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JORDAN MAPPINGS OF SEMIPRIME RINGS II

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We describe Jordan homomorphisms and Jordan triple homomorphisms onto 2-torsion free semiprime rings in which the annihilator of any ideal is a direct summand.

1. INTRODUCTION AND PRELIMINARIES

The aim of this paper is to complete some results from our work [**3**], concerned with Jordan homomorphisms (that is, additive mappings of rings satisfying $\theta(ab+ba) = \theta(a)\theta(b) + \theta(b)\theta(a)$) and Jordan triple homomorphisms (that is, additive mappings satisfying $\theta(aba) = \theta(a)\theta(b)\theta(a)$). First, let us look at two simple examples of Jordan homomorphisms.

EXAMPLE 1: Let R, U' and V' be rings, and let $\varphi\colon R \to U'$ and $\psi\colon R \to V'$ be a homomorphism and an antihomomorphism, respectively. Define $\theta\colon R \to U' \oplus V'$ by $\theta(r) = (\varphi(r), \psi(r))$. Then θ is a Jordan homomorphism.

The next example is, in fact, the special case of Example 1.

EXAMPLE 2: Let U, V, U' and V' be rings, and let $\varphi\colon U \to U'$ and $\psi\colon V \to V'$ be a homomorphism and an antihomomorphism, respectively. Then the mapping $\theta\colon U \oplus V \to U' \oplus V'$, $\theta(u, v) = (\varphi(u), \psi(v))$, is a Jordan homomorphism.

Note the important difference: in Example 2 the image of θ is an (associative) subring, while in Example 1 this need not be true.

Let θ be a Jordan homomorphism of a ring R onto a 2-torsion free semiprime ring R'. Baxter and Martindale showed that in this case there exists an essential ideal E of R such that the restriction of θ to E is of the same form as the mapping θ in Example 1 [**2**, Theorem 2.7]. Roughly speaking, in [**3**, Theorem 2.3] we generalised their result by showing that this restriction is rather of the form as in Example 2. Baxter and Martindale also proved that if R' is centrally closed (that is, its centroid coincides with its extended centroid) then the restriction to an essential ideal is unnecessary, thus θ is like in Example 1 [**2**, Theorem 3.8]. In this paper we show that in the case of centrally closed semiprime rings θ is rather like in Example 2. In fact, we prove that this is

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true for a more general class of semiprime rings, that is, semiprime rings in which the annihilator of any ideal is a direct summand.

Let I be an ideal of a semiprime ring R'. It can be easily shown that the left and right and two-sided annihilators $\mathrm{Ann}(I)$ of I coincide. Next, $I \cap \mathrm{Ann}(I) = 0$ and $I \oplus \mathrm{Ann}(I)$ is an essential ideal of R'. As we have mentioned, we will be concerned with semiprime rings R' in which the annihilator of any ideal is a direct summand; that is, $\mathrm{Ann}(I) \oplus \mathrm{Ann}(\mathrm{Ann}(I)) = R'$ for every ideal I of R'. Every prime ring trivially satisfies this condition. Moreover, the same is true for a direct sum of prime rings. Another important example is a semiprime Baer ring (see [4, Theorem 13]). Next we have

Lemma 1. *Let R' be a centrally closed semiprime ring. Then the annihilator of any ideal in R' is a direct summand.*

Proof: In [1] Amitsur defined the notion of a closed ideal in a semiprime ring (not to be confused with the notion of a centrally closed semiprime ring): an ideal U is closed if $U = \mathrm{Ann}(\mathrm{Ann}(U))$. We claim that the ideal U in any semiprime ring is closed if and only if it is the annihilator of some ideal. If U is closed then U is the annihilator of the ideal $\mathrm{Ann}(U)$. Conversely, let U be the annihilator of an ideal V. Then $V \subseteq \mathrm{Ann}(U)$ and so $\mathrm{Ann}(\mathrm{Ann}(U)) \subseteq \mathrm{Ann}(V) = U$; hence U is closed.

Now, let R' be a centrally closed semiprime ring, and let J be the annihilator of some ideal of R'. We want to show that J is a direct summand. By the above argument J is a closed ideal and so it follows from [1, Corollary 9] that $J = R' \cap Q_0\varepsilon$ where Q_0 is the ring of quotients of R' and ε is an idempotent contained in the extended centroid of R' (that is, the centre of Q_0). However, R' is centrally closed and so ε actually lies in the centroid of R', thus $R'\varepsilon \subseteq R'$. We claim that $J = R'\varepsilon$. Clearly $R'\varepsilon \subseteq J$. Conversely, since $J \subseteq Q_0\varepsilon$ we have $x = x\varepsilon$ for every $x \in J$, and therefore $J \subseteq R'\varepsilon$. Of course, $R' = R'\varepsilon \oplus R'(1-\varepsilon)$ which means that $J = R'\varepsilon$ is indeed a direct summand. ∎

2. The results

Our first theorem is an extension of Theorem 2.3 in [3]. Fortunately the same proof works, but we include it for the sake of completeness.

