalent to being residually hereditarily in a suitable class $\mathcal{C}$ underlies some of our reasoning, but our proofs also require a semiprojectivity condition. (See the proof of Lemma 5.13.) Proposition 5.19 gives some hope that the results might still be true for a general property of this form.

Following standard notation, if A is a C*-algebra, then $M(A)$ is its multiplier algebra and $Z(A)$ is its center.

Definition 5.1 Let X be a locally compact Hausdorff space. If A is a C*-algebra and $\iota\colon C_0(X) \to Z(M(A))$ is a homomorphism, we say that ι is *nondegenerate* if $\overline{\iota(C_0(X))A} = A$. A $C_0(X)$-*algebra* is a C*-algebra A together with a nondegenerate homomorphism $\iota\colon C_0(X) \to Z(M(A))$.

Unlike in Definition 2.1 of [24], we do not assume that ι is injective. This permits a hereditary subalgebra of A to also be a $C_0(X)$-algebra, without having to replace X by a closed subspace.

Notation 5.2 Let the notation be as in Definition 5.1. For an open set $U \subset X$, we identify $C_0(U)$ with the obvious ideal of $C_0(X)$. Then $\overline{\iota(C_0(U))A}$ is an ideal in A. For $x \in X$, we define $A_x = A\big/\overline{\iota(C_0(X \setminus \{x\}))A}$, and we let $\mathrm{ev}_x\colon A \to A_x$ be the quotient map. For a closed subset $L \subset X$, we define $A|_L = A/\overline{\iota(C_0(X \setminus L))A}$. We equip it with the $C_0(L)$-algebra structure that comes from the fact that $C_0(X \setminus L)$ is contained in the kernel of the composition $C_0(X) \to Z(M(A)) \to Z(M(A|_L))$.

Notation 5.2 entails in particular $A_x = A|_{\{x\}}$. Strictly speaking, A is the section algebra of a bundle and $A|_L$ is the section algebra of the restriction of this bundle to L, but the abuse of notation is convenient.

Lemma 5.3 *Let the notation be as in Definition 5.1 and Notation 5.2. Let $a \in A$. Then we have the following.*

(i) $\|a\| = \sup_{x \in X} \|\mathrm{ev}_x(a)\|$.
(ii) *For every $\varepsilon > 0$, the set $\{x \in X : \|\mathrm{ev}_x(a)\| \geq \varepsilon\} \subset X$ is compact.*
(iii) *The function $x \mapsto \|\mathrm{ev}_x(a)\|$ is upper semicontinuous.*
(iv) *For $f \in C_0(X)$ and $x \in X$, we have $\mathrm{ev}_x(\iota(f)a) = f(x)\mathrm{ev}_x(a)$.*

Proof When ι is injective, the first three parts are [24, Corollary 2.2], and the last part is contained in the proof of [24, Theorem 2.3]. (See [24, Lemma 1.1] for the notation.) In the general case, let $Y \subset X$ be the closed subset such that

$$\mathrm{Ker}(\iota) = \left\{ f \in C_0(X) : f|_Y = 0 \right\}.$$

Then A is a $C_0(Y)$-algebra in the obvious way. We have $A_x = 0$ for $x \notin Y$, and the function $x \mapsto \|\mathrm{ev}_x(a)\|$ associated with the $C_0(X)$-algebra structure is obtained by extending the one associated with the $C_0(Y)$-algebra structure to be zero on $X \setminus Y$. The first three parts then follow from those for the $C_0(Y)$-algebra structure, as does the last when $x \in Y$. The last part is trivial for $x \in X \setminus Y$. ∎

Definition 5.4 Let X be a locally compact Hausdorff space, and let A be a $C_0(X)$-algebra. We say that A is a *continuous $C_0(X)$-algebra* if for all $a \in A$, the map $x \mapsto \|\mathrm{ev}_x(a)\|$ of Lemma 5.3 (iii) is continuous.

Proposition 5.5 *Let X be a locally compact Hausdorff space and let A be a C^*-algebra. Then homomorphisms $\iota: C_0(X) \to Z(M(A))$ that make A into a continuous $C_0(X)$-algebra correspond bijectively to isomorphisms of A with the algebra of continuous sections vanishing at infinity of a continuous field of C^*-algebras over X, as in [9, 10.4.1].*

Proof This is essentially contained in [24, Theorem 2.3], referring to the definitions at the end of [24, §1]. ∎

We will also need to use results from [22], so we compare definitions.

Proposition 5.6 *Let X be a locally compact Hausdorff space.*

(i) *Let $(X, (\pi_x: A \to A_x)_{x \in X}, A)$ be a (not necessarily continuous) C^*-bundle in the sense of [22, Definition 1.1]. Then A is a $C_0(X)$-algebra, with structure map $\iota: C_0(X) \to Z(M(A))$ determined by the product in [22, Definition 1.1 (ii)], if and only if for every $a \in A$ the function $x \mapsto \|\pi_x(a)\|$ is upper semicontinuous and vanishes at infinity.*
(ii) *Let A be a $C_0(X)$-algebra. Then $(X, (\mathrm{ev}_x: A \to A_x)_{x \in X}, A)$ is a C^*-bundle in the sense of [22, Definition 1.1] that satisfies the semicontinuity condition in (i).*
(iii) *In (i) and (ii), A is a continuous $C_0(X)$-algebra if and only if the corresponding C^*-bundle is continuous in the sense of of [22, Definition 1.1 (iii)].*

Proof Theorem 2.3 of [24] and the preceding discussion give a one-to-one correspondence between $C_0(X)$-algebras with injective structure maps and upper semicontinuous bundles over X in the sense of the definitions at the end [24, §1] and for which the set of points in X with nonzero fibers is dense. By substituting our Lemma 5.3 for [24, Corollary 2.2] at appropriate places in the proof in [24], one sees that the proof still works if one simultaneously drops injectivity of the structure map and density of the points with nonzero fibers. (Some of the argument is also contained in [22, Lemma 2.1].)

The difference between [22, Definition 1.1] and the definition of [24] is that [22] omits the requirement (condition (iii) in [24]) that the set $\{x \in X : \|a(x)\| \geq r\}$ be compact for $a \in A$ and $r > 0$. It is easy to check that a function $f: X \to [0, \infty)$ is upper semicontinuous and vanishes at infinity if and only if for every $r > 0$ the set $\{x \in X : f(x) \geq r\}$ is compact. Thus, the definitions of [24] and [22] are equivalent. This completes the proofs of parts (i) and (ii).

Part (iii) is now immediate from the definitions. ∎

We prove results stating that if X is totally disconnected and the fibers of a $C_0(X)$-algebra A have a particular property, then so does A. These do not require continuity. We return to continuity later in this section when we prove that if a continuous $C_0(X)$-algebra with nonzero fibers has one of our properties, then X is totally disconnected. These results fail without continuity.

Lemma 5.7 *Let the notation be as in Definition 5.1 and Notation 5.2. Let $B \subset A$ be a hereditary subalgebra. Let $a \in A$. Then $a \in B$ if and only if $\mathrm{ev}_x(a) \in \mathrm{ev}_x(B)$ for all $x \in X$.*

Proof The forward implication is immediate.

