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ASYMPTOTIC BEHAVIOR OF NONOSCILLATORY SOLUTIONS OF A HIGHER ORDER FUNCTIONAL DIFFERENTIAL EQUATION

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The asymptotic behavior of nonoscillatory solutions of nth order nonlinear functional differential equations

$$\Big(r_{n-1}(t)\Big(r_{n-2}(t)\Big(\ldots\Big(r_2(t)\Big(r_1(t)y'(t)\Big)'\Big)'\Big)\ldots\Big)'\Big)'\Big)' + a(t)f\big(y\big(g(t)\big)\big) = b(t)$$

is investigated. Sufficient conditions are provided which ensure that all nonoscillatory solutions approach zero as $t \to \infty$.

1. Introduction

We consider the nth order functional differential equation with deviating argument

$$\Big(r_{n-1}(t)\Big(r_{n-2}(t)\Big(\ldots\Big(r_2(t)\Big(r_1(t)y'(t)\Big)'\Big)'\ldots\Big)'\Big)'\Big)' + a(t)f\big(y\big(g(t)\big)\big) = b(t) , \tag{1}$$

where $a(t)$, $b(t)$, $g(t)$, $r_1(t)$, $\ldots$, $r_{n-1}(t)$ are real-valued and continuous on $[T, \infty)$ and $f(y)$ is real-valued and continuous on $(-\infty, \infty)$.

The following conditions are assumed to hold throughout the paper:

(2a) $\lim_{t\to\infty} g(t) = \infty$;

(2b) $yf(y) > 0$ for $y \neq 0$;

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(2c) $r_i(t) > 0$ and $\lim_{t\to\infty} \rho_i(t) = 0$, where $\rho_i(t) = \int_t^\infty \frac{\rho_{i-1}(s)}{r_i(s)}\, ds$,

$i = 1, \ldots, n-1$, $(\rho_0(t) \equiv 1)$.

We note that the condition (2c) is satisfied if

$$\int_T^\infty \frac{dt}{r_i(t)} < \infty \ , \quad i = 1, \ldots, n-1 \ . \tag{3}$$

We restrict our consideration to those solutions $y(t)$ of (1) which exist on some ray $[T_y, \infty)$ and satisfy

$$\sup\{|y(t)| : t_0 \le t < \infty\} > 0$$

for any $t_0 \in [T_y, \infty)$. Such a solution is said to be oscillatory if it has arbitrary large zeros; otherwise, it is said to be nonoscillatory. It is important to find sufficient conditions in order that all nonoscillatory solutions of (1) tend to zero as $t \to \infty$. Many authors have studied this problem, for example, Hammett [3], Graef and Spikes [1], Grimmer [2], Kartsatos [4], Kusano and Onose ([5], [6]), Londen [7] and Singh [8]. In this paper we present some results on this problem.

2. Non-oscillation theorems

We use the following lemmas to prove our results.

LEMMA 1 [5]. *Consider the differential equation*

$$u'(t) - \frac{\rho'(t)}{\rho(t)} u(t) + \frac{\rho'(t)}{\rho(t)} \phi(t) = 0 \ , \tag{4}$$

where $\phi(t)$ *is continuous on* $[T, \infty)$, $\rho(t)$ *is continuously differentiable on* $[T, \infty)$ *and* $\rho(t) > 0$, $\rho'(t) < 0$, $\lim_{t\to\infty} \rho(t) = 0$. *Let* $u(t)$ *be the solution of* (4) *on* $[T, \infty)$ *satisfying* $u(T) = 0$. *Then* $\lim_{t\to\infty} \phi(t) = \infty$ [or $-\infty$] *implies* $\lim_{t\to\infty} u(t) = \infty$ [or $-\infty$] .

LEMMA 2 [5]. *Let* $\sigma(t)$ *be continuous on* $[T, \infty)$ *and let* $v(t)$ *be continuous differentiable on* $[T, \infty)$. *If the limit* $\lim_{t\to\infty} [\sigma(t)v'(t)+v(t)]$ *exists in the extended real line* $R^{\#}$, *then the limit* $\lim_{t\to\infty} v(t)$ *exists in*

$R^{\#}$.

THEOREM 1. *Let the condition* (3) *hold. Suppose that* $a(t) \geq 0$. *If*

$$\int^{\infty} \rho_{n-1}(t)a(t)dt = \infty , \tag{5}$$

$$\int^{\infty} |b(t)|dt < \infty , \tag{6}$$

then all nonoscillatory solutions of (1) *tend to zero as* $t \to \infty$.