Theorem 1. *Let θ be a Jordan homomorphism of a ring R onto a 2-torsion free semiprime ring R' in which the annihilator of every ideal is a direct summand. Then there exist ideals U and V of R and ideals U' and V' of R' such that $U \cap V = \mathrm{Ker}\,\theta$ and $U + V = R$, $U' \cap V' = 0$ and $U' \oplus V' = R'$, the restriction of θ to U is a homomorphism of U onto U', and the restriction of θ to V is an antihomomorphism of V onto V'.*

PROOF: We introduce the abbreviations $a^b = \theta(ab) - \theta(a)\theta(b)$, $a_b = \theta(ab) - \theta(b)\theta(a)$. By [**3**, Corollary 2.2] we have

$$a^b R' c_d = 0 \quad \text{for all} \quad a,\, b,\, c,\, d \in R. \tag{1}$$

Let V_0' be an ideal of R' generated by the set $\{a^b \mid a,\, b \in R\}$, and set $U' = \operatorname{Ann}(V_0')$, $V' = \operatorname{Ann}(U')$. By assumption, $U' \oplus V' = R'$. Of course, $\{a^b \mid a,\, b \in R\} \subseteq V_0' \subseteq V'$, and from (1) we see that $\{a_b \mid a,\, b \in R\} \subseteq U'$. Let $U = \theta^{-1}(U')$ and $V = \theta^{-1}(V')$. Take $u \in U$, $y \in R$, $x' \in R'$. By (1) we have

$$\begin{aligned} u^y x' u^y &= u^y x'(u^y - u_y) \\ &= u^y x'(\theta(y)\theta(u) - \theta(u)\theta(y)) \\ &= u^y x'\theta(y)\theta(u) - (u^y x'\theta(u))\theta(y) \\ &= 0 \end{aligned}$$

since $\theta(u) \in U' = \operatorname{Ann}(V_0')$. Hence $u^y = 0$ by the semiprimeness of R', that is, $\theta(uy) = \theta(u)\theta(y)$ for all $u \in U$, $y \in R$. By the definition of Jordan homomorphisms we then also have $\theta(yu) = \theta(y)\theta(u)$. The last two relations imply that U is an ideal of R'. Clearly $\theta(U) = U'$. Thus we have proved that the restriction of θ to U is a homomorphism of U onto U'. In a similar fashion one shows that V is an ideal of R, and that the restriction of θ to V is an antihomomorphism of V onto V'.

It is obvious that $\operatorname{Ker}\theta \subseteq U \cap V$. Conversely, $\theta(U \cap V) \subseteq \theta(U) \cap \theta(V) = 0$ and therefore $U \cap V = \operatorname{Ker}\theta$. Let us show that $U + V = R$. Given $x \in R$, we have $\theta(x) = u' + v'$ where $u' \in U'$, $v' \in V'$. Since $u' = \theta(u)$, $u \in U$, and $v' = \theta(v)$, $v \in V$, it follows that $x - u - v \in \operatorname{Ker}\theta = U \cap V$. Hence $U + V = R$. The proof of the theorem is complete. □

REMARK. Combining Theorem 1 and Lemma 1 we obtain, as outlined in the introduction, a generalisation of Theorem 3.8 in [**2**]. Note also that Theorem 3.9 in [**2**] can now be stated in a more general form.

We now want to prove the analogous result for Jordan triple homomorphisms. The proof is an adaption of the proof of Theorem 3.5 in [**3**].

THEOREM 2. *Let R be a ring with property $R^2 = R$, and let R' be a 2-torsion free semiprime ring in which the annihilator of any ideal is a direct summand. If θ is a Jordan triple homomorphism of R onto R' then there exist ideals U_1, U_2, U_3, U_4 of R, and ideals U_1', U_2', U_3', U_4' of R' such that*

(i) $U_i \cap U_j = \operatorname{Ker}\theta$, $i \neq j$, *and* $U_1 + U_2 + U_3 + U_4 = R$,
(ii) $U_i' \cap U_j' = 0$, $i \neq j$, *and* $U_1' \oplus U_2' \oplus U_3' \oplus U_4' = R'$,
(iii) *the restriction of θ to U_1 is a homomorphism of U_1 onto U_1',*

(iv) *the restriction of* θ *to* U_2 *is a negative of a homomorphism of* U_2 *onto* U_2',

(v) *the restriction of* θ *to* U_3 *is an antihomomorphism of* U_3 *onto* U_3',

(vi) *the restriction of* θ *to* U_4 *is a negative of an antihomomorphism of* U_4 *onto* U_4'.