For the reverse implication, we first claim that if $f \in C_0(X)$ and $b \in B$, then $\iota(f)b \in B$. To prove the claim, it suffices to consider the case $b \geq 0$. In this case, $\iota(f)b = b^{1/2}\iota(f)b^{1/2}$, and the claim follows from the fact that B is also a hereditary subalgebra in $M(A)$.

To prove the result, suppose that $a \in A$ satisfies $\mathrm{ev}_x(a) \in \mathrm{ev}_x(B)$ for all $x \in X$. It is enough to prove that for every $\varepsilon > 0$ there is $b \in B$ such that $\|a - b\| < \varepsilon$. So let $\varepsilon > 0$. Define $K \subset X$ by $K = \{x \in X : \|\mathrm{ev}_x(a)\| \geq \frac{\varepsilon}{2}\}$. For $x \in K$, choose $c_x \in B$ such that $\mathrm{ev}_x(c_x) = \mathrm{ev}_x(a)$, and define $U_x \subset X$ by $U_x = \{y \in X : \|\mathrm{ev}_y(c_x - a)\| < \frac{\varepsilon}{2}\}$. It follows from Lemma 5.3 (ii) that K is compact and from Lemma 5.3 (iii) that U_x is open for all $x \in K$. Choose $x_1, x_2, \dots, x_n \in K$ such that the sets $U_{x_1}, U_{x_2}, \dots, U_{x_n}$ cover K. Choose continuous functions $f_k: X \to [0,1]$ with compact support contained in U_{x_k} for $k = 1, 2, \dots, n$, and such that for $x \in K$ we have $\sum_{k=1}^n f_k(x) = 1$ and for $x \in X \setminus K$ we have $\sum_{k=1}^n f_k(x) \leq 1$. Define $b \in A$ by $b = \sum_{k=1}^n \iota(f_k)c_{x_k}$. Then $b \in B$ by the claim. Moreover, if $x \in K$, then, using Lemma 5.3 (iv) at the first step and $\|\mathrm{ev}_x(c_{x_k} - a)\| < \frac{\varepsilon}{2}$ whenever $f_k(x) \neq 0$ at the second step, we have

$$\|\mathrm{ev}_x(b - a)\| \leq \sum_{k=1}^n f_k(x)\|\mathrm{ev}_x(c_{x_k} - a)\| < \frac{\varepsilon}{2}.$$

Define $f(x) = 1 - \sum_{k=1}^{n} f_k(x)$ for $x \in X$. For $x \in X \smallsetminus K$, similar reasoning gives

$$\begin{aligned}\|\mathrm{ev}_x(b-a)\| &\leq \|\mathrm{ev}_x(b - [1-f(x)]a)\| + \|f(x)\mathrm{ev}_x(a)\| \\ &\leq \sum_{k=1}^{n} f_k(x)\|\mathrm{ev}_x(c_{x_k} - a)\| + f(x)\|\mathrm{ev}_x(a)\| \\ &\leq [1-f(x)]\frac{\varepsilon}{2} + f(x)\|\mathrm{ev}_x(a)\| \leq \frac{\varepsilon}{2}.\end{aligned}$$

It now follows from Lemma 5.3 (i) that $\|b - a\| < \varepsilon$. This completes the proof. ■

Corollary 5.8 *Let X be a locally compact Hausdorff space, let A be a $C_0(X)$-algebra with structure map $\iota\colon C_0(X) \to Z(M(A))$, and let $B \subset A$ be a hereditary subalgebra. Then there is a homomorphism $\mu\colon C_0(X) \to Z(M(B))$ that makes B a $C_0(X)$-algebra and such that for all $b \in B$ and $f \in C_0(X)$ we have $\mu(f)b = \iota(f)b$. Moreover, $B_x = \mathrm{ev}_x(B)$ for all $x \in X$.*

Proof It follows from Lemma 5.7 that if $f \in C_0(X)$ and $b \in B$, then $\iota(f)b \in B$. For $f \in C_0(X)$, we define $T_f\colon B \to B$ by $T_f(b) = \iota(f)b$ for $b \in B$. It is easy to check that (T_f, T_f) is a double centralizer of B, and that $f \mapsto (T_f, T_f)$ defines a homomorphism $\mu\colon C_0(X) \to Z(M(B))$. Nondegeneracy of μ follows from nondegeneracy of ι. The relations $\mu(f)b = \iota(f)b$ and $B_x = \mathrm{ev}_x(B)$ hold by construction. ■

Lemma 5.9 *Let X be a locally compact Hausdorff space, let A be a $C_0(X)$-algebra with structure map $\iota\colon C_0(X) \to Z(M(A))$, let $F \subset A$ be a finite set, and let $\varepsilon > 0$. Then there is $f \in C_c(X)$ such that $0 \leq f \leq 1$ and $\|\iota(f)a - a\| < \varepsilon$ for all $a \in F$.*

Proof Define $K \subset X$ by $K = \{x \in X : \text{there is } a \in F \text{ such that } \|\mathrm{ev}_x(a)\| \geq \frac{\varepsilon}{3}\}$. It follows from Lemma 5.3 (ii) that K is compact. Choose $f \in C_c(X)$ such that $0 \leq f \leq 1$ and $f(x) = 1$ for all $x \in K$.

Fix $a \in F$. Let $x \in X$. If $x \in K$, then, using Lemma 5.3 (iv), $\|\mathrm{ev}_x(\iota(f)a - a)\| = 0$. Otherwise, again using Lemma 5.3 (iv),

$$\|\mathrm{ev}_x(\iota(f)a - a)\| \leq f(x)\|\mathrm{ev}_x(a)\| + \|\mathrm{ev}_x(a)\| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3}.$$

Clearly $\sup_{x \in X} \|\mathrm{ev}_x(\iota(f)a - a)\| \leq \frac{2\varepsilon}{3} < \varepsilon$. So $\|\iota(f)a - a\| < \varepsilon$ by Lemma 5.3 (i). ■

Lemma 5.10 *Let X be a locally compact Hausdorff space, let A be a $C_0(X)$-algebra with structure map $\iota\colon C_0(X) \to Z(M(A))$, let $z \in X$, let $F \subset \mathrm{Ker}(\mathrm{ev}_z)$ be a finite set, and let $\varepsilon > 0$. Then there is $f \in C_c(X \smallsetminus \{z\})$ such that $0 \leq f \leq 1$ and $\|\iota(f)a - a\| < \varepsilon$ for all $a \in F$.*

Proof The proof is essentially the same as that of Lemma 5.9. We define K as there, observe that $z \notin K$, and require that $\mathrm{supp}(f)$, in addition to being compact, be contained in $X \smallsetminus \{z\}$. ■

Lemma 5.11 *Let X be a locally compact Hausdorff space, let A be a $C_0(X)$-algebra with structure map $\iota\colon C_0(X) \to Z(M(A))$, and let $I \subset A$ be an ideal. Let $\pi\colon A \to A/I$ be the quotient map. Then there is a homomorphism $\mu\colon C_0(X) \to Z(M(A/I))$ that makes*

A/I a $C_0(X)$-algebra and such that for all $a \in A$ and $f \in C_0(X)$ we have $\mu(f)\pi(a) = \pi(\iota(f)a)$. Moreover, giving I the $C_0(X)$-algebra from Corollary 5.8, for every $x \in X$, we have $(A/I)_x \cong A_x/I_x$.