Proof. Let $y(t)$ be a nonoscillatory solution of (1). We may suppose that $y(g(t)) > 0$ for $t \geq t_1$. We define

$$G_0(t) = y(t) , \quad G_i(t) = r_i(t)G'_{i-1}(t) , \quad i = 1, \ldots, n-1 , \tag{7}$$

$$u_{k-1}(t) \equiv \int_{t_1}^{t} \rho_{n-k}(s)G'_{n-k}(s)ds \quad \text{for } k = 1, 2, \ldots, n , \tag{8}$$

which implies

$$u_{k-1}(t) = -\frac{\rho_{n-k}(t)}{\rho'_{n-k}(t)} u'_k(t) + u_k(t) - \rho_{n-k}(t_1)G_{n-k}(t_1) .$$

This shows that $u_k(t)$ satisfies the differential equation

$$\frac{\rho_{n-k}(t)}{\rho'_{n-k}(t)} u' - u + \phi_k(t) = 0 , \tag{9}$$

or equivalently,

$$u' - \frac{\rho'_{n-k}(t)}{\rho_{n-k}(t)} u + \frac{\rho'_{n-k}(t)}{\rho_{n-k}(t)} \phi_k(t) = 0 , \tag{10}$$

where

$$\phi_k(t) = u_{k-1}(t) + \rho_{n-k}(t_1)G_{n-k}(t_1) .$$

Since $u_k(t_1) = 0$ by (8) and since $\rho_{n-k}(t) > 0$, $\rho'_{n-k}(t) < 0$, $\lim_{t\to\infty} \rho_{n-k}(t) = 0$ by (2c), we apply Lemma 1 to (10) to conclude that $\lim_{t\to\infty} u_{k-1}(t) = \infty$ [or $-\infty$] implies that $\lim_{t\to\infty} u_k(t) = \infty$ [or $-\infty$] . Moreover,

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applying Lemma 2 to (9), we conclude that $\lim_{t\to\infty} u_k(t)$ exists in $R^\#$ whenever $\lim_{t\to\infty} u_{k-1}(t)$ exists in $R^\#$. From (1) we obtain

$$(11) \qquad G_{n-1}(t) - G_{n-1}(t_1) + \int_{t_1}^{t} a(s)f(y(g(s)))ds = \int_{t_1}^{t} b(s)ds .$$

Since the first integral of (11) is positive and, by (6), the second integral is bounded, there exist a constant K_{n-1} such that

$$G_{n-1}(t) = r_{n-1}(t)G'_{n-2}(t) \le K_{n-1} \quad \text{for} \quad t \ge t_2 \ge t_1 .$$

Dividing the inequality by $r_{n-1}(t)$ and integrating from t_2 to t, we get

$$G_{n-2}(t) - G_{n-2}(t_1) \le K_{n-1} \int_{t_2}^{t} \frac{ds}{r_{n-1}(s)} \quad \text{for} \quad t \ge t_2 ,$$

which shows, in view of (3), that there exists a constant K_{n-2} such that

$$G_{n-2}(t) = r_{n-2}(t)G'_{n-3}(t) \le K_{n-2} \quad \text{for} \quad t \ge t_2 .$$

Applying the above argument repeatedly, we have

$$G_{n-3}(t) \le K_{n-3}, \ \ldots, \ G_1(t) \le K_1 , \quad G_0(t) \le K_0 \quad \text{for} \quad t \ge t_2 ,$$

where $K_{n-3}, \ldots, K_1, K_0$ are constants. It follows that $G_0(t) \equiv y(t)$ is bounded above for $t \ge t_2$. We now multiply both sides of (1) by $\rho_{n-1}(t)$ and integrate it over $[t_2, t]$. Then we have

$$(12) \quad \int_{t_2}^{t} \rho_{n-1}(s)G'_{n-1}(s)ds + \int_{t_2}^{t} \rho_{n-1}(s)a(s)f(y(g(s)))ds = \int_{t_2}^{t} \rho_{n-1}(s)b(s)ds .$$

Noting that on account of (6) the right hand of (12) tends to a finite limit as $t \to \infty$, we can deduce from (12) that

(13) $$\int_{t_2}^{\infty} \rho_{n-1}(t)a(t)f(y(g(t)))dt < \infty ,$$

since otherwise we could use Lemma 1 to obtain $\lim_{t\to\infty} u_k(t) = -\infty$ for $k = 0, 1, \ldots, n-1$, which implies $\lim_{t\to\infty} y(t) = -\infty$, a contradiction. Next, using (12), (13), the boundedness of $y(t)$ and applying Lemma 2, we can find that $\lim_{t\to\infty} u_k(t)$ is finite for each $k = 0, 1, \ldots, n-1$. Thus we see that $\lim_{t\to\infty} y(t)$ exists and finite. Namely, $\lim_{t\to\infty} y(t) = c$, where c is a finite nonnegative constant. If $c > 0$, then we have

$c/2 \le y(g(t)) \le 2c$ for sufficiently large t, say $t \ge t_3 \ge t_2$.

