

$$\frac{1}{2} \int_0^{\pi/2} r_2^2 d\theta - \frac{1}{2} \int_0^{\pi/2} r_1^2 d\theta = \frac{1}{2} \int_0^{\pi/2} (r_2^2 - r_1^2) d\theta = \frac{1}{2} \int_0^{\pi/2} (2 \cos \theta + 3 \cos^2 \theta) d\theta = \frac{3}{8} \pi + 1.$$

Now consider the area of region *B*. One endpoint is *Q* with polar coordinates $(1, \frac{1}{2}\pi)$. The other is given by $r_1 = 0 = r_2$. The left-hand side solves to give $\theta = \pi$ and the right-hand side gives $\theta = \frac{2}{3}\pi$: if you like, the pole *P* has rival polar coordinates $(0, \pi)$ and $(0, \frac{2}{3}\pi)$ on the two curves. The area of *B* is then given by two separate integrals that cannot be combined:

$$\begin{aligned} & \frac{1}{2} \int_{\pi/2}^{\pi} r_1^2 d\theta - \frac{1}{2} \int_{\pi/2}^{2\pi/3} r_2^2 d\theta \\ &= \frac{1}{2} \int_{\pi/2}^{\pi} (1 + 2 \cos \theta + \cos^2 \theta) d\theta - \frac{1}{2} \int_{\pi/2}^{2\pi/3} (1 + 4 \cos \theta + 4 \cos^2 \theta) d\theta \\ &= \left(\frac{3\pi}{8} - 1 \right) - \left(\frac{\pi}{4} + \frac{3\sqrt{3}}{4} - 2 \right) = \frac{\pi}{8} - \frac{3\sqrt{3}}{4} + 1. \end{aligned}$$

The ambiguity of endpoints exhibited at *P* can only occur when the curves meet at the pole. This example is a fine one for class discussion and similar ones are also worth investigating: for example, the region corresponding to *B* for the curves $r_1 = 1 + \sqrt{2} \cos \theta$ and $r_2 = 1 + 2 \cos \theta$ has the “ π -less” area $\frac{11}{4} - \sqrt{2} - \frac{3\sqrt{3}}{4}$.

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Another appearance of the golden ratio

Professor Anne Watson in her inspirational plenary talk “What school mathematics can be ... really” at the 2022 MA Conference mentioned the following problem as a rich one for a discussion of problem-solving strategies and approaches.

Find the area of right-angled triangle *ABC*, situated in a quadrant of the unit circle as in Figure 1(a).

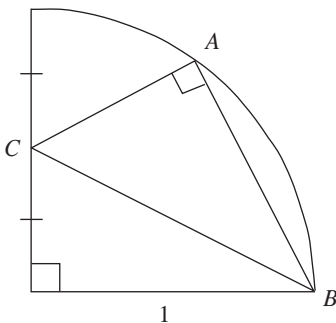


FIGURE 1(a)

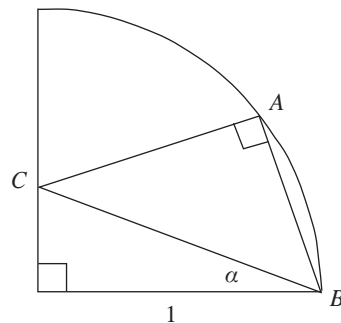


FIGURE 1(b)



In this Note, we look at the more general configuration in Figure 1(b) and will find an unexpected appearance of the golden ratio when finding the angle α with $0 \leq \alpha \leq \frac{1}{4}\pi$ that maximises the area of triangle ABC .

Extending AC gives Figure 2 which embeds Figure 1(b) in a semicircle. Then the area, $f(\alpha)$, of triangle ABC is given by “area ABD – area BCD ” which evaluates as

$$f(\alpha) = \frac{1}{2}(2 \sin \alpha)(2 \cos \alpha) - \frac{1}{2} \times 2 \times \tan \alpha = \sin 2\alpha - \tan \alpha.$$

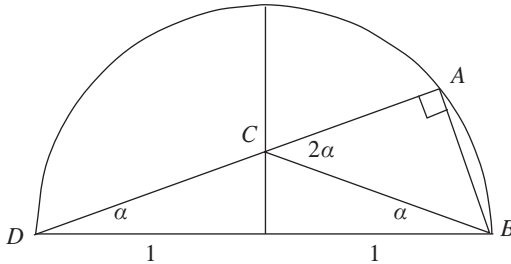


FIGURE 2

In the original problem of Figure 1(a), $\tan \alpha = \frac{1}{2}$, so $f(\alpha) = 2 \cdot \frac{1}{\sqrt{5}} \cdot \frac{2}{\sqrt{5}} - \frac{1}{2} = \frac{3}{10}$. And, since $\sin 2\alpha = \frac{4}{5}$, triangle ABC is then similar to a 3, 4, 5 triangle.

The maximum area of triangle ABC occurs when

$$0 = f'(\alpha) = 2 \cos 2\alpha - \sec^2 \alpha = 2 \cos 2\alpha - \frac{2}{1 + \cos 2\alpha}.$$

This leads to $\cos^2 \alpha + \cos 2\alpha - 1 = 0$ so that $\cos 2\alpha = \frac{1}{2}(\sqrt{5} - 1) = \frac{1}{\phi}$, where $\phi = \frac{1}{2}(\sqrt{5} + 1)$ is the golden ratio. This means that triangle ABC is then similar to the so-called *Kepler triangle* with sides $1, \sqrt{\phi}, \phi$.

Now,

$$\begin{aligned} f(\alpha) &= \sin 2\alpha - \tan \alpha = 2 \sin \alpha \cos \alpha - \tan \alpha \\ &= \tan \alpha (2 \cos^2 \alpha - 1) = \tan \alpha \cos 2\alpha. \end{aligned}$$

But if $\cos 2\alpha = \frac{1}{\phi}$, then $\frac{1 - \tan^2 \alpha}{1 + \tan^2 \alpha} = \frac{1}{\phi}$ and, since $\phi^2 = \phi + 1$, $\tan^2 \alpha = \frac{\phi - 1}{\phi + 1} = \frac{\phi^{-1}}{\phi^2} = \frac{1}{\phi^3}$. Thus the maximum value of $f(\alpha)$ is $\frac{1}{\phi^{3/2}} \cdot \frac{1}{\phi} = \frac{1}{\phi^{5/2}} \approx 0.30028$, attained when $\tan \alpha = \frac{1}{\phi^{3/2}} \approx 0.48587$: these values are actually very close to the corresponding values of 0.3 and 0.5 in Figure 1(a).

Finally, it is worth noting that maximising the perimeter of triangle ABC is equivalent to maximising the sum of the lengths $DA + AB$ in Figure 2. This occurs when $\alpha = \frac{1}{4}\pi$ and the triangle has degenerated into a line segment, a nice example of an endpoint maximum. This can be seen analytically since the perimeter is given by

$$DA + AB = 2(\cos \alpha + \sin \alpha) = 2\sqrt{2}\cos\left(\alpha - \frac{1}{4}\pi\right).$$

But it can also be seen geometrically by imagining ellipses with foci at D and B generated by a string of length $DA + AB$. As A moves up from B on the circle, the respective ellipses are disjoint and correspond to increasing string-length constants. The maximum such length occurs when $\alpha = \frac{1}{4}\pi$, with the ellipse tangential to the circle.

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