Proof Let $\overline{\pi}\colon M(A) \to M(A/I)$ be the map on multiplier algebras induced by $\pi\colon A \to A/I$. Define $\mu = \overline{\pi} \circ \iota$. It is clear that μ is a homomorphism to $Z(M(A/I))$. We prove nondegeneracy. So let $b \in A/I$ and let $\varepsilon > 0$. Choose $a \in A$ such that $\pi(a) = b$. Use Lemma 5.9 to choose $f \in C_c(X)$ such that $0 \le f \le 1$ and $\|\iota(f)a - a\| < \varepsilon$. Then

$$\|\mu(f)b - b\| = \big\|\pi\big(\iota(f)a - a\big)\big\| < \varepsilon.$$

This completes the proof of nondegeneracy.

It remains to prove the last statement. Let $x \in X$. Let $\mathrm{ev}_x\colon A \to A_x$ be as in Notation 5.2, and let $\overline{\mathrm{ev}}_x\colon A/I \to (A/I)_x$ be the corresponding map with A/I in place of A. Also let $\pi_x\colon A_x \to A_x/I_x$ be the quotient map. Then $\pi_x \circ \mathrm{ev}_x$ and $\overline{\mathrm{ev}}_x \circ \pi$ are surjective, so it suffices to show that they have the same kernel.

Let $a \in A$. Suppose first $(\pi_x \circ \mathrm{ev}_x)(a) = 0$. Let $\varepsilon > 0$. We will prove that $\|(\overline{\mathrm{ev}}_x \circ \pi)(a)\| < \varepsilon$. We have $\mathrm{ev}_x(a) \in I_x$. So there is $b \in I$ such that $\mathrm{ev}_x(b) = \mathrm{ev}_x(a)$. Then $\mathrm{ev}_x(a - b) = 0$. So Lemma 5.10 provides $f \in C_c(X \smallsetminus \{x\})$ such that $0 \le f \le 1$ and

$$\|\iota(f)(a-b) - (a-b)\| < \varepsilon.$$

By Corollary 5.8, we have $\iota(f)b \in I$. So $\pi(\iota(f)b) = 0$. We already know that $\pi(b) = 0$, so

$$\|\mu(f)\pi(a) - \pi(a)\| = \big\|\pi\big(\iota(f)(a-b) - (a-b)\big)\big\| < \varepsilon.$$

Since $\overline{\mathrm{ev}}_x(\mu(f)\pi(a)) = 0$, it follows that $\|(\overline{\mathrm{ev}}_x \circ \pi)(a)\| < \varepsilon$.

Now assume that $(\overline{\mathrm{ev}}_x \circ \pi)(a) = 0$. Let $\varepsilon > 0$. We prove that $\|(\pi_x \circ \mathrm{ev}_x)(a)\| < \varepsilon$. Apply Lemma 5.10 to the $C_0(X)$-algebra A/I, getting $f \in C_c(X \smallsetminus \{x\})$ such that $0 \le f \le 1$ and $\|\mu(f)\pi(a) - \pi(a)\| < \varepsilon$. Thus $\|\pi(\iota(f)a - a)\| < \varepsilon$. Choose $b \in I$ such that $\big\|[\iota(f)a - a] - b\big\| < \varepsilon$. It follows that $\|(\pi_x \circ \mathrm{ev}_x)(\iota(f)a - a - b)\| < \varepsilon$. Since $\mathrm{ev}_x(\iota(f)a) = 0$ and $(\pi_x \circ \mathrm{ev}_x)(b) = 0$, it follows that $\|(\pi_x \circ \mathrm{ev}_x)(a)\| < \varepsilon$, as desired. ∎

The following result is closely related to [22].

Lemma 5.12 *Let X be a locally compact Hausdorff space, let A be a $C_0(X)$-algebra with structure map $\iota\colon C_0(X) \to Z(M(A))$, and let D be a C^*-algebra. Then there is a homomorphism $\mu\colon C_0(X) \to Z(M(D \otimes_{\max} A))$ that makes $D \otimes_{\max} A$ a $C_0(X)$-algebra and such that for all $a \in A$, $d \in D$, and $f \in C_0(X)$ we have $\mu(f)(d \otimes a) = d \otimes \iota(f)a$. Moreover, for every $x \in X$, we have $(D \otimes_{\max} A)_x \cong D \otimes_{\max} A_x$.*

Proof The family

$$(X, (\mathrm{id}_D \otimes_{\max} \pi_x\colon D \otimes_{\max} A \to D \otimes_{\max} A_x)_{x \in X}, D \otimes_{\max} A)$$

is a C*-bundle in the sense of [22, Definition 1.1]. (See (2) in [22, p. 678].) In particular, for $f \in C_0(X)$ and $b \in D \otimes_{\max} A$, the product $f \cdot b$ is defined, and for $a \in A$ and $d \in D$ it satisfies $f \cdot (d \otimes a) = d \otimes \iota(f)a$.

Let $x \in X$. We know (see Notation 5.2) that the sequence

$$0 \longrightarrow \overline{\iota(C_0(X \setminus \{x\}))A} \longrightarrow A \xrightarrow{\mathrm{ev}_x} A_x \longrightarrow 0$$

is exact. The functor $D \otimes_{\max} (-)$ is exact, so the sequence

$$0 \longrightarrow D \otimes_{\max} \overline{\iota(C_0(X \setminus \{x\}))A} \longrightarrow D \otimes_{\max} A \xrightarrow{\mathrm{id}_D \otimes \mathrm{ev}_x} D \otimes_{\max} A_x \longrightarrow 0 \tag{5.1}$$

is exact. Now let $d \in D$ and let $a \in \overline{\iota(C_0(X \setminus \{x\}))A}$. We claim that the image b of $d \otimes a$ in $D \otimes_{\max} A$ is actually in $\overline{C_0(X \setminus \{x\})(D \otimes_{\max} A)}$. To prove the claim, let $\varepsilon > 0$ and use Lemma 5.10 to choose $f \in C_c(X \setminus \{x\})$ such that $0 \leq f \leq 1$ and

$$\|\iota(f)a - a\| < \frac{\varepsilon}{\|d\| + 1}.$$

Then $\|f \cdot b - b\| \leq \|d\| \|\iota(f)a - a\| < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, the claim follows. Using exactness of (5.1), we conclude that

$$\mathrm{Ker}(\mathrm{id}_D \otimes \mathrm{ev}_x) \subset \overline{C_0(X \setminus \{x\})(D \otimes_{\max} A)}. \tag{5.2}$$

The reverse inclusion is clear. Combining equality in (5.2) with exactness of (5.1), we get the exact sequence

$$0 \longrightarrow \overline{C_0(X \setminus \{x\})(D \otimes_{\max} A)} \longrightarrow D \otimes_{\max} A \xrightarrow{\mathrm{id}_D \otimes \mathrm{ev}_x} D \otimes_{\max} A_x \longrightarrow 0.$$

Since this sequence is exact for all $x \in X$, [22, Lemma 2.3] implies that for all $b \in D \otimes_{\max} A$, the function $x \mapsto \|\mathrm{ev}_x(b)\|$ is upper semicontinuous. It is clear that for $d \in D$ and $a \in A$ the function $x \mapsto \|\mathrm{ev}_x(d \otimes a)\|$ vanishes at infinity, and it then follows from density that for all $b \in D \otimes_{\max} A$ the function $x \mapsto \|\mathrm{ev}_x(b)\|$ vanishes at infinity. Now apply Proposition 5.6. ■

Lemma 5.13 *Let X be a totally disconnected locally compact Hausdorff space, let A be a $C_0(X)$-algebra with structure map $\iota: C_0(X) \to Z(M(A))$, and let $x \in X$.*

(i) *Let $p \in A_x$ be a projection. Then there is a projection $e \in A$ such that $\mathrm{ev}_x(e) = p$.*
(ii) *Let $p \in A_x$ be an infinite projection. Then there is an infinite projection $e \in A$ such that $\mathrm{ev}_x(e) = p$.*

The proof is a semiprojectivity argument. It is slightly indirect, because we do not know that there is a countable neighborhood base at x.