From (13) and $f(y)$ is continuous, we have a contradiction that

$$\infty > \int_{t_3}^{\infty} \rho_{n-1}(t)a(t)f(y(g(t)))dt \ge K^* \int_{t_3}^{\infty} \rho_{n-1}(t)a(t)dt = \infty ,$$

where

$$K^* = \operatorname*{Min}_{(c/2)\le y\le 2c} f(y) > 0 .$$

Therefore, we conclude that $y(t)$ tends to zero as $t \to \infty$.

REMARK. Kusano and Onose [5] obtain the same conclusion with the additional assumption $\liminf_{y\to\infty} f(y) > 0$ and $\limsup_{y\to-\infty} f(y) < 0$.

THEOREM 2. *Let the condition* (3) *hold. Suppose that* $a(t) \ge 0$, $\liminf_{y\to\infty} f(y) > 0$ *and* $\limsup_{y\to-\infty} f(y) < 0$. *If*

(14) $$\int^{\infty} \rho_{n-1}(t)a(t)dt = \infty ,$$

(15) $$\int^{\infty} \rho_{n-1}(t)|b(t)|dt < \infty ,$$

then all nonoscillatory solutions of (1) *tends to zero as* $t \to \infty$.

Proof. Let $y(t)$ be a nonoscillatory solution of (1). We may suppose that $y(g(t)) > 0$ for $t \ge t_1$. Define $G_i(t)$ and $u_k(t)$ by (7) and (8). We now multiply both sides of (1) by $\rho_{n-1}(t)$ and integrate it over $[t_1, t]$. Then we have

$$(12)\quad \int_{t_1}^{t} \rho_{n-1}(s)G'_{n-1}(s)ds + \int_{t_1}^{t} \rho_{n-1}(s)a(s)f(y(g(s)))ds = \int_{t_1}^{t} \rho_{n-1}(s)b(s)ds .$$

By using (12) and (15) we can deduce that

$$(13)\quad \int_{t_1}^{\infty} \rho_{n-1}(t)a(t)f(y(g(t)))dt < \infty ,$$

since otherwise we could use Lemma 1 to obtain $\lim_{t\to\infty} y(t) = -\infty$, a contradiction. Next, using (12), (13) and applying Lemma 2, we can find that $\lim_{t\to\infty} u_k(t)$ $(k = 0, 1, \ldots, n-1)$ exist as definite limit finite or ∞. Thus we see $\lim_{t\to\infty} y(t) = \infty$ or $\lim_{t\to\infty} y(t) = c$, where c is a finite and nonnegative constant. If $\lim_{t\to\infty} y(t) = \infty$, then we have $\liminf_{t\to\infty} f(y(g(t))) > 0$ by assumption, which and (14) lead to a contradiction that $\int_{t_1}^{\infty} \rho_{n-1}(t)a(t)f(y(g(t)))dt = \infty$.

If $\lim_{t\to\infty} y(t) = c > 0$, then also we have a contradiction:

$$\int_{t}^{\infty} \rho_{n-1}(t)a(t)f(y(g(t)))dt = \infty .$$

Therefore we conclude that $y(t)$ tends to zero as $t \to \infty$. //

REMARK. Theorem 2 contains the result of Kusano and Onose ([5], Theorem 3).

EXAMPLE 1. Consider the equation

$$(16)\quad \left(t^2\left(t^2\left(t^2y'(t)\right)'\right)'\right)' + t^7y^3(\gamma t) = \gamma^{-6}t , \quad t > 0 ,$$

where γ is a positive constant. In this case we have $\rho_1(t) = t^{-1}$, $\rho_2(t) = (1/2)t^{-2}$, $\rho_3(t) = (1/6)t^{-3}$. Since all assumptions of Theorem 2 are satisfied, every nonoscillatory solution of (16) approaches zero as $t \to \infty$. This equation has a nonoscillatory solution $y(t) = t^{-2}$.

EXAMPLE 2. Consider the equation

$$(17)\quad \left(e^t\left(e^t\left(e^ty'(t)\right)'\right)'\right)' + e^{5t}y(t+\theta) = 24e^{-t} + e^{-4\theta}e^t, \quad t \geq 0,$$

where θ is a constant. This equation possesses $y(t) = e^{-4t}$ as a nonoscillatory solution tending to zero as $t \to \infty$. It is easy to verify that $\rho_1(t) = e^{-t}$, $\rho_2(t) = (1/2)e^{-2t}$, $\rho_3(t) = (1/6)e^{-3t}$ and the conclusions of Theorem 2 are satisfied. Therefore all nonoscillatory solutions of (17) also tend to zero as $t \to \infty$.

REMARK. These examples cannot be covered by Kusano and Onose ([5], Theorem 3).

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