PROOF: We write $S(a, b, c) = \theta(abc) - \theta(a)\theta(b)\theta(c)$ and $T(a, b, c) = \theta(abc) - \theta(c)\theta(b)\theta(a)$. By [**3**, Corollary 3.2] we have

$$T(a_1, a_2, a_3)R'S(b_1, b_2, b_3) = 0 \quad \text{for all } a_i, b_i \in R, \quad i = 1, 2, 3. \tag{2}$$

Let I_0' be an ideal of R' generated by the set $\{S(a, b, c) \mid a, b, c \in R\}$ and let $U' = \operatorname{Ann}(I_0')$, $V' = \operatorname{Ann}(U')$. By assumption, $U' \oplus V' = R'$. Of course, $\{S(a, b, c) \mid a, b, c \in R\} \subseteq V'$, and by (2) we have $\{T(a, b, c) \mid a, b, c \in R\} \subseteq U'$. We set $U = \theta^{-1}(U')$ and $V = \theta^{-1}(V')$. Take $x, y, x \in R$ with at least one in U. According to (2) for any $x' \in R'$ we have

$$\begin{aligned} &S(x, y, z)x'S(x, y, z) \\ &\quad = S(x, y, z)x'(S(x, y, z) - T(x, y, z)) \\ &\quad = S(x, y, z)x'(\theta(z)\theta(y)\theta(x) - \theta(x)\theta(y)\theta(z)) \\ &\quad = 0 \end{aligned}$$

since at least one of $\theta(x)$, $\theta(y)$, $\theta(z)$ lies in U'. Consequently $S(x, y, z) = 0$. Since $R^2 = R$ we have $\theta(UR) = \theta(URR) = \theta(U)\theta(R)\theta(R) \subseteq U'$ which means that U is a right ideal of R. Similarly we see that U is a left ideal.

Analogously one shows that V is an ideal of R and that $T(v, x, y) = T(x, v, y) = T(x, y, v) = 0$ holds for all $x, y \in R$, $v \in V$.

Consider $\theta(uxyux)$ where $x, y \in R$ and $u \in U$. On the one hand we have

$$\begin{aligned} \theta(uxyux) &= \theta(u(xyu)x) \\ &= \theta(u)\theta(xyu)\theta(x) \\ &= \theta(u)\theta(x)\theta(y)\theta(u)\theta(x), \end{aligned}$$

and on the other hand,

$$\begin{aligned} \theta(uxyux) &= \theta((ux)y(ux)) \\ &= \theta(ux)\theta(y)\theta(ux). \end{aligned}$$

Comparing the last two relations we arrive at $P(u, x)\theta(y)Q(u, y) + Q(u, x)\theta(y)P(u, x) = 0$ where $P(r, s)$ denotes the element $\theta(rs) - \theta(r)\theta(s)$, and $Q(r, s)$ denotes $\theta(rs) +$

$\theta(r)\theta(s)$. Since θ is onto it follows from [**3**, Lemma 1.1] that $P(u, x)R'Q(u, x) = 0$ for all $u \in U$, $x \in R$. But then [**3**, Lemma 1.2] tells us that

$$P(u, x)R'Q(w, y) = 0 \text{ for all } u, w \in U, x, y \in R. \tag{3}$$

Let I_1' be an ideal of R' generated by the set $\{P(u, x) \mid u \in U, x \in R\}$. Since U is an ideal we see that $P(u, x)$ and $Q(u, x)$ lie in U' if $u \in U$. Therefore I_1' is contained in U'. Let $U_1' = \operatorname{Ann}(I_1') \cap U'$, $U_2' = \operatorname{Ann}(U_1') \cap U'$.