Proof of Lemma 5.13 We prove (i). Since $\mathbb{C}$ is semiprojective, there is $\varepsilon > 0$ such that if B and C are C*-algebras, $\varphi: B \to C$ is a homomorphism, $b \in B$ satisfies $\|b^* - b\| < \varepsilon$, $\|b^2 - b\| < \varepsilon$, and $\varphi(b)$ is a projection, then there exists a projection $e \in B$ such that $\varphi(e) = \varphi(b)$. Since ev_x is surjective, there is $a \in A$ such that $\mathrm{ev}_x(a) = p$. By Lemma 5.3 (iii), there is an open set $U \subset X$ with $x \in U$ such that for all $y \in U$ we have $\|\mathrm{ev}_y(a^* - a)\| < \frac{\varepsilon}{2}$ and $\|\mathrm{ev}_y(a^2 - a)\| < \frac{\varepsilon}{2}$. Since X is totally disconnected, there is a compact open set $K \subset X$ such that $x \in K \subset U$. Define $b = \iota(\chi_K)a$. Using Lemma 5.3 (iv), we get $\|\mathrm{ev}_y(b^* - b)\| < \frac{\varepsilon}{2}$ and $\|\mathrm{ev}_y(b^2 - b)\| < \frac{\varepsilon}{2}$ when $y \in K$, and $\mathrm{ev}_y(b^* - b) = \mathrm{ev}_y(b^2 - b) = 0$ when $y \in X \setminus K$. It follows from Lemma 5.3 (i) that $\|b^* - b\| \leq \frac{\varepsilon}{2} < \varepsilon$ and $\|b^2 - b\| \leq \frac{\varepsilon}{2} < \varepsilon$. Now obtain e by using the choice of ε with $B = A$ and $C = A_x$.

We describe the changes needed for the proof of (ii). Let T be the Toeplitz algebra, generated by an isometry s (so $s^*s = 1$, but $ss^* \neq 1$). By hypothesis, there is a homomorphism $\varphi_0: T \to A_x$ such that $\varphi_0(1) = p$ and $\varphi_0(1 - ss^*) \neq 0$. Since T is semiprojective, an argument similar to that in the proof of (i) shows that there is a homomorphism $\varphi: T \to A$ such that $\mathrm{ev}_x \circ \varphi = \varphi_0$. Set $e = \varphi(1)$. Then $\varphi(s)^*\varphi(s) = e$ and $\varphi(s)\varphi(s)^* \leq e$. We have $e - \varphi(s)\varphi(s)^* \neq 0$ because $\mathrm{ev}_x(e - \varphi(s)\varphi(s)^*) \neq 0$. So e is an infinite projection. ∎

Theorem 5.14 *Let X be a totally disconnected locally compact Hausdorff space and let A be a $C_0(X)$-algebra.*

(i) *Assume that A_x has residual (SP) for all $x \in X$. Then A has residual (SP).*
(ii) *Assume that A_x is purely infinite and has the ideal property for all $x \in X$. Then A is purely infinite and has the ideal property.*
(iii) *Assume that A_x has the weak ideal property for all $x \in X$. Then A has the weak ideal property.*
(iv) *Assume that A is separable and A_x has topological dimension zero for all $x \in X$. Then A has topological dimension zero.*

Proof We prove (i). Recall [33, Definition 7.1] that a C*-algebra D has residual (SP) if and only if D is residually hereditarily in the class $\mathcal{C}$ of all C*-algebras that contain a nonzero projection. (See (4) in the introduction.)

We verify the definition directly. So let $I \subset A$ be an ideal such that $A/I \neq 0$, and let $B \subset A/I$ be a nonzero hereditary subalgebra. Combining Lemma 5.11 and Corollary 5.8, we see that B is a $C_0(X)$-algebra. Since $B \neq 0$, Lemma 5.3 (i) provides $x \in X$ such that $B_x \neq 0$. Let $\overline{\mathrm{ev}}_x: A/I \to (A/I)_x$ be the map of Notation 5.2 for the $C_0(X)$-algebra A/I. Then $B_x = \overline{\mathrm{ev}}_x(B)$ by Corollary 5.8 and $(A/I)_x \cong A_x/I_x$ by Lemma 5.11. Thus B_x is isomorphic to a nonzero hereditary subalgebra of A_x/I_x. Since A_x has residual (SP), it follows that there is a nonzero projection $p \in B_x$. Lemma 5.13 (i) provides a projection $e \in B$ such that $\overline{\mathrm{ev}}_x(e) = p$. Then $e \neq 0$ since $\overline{\mathrm{ev}}_x(e) \neq 0$. We have thus verified that A has residual (SP).

We next prove (ii). Let $\mathcal{C}$ be the class of all C*-algebras that contain an infinite projection. By the equivalence of conditions (ii) and (iv) of Proposition 2.11 of [35] (valid, as shown there, even when A is not separable), a C*-algebra D is purely infinite and has the ideal property if and only if D is residually hereditarily in $\mathcal{C}$. (See (1) in the introduction.) The argument is now the same as for (i), except using Lemma 5.13 (ii) in place of Lemma 5.13 (i).

Now we prove (iii). Let $\mathcal{C}$ be the class of all C*-algebras B such that $K \otimes B$ contains a nonzero projection. It was shown at the beginning of the proof of [33, Theorem 8.5] that a C*-algebra D has the weak ideal property if and only if D is residually hereditarily in $\mathcal{C}$. (See (5) in the introduction.)

We verify that A satisfies this condition. So let $I \subset A$ be an ideal such that $A/I \neq 0$, and let $B \subset A/I$ be a nonzero hereditary subalgebra. As in the proof of (i), B is a $C_0(X)$-algebra and there is $x \in X$ such that B_x is isomorphic to a nonzero hereditary subalgebra of A_x/I_x. Therefore $K \otimes B_x$ contains a nonzero projection p. Since K is nuclear, Lemma 5.12 implies that $K \otimes B$ is a $C_0(X)$-algebra with $(K \otimes B)_x \cong K \otimes B_x$. Let $\overline{\mathrm{ev}}_x: K \otimes B \to (K \otimes B)_x$ be the evaluation map at x for the $C_0(X)$-algebra $K \otimes B$, as in Notation 5.2. Lemma 5.13 (i) provides a projection $e \in K \otimes B$ such that $\overline{\mathrm{ev}}_x(e) = p$. Then $e \neq 0$ since $\overline{\mathrm{ev}}_x(e) \neq 0$. This shows that A is residually hereditarily in $\mathcal{C}$, as desired.

Finally we prove (iv). Since A is separable, by the equivalence of conditions (i) and (ix) in Theorem 2.10, it suffices to show that A is residually hereditarily in the class $\mathcal{C}$ of all C*-algebras D such that $\mathcal{O}_2 \otimes D$ contains a nonzero projection. Also, for every $x \in X$, the algebra A_x is separable. So Theorem 2.10 implies that A_x is residually hereditarily in $\mathcal{C}$. The proof is now the same as for (iii), except using $\mathcal{O}_2$ in place of K. ∎

We will next show that when the $C_0(X)$-algebra is continuous, the fibers are all nonzero, and the algebra is separable, then the algebra has one of our properties if and only if all the fibers have this property and X is totally disconnected.

Separability should not be necessary.

Having nonzero fibers is necessary. The zero C*-algebra is a $C_0(X)$-algebra for any X, and it certainly has all our properties. For a less trivial example, let X_0 be the Cantor set, take $X = X_0 \amalg [0,1]$, and make $C(X_0, \mathcal{O}_2)$ a $C(X)$-algebra via restriction of functions in $C(X)$ to X_0.

Continuity is also necessary. The following important example was suggested by the referee; our original example, the C*-algebra product $A = \prod_{x \in [0,1]} \mathcal{O}_2$, was not separable.

Example 5.15 Let Y be the Cantor set, set $A = C(Y, \mathcal{O}_2)$, and let

$$\iota_0 \colon C(Y) \to Z(M(A)) = Z(A)$$

be the obvious isomorphism, sending $f \in C(Y)$ to the function $y \mapsto f(y) \cdot 1_{\mathcal{O}_2}$. Set $X = [0,1]$. Let $h \colon Y \to X$ be a surjective continuous function, and define $\psi \colon C(X) \to C(Y)$ by $\psi(f) = f \circ h$ for $f \in C(X)$. Then define $\iota = \iota_0 \circ \psi \colon C(X) \to Z(M(A))$. This map is clearly nondegenerate, so A becomes a $C(X)$-algebra (but not a continuous $C(X)$-algebra). Also, ι is injective.

We identify the fibers. Let $x \in X$. Then

$$\psi(C_0(X \smallsetminus \{x\})) = \{ f \circ h : f \in C(X) \text{ and } f(x) = 0 \}.$$

Thus, all functions in $\psi(C_0(X \smallsetminus \{x\}))$ vanish on $h^{-1}(\{x\})$. But for every point $y \in Y \smallsetminus h^{-1}(\{x\})$ there is some $f \in C_0(X \smallsetminus \{x\})$ such that $\psi(f)(y) \neq 0$. It follows from the locally compact version of the Stone–Weierstrass Theorem that

$$\overline{\psi\big(C_0(X \smallsetminus \{x\})\big) C(Y)} = C_0\big(Y \smallsetminus h^{-1}(\{x\})\big).$$

It is now easy to see that

$$\overline{\psi\big(C_0(X \smallsetminus \{x\})\big) A} = C_0(Y \smallsetminus h^{-1}(\{x\}), \mathcal{O}_2),$$

so the fiber A_x is $A_x = C(h^{-1}(\{x\}), \mathcal{O}_2)$. Since $h^{-1}(\{x\})$ is compact and totally disconnected (being a closed subset of the Cantor set Y) and $\mathcal{O}_2$ is purely infinite and has the ideal property, the weak ideal property, residual (SP), and topological dimension zero, Theorem 5.14 implies that A_x also has all these properties.

However, $X = [0,1]$ is not totally disconnected. Thus, without continuity of the $C(X)$-algebra structure, all four parts of Theorem 5.17 will fail.

Lemma 5.16 *Let X be a second countable locally compact Hausdorff space, and let A be a separable continuous $C_0(X)$-algebra such that $A_x \neq 0$ for all $x \in X$. If A has topological dimension zero, then X is totally disconnected.*

We assume that X is second countable because we need A to be separable in Theorem 2.9. Example 5.15 shows that continuity of the $C_0(X)$-algebra is necessary. In fact, a much simpler version of Example 5.15 shows this: a surjective continuous map from the Cantor set Y to $[0,1]$ gives an injective map from $C([0,1])$ to $C(Y)$ that makes $C(Y)$ a $C([0,1])$-algebra whose fibers are all nonzero and which has topological dimension zero.

Proof of Lemma 5.16 As in Proposition 5.6, we identify continuous $C_0(X)$-algebras and continuous C*-bundles. Now use [22, Corollary 2.8] to see that $\mathcal{O}_2 \otimes A$ is a continuous $C_0(X)$-algebra. It follows from Theorem 2.9 that $\mathcal{O}_2 \otimes A$ has the ideal property. Since the set of points with nonzero fibers is all of X, [30, Theorem 2.1] implies that X is totally disconnected. ■

Theorem 5.17 *Let X be a second countable locally compact Hausdorff space, and let A be a separable continuous $C_0(X)$-algebra such that $A_x \neq 0$ for all $x \in X$.*

(i) *A has residual (SP) if and only if X is totally disconnected and A_x has residual (SP) for all $x \in X$.*
(ii) *A is purely infinite and has the ideal property if and only if X is totally disconnected and A_x is purely infinite and has the ideal property for all $x \in X$.*
(iii) *A has the weak ideal property if and only if X is totally disconnected and A_x has the weak ideal property for all $x \in X$.*
(iv) *A has topological dimension zero if and only if X is totally disconnected and A_x has topological dimension zero for all $x \in X$.*

Proof In all four parts, the reverse implications follow from Theorem 5.14. Also, in all four parts, the fact that A_x has the appropriate property for all $x \in X$ follows from the general fact that the property passes to arbitrary quotients. See [33, Theorem 7.4 (7)] for residual (SP), [33, Theorem 6.8 (7)] for the combination of purely infiniteness and the ideal property, [33, Theorem 8.5 (5)] for the weak ideal property, and combine [33, Proposition 5.8] with the equivalence of conditions (i) and (ix) in Theorem 2.10 for the weak ideal property.

It remains to show that all four properties imply that X is totally disconnected. All four properties imply topological dimension zero (using as necessary Theorem 2.8 and the fact that residual (SP) implies the weak ideal property), so this follows from Lemma 5.16. ■

The proofs in this section depend on properties of projections, and so do not work for a general property defined by being residually hereditarily in an upwards directed class of C*-algebras. However, we know of no counterexamples to either version of the following question, and Proposition 5.19 gives hope that something along these lines might be true.

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Question 5.18 Let $\mathcal{C}$ be an upwards directed class of C*-algebras, let X be a totally disconnected locally compact space, and let A be a $C_0(X)$-algebra such that A_x is residually hereditarily in $\mathcal{C}$ for all $x \in X$. Does it follow that A is residually hereditarily in $\mathcal{C}$? What if we assume that A is a continuous $C_0(X)$-algebra?

Proposition 5.19 *Let $\mathcal{C}$ be an upwards directed class of C*-algebras. Let A be a C*-algebra that is residually hereditarily in $\mathcal{C}$ and let X be a totally disconnected locally compact metric space. Then $C_0(X, A)$ is residually hereditarily in $\mathcal{C}$.*

Proof It is well known that $C_0(X)$ is an AF algebra. Being residually hereditarily in $\mathcal{C}$ is preserved by tensoring with matrix algebras [33, Proposition 5.11 (2)], finite direct sums [33, Proposition 5.8], and direct limits [33, Proposition 5.9 (2)]. Therefore, being residually hereditarily in $\mathcal{C}$ is preserved by tensoring with AF algebras. ■

6 Strong Pure Infiniteness for Bundles

It seems to be unknown whether $C_0(X) \otimes A$ is purely infinite when X is a locally compact Hausdorff space and A is a general purely infinite C*-algebra, even when A is additionally assumed to be simple. (To apply [20, Theorem 5.11], one also needs to know that A is approximately divisible.) Efforts to prove this by working locally on X seem to fail. Even in cases in which they work, such methods are messy. It therefore seems worthwhile to give the following result, which, given what is already known, has a simple proof.