We claim that $U_1' \oplus U_2' = U'$. Clearly $U_1' \cap U_2' = 0$. By assumption, $\operatorname{Ann}(I_1')$ is a direct summand, thus $R' = \operatorname{Ann}(I_1') \oplus W'$ for some ideal W' of R'. Then $W' = \operatorname{Ann}(\operatorname{Ann}(I_1')) \subseteq \operatorname{Ann}(\operatorname{Ann}(I_1') \cap U') = \operatorname{Ann}(U_1')$, and, since $I_1' \subseteq U'$, $W' = \operatorname{Ann}(\operatorname{Ann}(I_1')) \subseteq \operatorname{Ann}(\operatorname{Ann}(U')) = \operatorname{Ann}(V') = U'$. Thus $W' \subseteq U_2'$ (in fact, $W' = U_2'$). Given $u' \in U'$, we then have $u' = z' + w'$ for some $z' \in \operatorname{Ann}(I_1')$, $w' \in W'$. Since $W' \subseteq U'$ we then have $z' \in U'$; that is, $z' \in \operatorname{Ann}(I_1') \cap U' = U_1'$. Hence $U' = U_1' + U_2'$.

We set $U_1 = \theta^{-1}(U_1')$ and $U_2 = \theta^{-1}(U_2')$. Take $u_1 \in U_1$, $x \in R$, $y' \in R'$. By (3) we have

$$\begin{aligned} P(u_1, x)y'P(u_1, x) &= P(u_1, x)y'(P(u_1, x) - Q(u_1, x)) \\ &= -2P(u_1, x)y'\theta(u_1)\theta(x) \\ &= 0 \end{aligned}$$

since $\theta(u_1) \in U_1'$. Thus $P(u_1, x) = 0$ by the semiprimeness of R'. This means that the restriction of θ to U_1 is a homomorphism of U_1 onto U_1'. The last relation also implies that U_1 is a right ideal of R. In order to prove that U_1 is a left ideal we will show that $P(x, u_1)$ with $x \in R$, $u_1 \in U_1$, is zero as well. Take $u_1 \in U_1$, $x, y, z \in R$. Since $xu_1 \in U$ we have $\theta((xu_1)yz) = \theta(xu_1)\theta(y)\theta(z)$. But on the other hand, using $u_1y \in U$ and $P(u_1, y) = 0$ we obtain $\theta(x(u_1y)z) = \theta(x)\theta(u_1y)\theta(z) = \theta(x)\theta(u_1)\theta(y)\theta(z)$. Comparing, we arrive at $(\theta(xu_1) - \theta(x)\theta(u_1))\theta(y)\theta(z) = 0$. But then, since θ is onto and R is semiprime, it follows that $\theta(xu_1) = \theta(x)\theta(u_1)$.

Note that (3) implies $\{Q(u, x) \mid u \in U, x \in R\} \subseteq U_1'$. Using similar arguments as above one then verifies that $Q(u_2, x) = 0$ and $Q(x, u_2) = 0$ for all $u_2 \in U_2$, $x \in R$. This implies that U_2 is an ideal of R and that the restriction of θ of U_2 is a negative of a homomorphism of U_2 onto U_2'.

In an analogous way one shows that there exist ideals U_3', U_4' of R' such that $U_3' \oplus U_4' = V'$, and that $U_3 = \theta^{-1}(U_3')$, $U_4 = \theta^{-1}(U_4')$ are ideals of R satisfying $\theta(u_3x) = \theta(x)\theta(u_3)$, $\theta(xu_3) = \theta(u_3)\theta(x)$, $\theta(u_4x) = -\theta(x)\theta(u_4)$, $\theta(xu_4) = -\theta(u_4)\theta(x)$ for all $x \in R$, $u_3 \in U_3$, $u_4 \in U_4$. Then, of course, the restriction of θ to U_3 is an antihomomorphism of U_3 onto U_3', and the restriction of θ to U_4 is a negative of an antihomomorphism of U_4 onto U_4'.

Clearly $U_i' \cap U_j' = 0$, $i \neq j$, and $U_1' \oplus U_2' \oplus U_3' \oplus U_4' = R'$. It remains to prove (i). It is obvious that $\operatorname{Ker}\theta \subseteq U_i$, and therefore $\theta(U_i \cap U_j) \subseteq \theta(U_i) \cap \theta(U_j) = U_i' \cap U_j' = 0$, $i \neq j$, implies $U_i \cap U_j = \operatorname{Ker}\theta$, $i \neq j$. Let us show that $U + V = R$. Given $x \in R$, the element $\theta(x)$ can be written in the form $u' + v'$, $u' \in U'$, $v' \in V'$. There exist $u \in U$, $v \in V$ such that $\theta(u) = u'$ and $\theta(v) = v'$. Hence $x - u - v \in \operatorname{Ker}\theta = U \cap V$ which means that $U + V = R$. Similarly we verify that $U_1 + U_2 = U$ and $U_3 + U_4 = V$. Consequently $U_1 + U_2 + U_3 + U_4 = R$. The proof of the theorem is complete. $\square$

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