Theorem 6.1 *Let X be a locally compact Hausdorff space, and let A be a locally trivial $C_0(X)$-algebra whose fibers A_x are strongly purely infinite [21, Definition 5.1]. Then A is strongly purely infinite.*

Since X is locally compact, local triviality is equivalent to the requirement that every point $x \in X$ has a compact neighborhood L such that, using the $C(L)$-algebra structure on $A|_L$ (Notation 5.2) and the obvious $C(L)$-algebra structure on $C(L, A_x)$, these two algebras are isomorphic as $C(L)$-algebras. In this case we say that $A|_L$ is trivial.

Proof of Theorem 6.1 Let $\iota\colon C_0(X) \to Z(M(A))$ be the structure map.

We first prove the result when X is compact, by induction on the least $n \in \mathbb{Z}_{>0}$ for which there are open sets $U_1, U_2, \dots, U_n \subset X$ that cover X and such that $A|_{\overline{U_j}}$ is trivial for $j = 1, 2, \dots, n$. If $n = 1$, there is a strongly purely infinite C*-algebra B such that $A \cong C(X, B)$, and A is strongly purely infinite [17, Corollary 5.3]. Assume the result is known for some $n \in \mathbb{Z}_{>0}$, and suppose that there are open sets $U_1, U_2, \dots, U_{n+1} \subset X$ that cover X and such that $A|_{\overline{U_j}}$ is trivial for $j = 1, 2, \dots, n+1$. Define $U = \bigcup_{j=1}^{n} U_j$. If $X \smallsetminus U = \varnothing$, then the induction hypothesis applies directly. Otherwise, use

$$X \smallsetminus U \subset U_{n+1}$$

to choose an open set $W \subset X$ such that $X \smallsetminus U \subset W \subset \overline{W} \subset U_{n+1}$. Define $Y = X \smallsetminus W$ and $L = \overline{W}$. Then $L \cup Y = X$, $X \smallsetminus L \subset Y$, $Y \subset U$, and $L \subset U_{n+1}$. Since $L \subset \overline{U_{n+1}}$, there is a strongly purely infinite C*-algebra B such that $A|_L \cong C(L, B)$. By definition

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(Notation 5.2), there is a short exact sequence

$$0 \longrightarrow \overline{\iota(C_0(X \smallsetminus L))A} \longrightarrow A \longrightarrow A|_L \longrightarrow 0.$$

We can identify the algebra $\overline{\iota(C_0(X \smallsetminus L))A}$ with an ideal in $A|_Y$. Consideration of the sets $U_1 \cap Y, U_2 \cap Y, \ldots, U_n \cap Y$ shows that the induction hypothesis applies to $A|_Y$, which is therefore strongly purely infinite. So $\overline{\iota(C_0(X \smallsetminus L))A}$ is strongly purely infinite [21, Proposition 5.11 (ii)]. Also $A|_L$ is strongly purely infinite [17, Corollary 5.3], so A is strongly purely infinite [17, Theorem 1.3]. This completes the induction step and the proof of the theorem when X is compact.

We now prove the general case. Let $(U_\lambda)_{\lambda \in \Lambda}$ be an increasing net of open subsets of X such that $\overline{U_\lambda}$ is compact for all $\lambda \in \Lambda$ and $\bigcup_{\lambda \in \Lambda} U_\lambda = X$. For $\lambda \in \Lambda$, the algebra $A|_{\overline{U_\lambda}}$ is strongly purely infinite by the case already done. So its ideal $\overline{\iota(C_0(U_\lambda))A}$ is strongly purely infinite [21, Proposition 5.11 (ii)]. Using Lemma 5.9, one checks that $A \cong \varinjlim_{\lambda \in \Lambda} \overline{\iota(C_0(U_\lambda))A}$, so A is strongly purely infinite [21, Proposition 5.11 (iv)]. ■

Lemma 6.2 *Let A be a separable C*-algebra. Then the following are equivalent.*

(i) *A is purely infinite and has topological dimension zero.*

(ii) *A is strongly purely infinite and has the ideal property.*

Proof Condition (ii) implies condition (i) because strong pure infiniteness implies pure infiniteness [21, Proposition 5.4], the ideal property implies the weak ideal property, and the weak ideal property implies topological dimension zero (Theorem 2.8).

Now assume (i). Then A has the ideal property by Theorem 2.9. Apply [35, Proposition 2.14]. ■

Corollary 6.3 *Let X be a locally compact Hausdorff space, and let A be a locally trivial $C_0(X)$-algebra whose fibers A_x are all purely infinite, separable, and have topological dimension zero. Then A is strongly purely infinite.*

Proof Lemma 6.2 implies that the fibers are all strongly purely infinite, so that Theorem 6.1 applies. ■

7 When Does the Weak Ideal Property Imply the Ideal Property?

The weak ideal property seems to be the property most closely related to the ideal property that has good behavior on passing to hereditary subalgebras, fixed point algebras, and extensions. (Example 2.7 of [32] gave a separable unital C*-algebra A with the ideal property and an action of $\mathbb{Z}_2$ on A such that the fixed point algebra does not have the ideal property. Example 2.8 of [32] gave a separable unital C*-algebra A such that $M_2(A)$ has the ideal property, but A does not have the ideal property. Theorem 5.1 of [26] gave an extension of separable C*-algebras with the ideal property such that the extension does not have the ideal property.) On the other hand, the ideal property came first, and in some ways seems more natural. Accordingly, it seems interesting to find conditions under which the weak ideal property implies the ideal property. Our main result in this direction is Theorem 7.15. It covers, in particular, separable locally AH algebras (Definition 7.9). We also prove (Proposition 7.16) that the weak ideal property implies the ideal property for stable C*-algebras with Hausdorff primitive ideal space. We give an example to show that this implication can fail for $\mathcal{Z}$-stable C*-algebras.

In the introduction, we illustrated the importance of the ideal property with several theorems in which it is a hypothesis. We start by showing that two of these results can otherwise fail: Theorem 4.1 of [25] (stable rank one for AH algebras with slow dimension growth) in Example 7.1, and Theorem 3.6 of [14] (AT structure for AH algebras with very slow dimension growth and torsion free K-theory) in Example 7.2. In both cases, however, Theorem 7.15 implies that one can replace the the ideal property with the weak ideal property.

Example 7.1 Let D be the 2^∞ UHF algebra. Then $C([0,1]^2, D)$ is an AH algebra, even in the somewhat restrictive sense of [25, Definition 2.2], which has no dimension growth. It follows from [23, Proposition 5.3] that $C([0,1]^2, D)$ does not have stable rank one. Thus, [25, Theorem 4.1] fails without the ideal property.

Example 7.2 Let D be the 3^∞ UHF algebra, and let $X = [0,1]^5$. Then $C(X, D)$ is an AH algebra with no dimension growth. We show that $C(X, D)$ has torsion free K-theory and is not an AT algebra. Thus, [14, Theorem 3.6] fails without the ideal property.

We have $K_0(C(X,D)) \cong \mathbb{Z}[\frac{1}{3}]$ and $K_1(C(X,D)) = 0$. Thus $K_*(C(X,D))$ is torsion free. Since the real projective space $\mathbb{R}P^2$ is a compact 2-dimensional manifold, there is a closed subspace $Y \subset X$ such that $Y \cong \mathbb{R}P^2$. By [1, Proposition 2.7.7], $K^0(\mathbb{R}P^2) \cong \mathbb{Z} \oplus \mathbb{Z}_2$. Therefore $K_0(C(Y,D)) \cong \mathbb{Z}[\frac{1}{3}] \otimes (\mathbb{Z} \oplus \mathbb{Z}_2) \cong \mathbb{Z}[\frac{1}{3}] \oplus \mathbb{Z}_2$. Since this group has torsion, $C(Y, D)$ is not an AT algebra. Since $C(Y, D)$ is a quotient of $C(X, D)$, it follows that $C(X, D)$ is not an AT algebra.

In fact, with D as in Example 7.2, even $C([0,1]^2, D)$ is not an AT algebra. (We are grateful to the referee for pointing this out.) We use the implication from (i) to (iii) of [44, Theorem 1.1], with $A = D$ and with $n = 2$. Using the notation in the diagram in condition (iii) there, if φ as described there exists, then $(r \circ \varphi)_* \colon K_1(C(S^1)) \to K_1(C(S^1, D))$ must be the zero map, while $\iota_* \colon K_1(C(S^1)) \to K_1(C(S^1, D))$ is injective, hence nonzero, a contradiction.

It is convenient to work with the following class of C*-algebras.

Notation 7.3 We denote by $\mathcal{P}$ the class of all separable C*-algebras for which topological dimension zero, the ideal property, and the weak ideal property are all equivalent.

That is, a separable C*-algebra A is in $\mathcal{P}$ exactly when either A has all of the properties topological dimension zero, the ideal property, and the weak ideal property, or none of them.

The class $\mathcal{P}$ is not particularly interesting in its own right. (For example, all cones over nonzero C*-algebras are in $\mathcal{P}$, because they have none of the three properties.) However, proving results about it will make possible a result to the effect that these properties are all equivalent for the smallest class of separable C*-algebras that contains the separable AH algebras (as well as some others) and is closed under certain operations.

The following lemma isolates, for convenient reference, what we actually need to prove to show that a separable C*-algebra is in $\mathcal{P}$.

Lemma 7.4 *Let A be a separable C*-algebra for which topological dimension zero implies the ideal property. Then $A \in \mathcal{P}$.*

Proof The ideal property implies the weak ideal property [33, Proposition 8.2]. The weak ideal property implies topological dimension zero by Theorem 2.8. ∎

We prove two closure properties for the class $\mathcal{P}$. What can be done here is limited by the failure of other closure properties for the class of C*-algebras with the ideal property. See the introduction to this section. (It is hopeless to try to prove results about quotients of algebras in $\mathcal{P}$, since the cone over every C*-algebra is in $\mathcal{P}$).

Lemma 7.5 *Let $(A_\lambda)_{\lambda\in\Lambda}$ be a countable family of C*-algebras in $\mathcal{P}$. Then*

$$\bigoplus_{\lambda\in\Lambda} A_\lambda \in \mathcal{P}.$$

Proof Set $A = \bigoplus_{\lambda\in\Lambda} A_\lambda$. Then A is separable, since Λ is countable and A_λ is separable for all $\lambda \in \Lambda$. By Lemma 7.4, we need to show that if A has topological dimension zero, then A has the ideal property. For $\lambda \in \Lambda$, the algebra A_λ is a quotient of A, so has topological dimension zero [6, Proposition 2.6], [32, Lemma 3.6]. Therefore A_λ has the ideal property by hypothesis.

It is clear that arbitrary direct sums of C*-algebras with the ideal property also have the ideal property, so it follows that A has the ideal property. ∎

Lemma 7.6 *Let A and B be C*-algebras in $\mathcal{P}$. Assume that A is exact. Then*

$$A \otimes_{\min} B \in \mathcal{P}.$$

Proof Since the zero C*-algebra is in $\mathcal{P}$, we may assume that A and B are nonzero. The algebra $A \otimes_{\min} B$ is separable because A and B are. By Lemma 7.4, we need to show that if $A \otimes_{\min} B$ has topological dimension zero, then $A \otimes_{\min} B$ has the ideal property. Now A and B have topological dimension zero by Theorem 4.4, and so have the ideal property by hypothesis. It now follows that $A \otimes_{\min} B$ has the ideal property [34, Corollary 1.3]. ∎

We now identify a basic collection of C*-algebras in $\mathcal{P}$. The main point of the first class we consider is that it contains the separable AH algebras (as described below), but in fact it is much larger.

There are conflicting definitions of AH algebras in the literature. We follow [2, Definition 2.1]. (See the discussion after [2, Definition 1.2] for the meaning of *locally homogeneous*.) This definition does not assume the direct limit algebras are separable or unital. It is quite general, excluding only uncountable direct systems and terms in the direct system with nontrivial Dixmier–Douady invariant. We rewrite this definition without using direct sums by not requiring that the projections defining corners have constant rank.

Definition 7.7 ([2, Definition 2.1]) Let A be a C*-algebra. We say that A is an *AH algebra* if A is a direct limit of a sequence $(A_n)_{n\in\mathbb{Z}_{\geq 0}}$ of C*-algebras A_n, each of which has the form $pC(X, M_k)p$ for a compact Hausdorff space X, $k \in \mathbb{Z}_{>0}$, and a projection $p \in C(X, M_k)$, all depending on n.

Substituting compact metric spaces for compact Hausdorff spaces, one gets the definition in the introduction to the simple and no dimension growth classification paper [12], and in the introduction to [14]. This definition is probably the most common one. As we will see in Proposition 7.8, it covers all separable algebras given in Definition 7.7. An even more restrictive definition of an AH algebra is found in the introductions to [8, 25, 26], in which the spaces are required to be finite CW complexes. As pointed out in [8], Proposition 2.3 of [2] shows that this definition actually gives the same algebras as when one uses compact metric spaces.

We want all the spaces to have only finitely many connected components and all the maps to be injective. One might call such an algebra a "restricted AH algebra". In the separable case, it is already known that AH algebras are automatically of this form.

Proposition 7.8 *Let A be an AH algebra (as in Definition 7.7) that is also separable. Then A is a direct limit of a sequence $(A_n)_{n\in\mathbb{Z}_{\geq 0}}$ of C*-algebras A_n, each of which has the form $pC(X, M_k)p$ for a finite simplicial complex X, $k \in \mathbb{Z}_{>0}$, and a projection $p \in C(X, M_k)$, all depending on n, and in which the maps $A_n \to A_{n+1}$ are all injective.*

Proof Proposition 2.3 of [2] shows that we can require that every space X appearing in the system be a finite disjoint union of polyhedra. It now follows from [11, Theorem 2.1] that there is a direct system with direct limit A in which, in addition, all the maps of the system are injective. ■

The following definition is standard.

Definition 7.9 Let A be a C*-algebra. We say that A is a *locally AH algebra* if for every finite set $F \subset A$ and every $\varepsilon > 0$, there exist a subalgebra $B \subset A$ that is isomorphic to an AH algebra and such that for all $a \in F$ there is $b \in B$ with $\|b - a\| < \varepsilon$.

In particular, AH algebras are locally AH algebras.

Lemma 7.10 *Let A be a separable C*-algebra. Then A is a locally AH algebra if and only if for every finite set $F \subset A$ and every $\varepsilon > 0$ there exist a finite simplicial complex X, $k \in \mathbb{Z}_{>0}$, a projection $p \in C(X, M_k)$, and an injective homomorphism $\varphi\colon pC(X, M_k)p \to A$ such that for all $a \in F$ there is $b \in pC(X, M_k)p$ with $\|\varphi(b) - a\| < \varepsilon$.*

Proof The algebra B in Definition 7.9 must be separable, so that Proposition 7.8 can be applied. ■

Definition 7.11 Let A be a C*-algebra.

We say that A has the *projection slicing property* ("A is standard" in [7, Definition 2.7]) if A is unital and if, whenever B is a simple unital C*-algebra and $J \subset A \otimes_{\min} B$ is an ideal that is generated as an ideal by its projections, there is an ideal $I \subset A$ that is generated as an ideal by its projections and such that $J = I \otimes_{\min} B$.

We say that A is an *LS algebra* [7, Definition 2.13] if for every finite set $F \subset A$ and every $\varepsilon > 0$, there exists a C*-algebra D with the projection slicing property and an injective homomorphism $\varphi: D \to A$ such that for all $a \in F$ there is $b \in D$ with $\|\varphi(b) - a\| < \varepsilon$.

Lemma 7.12 (i) *Let X be a compact Hausdorff space with only finitely many connected components, let $k \in \mathbb{Z}_{>0}$, and let $p \in C(X, M_k)$ be a projection. Then $pC(X, M_k)p$ has the projection slicing property.*

(ii) *If A is a separable locally AH algebra, then A is an LS algebra.*

Proof Part (i) is a special case of [7, Remark 2.9 (2)]. Part (ii) is immediate from part (i) and Lemma 7.10. ∎

There are many more C*-algebras with the projection slicing property than in Lemma 7.12 (i), and therefore many more LS algebras than in Lemma 7.12 (ii). For example, in Definition 7.7 replace $pC(X, M_k)p$ by a finite direct sum of C*-algebras of the form $pC(X, D)p$ for connected compact Hausdorff spaces X, simple unital C*-algebras D, and projections $p \in C(X, D)$. Such a C*-algebra has the projection slicing property [7, Remark 2.9 (2)], so a direct limit of a system of such algebras with injective maps is an LS algebra. (When all the algebras D that occur are exact and the direct system is countable, but the maps of the system are not necessarily injective, such a direct limit is called an exceptional GAH algebra [28, Definitions 2.9 and 2.7].)

Lemma 7.13 (Definition 7.11) *Let A be a separable LS algebra. Then $A \in \mathcal{P}$.*

Proof As usual, we use Lemma 7.4. Assume A has topological dimension zero. By the implication (i) ⇒ (iii) in Theorem 2.10, the algebra $\mathcal{O}_2 \otimes A$ has the ideal property. Apply [7, Lemma 2.11] with $B = \mathcal{O}_2$ to conclude that A has the ideal property. ∎

Extending the list of properties in the discussion of type I C*-algebras in [30, Remark 2.12] (and using essentially the same proof as there), we get the following longer list of equivalent conditions on a separable type I C*-algebra.

Proposition 7.14 *Let A be a separable type I C*-algebra. Then the following are equivalent.*

(i) *A has topological dimension zero.*
(ii) *A has the weak ideal property.*
(iii) *A has the ideal property.*
(iv) *A has the projection property (every ideal in A has an increasing approximate identity consisting of projections [27, Definition 1]).*
(v) *A has real rank zero.*
(vi) *A is an AF algebra.*

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Proof It is clear that every condition on the list implies the previous one. So we need only show that (i) implies (vi). Use [32, Lemma 3.6] to see that $\mathrm{Prim}(A)$ has a base for its topology consisting of compact open sets. Then the theorem in Section 7 of [4] implies that A is AF. ∎

Theorem 7.15 *Let $\mathcal{W}$ be the smallest class of separable C*-algebras that contains the separable LS algebras (including the separable locally AH algebras), the separable type I C*-algebras, and the separable purely infinite C*-algebras, and is closed under finite and countable direct sums and under minimal tensor products when one tensor factor is exact. Then for any C*-algebra in $\mathcal{W}$, topological dimension zero, the weak ideal property, and the ideal property are all equivalent.*

Proof Combine Lemmas 7.5, 7.6, 7.13, Lemma 7.12 (ii), Proposition 7.14, and Theorem 2.9. ∎

Proposition 7.16 *Let A be a C*-algebra such that* $\mathrm{Prim}(A)$ *is Hausdorff. If A has the weak ideal property, then $K \otimes A$ has the ideal property.*

In particular, the weak ideal property implies the ideal property for stable C*-algebras with Hausdorff primitive ideal space.

Proof of Proposition 7.16 Arguing as in the proof of Proposition 4.11, we see that $K \otimes A$ is a continuous $C_0(\mathrm{Prim}(A))$-algebra, with fibers $(K \otimes A)_P \cong K \otimes (A/P)$ for $P \in \mathrm{Prim}(A)$. Moreover, $\mathrm{Prim}(A)$ is totally disconnected, and for every $P \in \mathrm{Prim}(A)$, the quotient A/P is simple and has the weak ideal property.

For $P \in \mathrm{Prim}(A)$, it follows that $K \otimes (A/P)$ is simple and has a nonzero projection, so has the ideal property. This is true for all $P \in \mathrm{Prim}(A)$, so $K \otimes A$ has the ideal property [30, Theorem 2.1]. ∎

Let Z be the Jiang–Su algebra. It is unfortunately not true that the weak ideal property implies the ideal property for Z-stable C*-algebras.

Example 7.17 We give a separable C*-algebra A such that A and $Z \otimes A$ have the weak ideal property, but such that neither A nor $Z \otimes A$ has the ideal property.

Let D be a Bunce–Deddens algebra, and let the extension

$$0 \longrightarrow K \otimes D \longrightarrow A \longrightarrow \mathbb{C} \longrightarrow 0$$

be as in the proof of [26, Theorem 5.1]. (The extension is as in the first paragraph of that proof, using the choices suggested in the second paragraph.) In particular, as proved in [26], A does not have the ideal property, and the connecting homomorphism $\exp\colon K_0(\mathbb{C}) \to K_1(K \otimes D)$ is injective. Since $K \otimes D$ and $\mathbb{C}$ have the weak ideal property (for trivial reasons), it follows that A has the weak ideal property [33, Theorem 8.5 (5)]. Clearly $Z \otimes K \otimes D$ and $Z \otimes \mathbb{C}$ have the ideal property. However, it was shown in the proof of [31, Theorem 2.9] that $Z \otimes A$ does not have the ideal property.

The following question was motivated by a discussion with Guihua Gong.

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Question 7.18 Let A be a separable C*-algebra that is a direct limit of recursive subhomogeneous C*-algebras. If A has the weak ideal property, does A have the ideal property?

We suspect that the answer is no, but we do not have a counterexample.

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