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# CHARACTERISATION OF NILPOTENT-BY-FINITE GROUPS

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Let $G$ be a finitely generated soluble group. The main result of this note is to prove that $G$ is nilpotent-by-finite if, and only if, for every pair $X, Y$ of infinite subsets of $G$, there exist an $x$ in $X$, $y$ in $Y$ and two positive integers $m = m(x,y)$, $n = n(x,y)$ satisfying $[x,_n y^m] = 1$. We prove also that if $G$ is infinite and if $m$ is a positive integer, then $G$ is nilpotent-by-(finite of exponent dividing $m$) if, and only if, for every pair $X$, $Y$ of infinite subsets of $G$, there exist an $x$ in $X$, $y$ in $Y$ and a positive integer $n = n(x,y)$ satisfying $[x,_n y^m] = 1$.

## INTRODUCTION AND RESULTS

Following questions of Erdös, B.H. Neumann proved in [**9**] that a group is centre-by-finite if, and only if, every infinite subset contains a commuting pair of distinct elements. From this, as was observed in [**7**], it is easy to show that if $G$ is an infinite group such that for every pair $X$, $Y$ of infinite subsets of $G$, there exist an $x$ in $X$ and $y$ in $Y$ that commute, then $G$ is Abelian. Endimioni [**2,** Theorem 2] extended this result, by proving that if $G$ is an infinite finitely generated soluble group such that for every pair $X$, $Y$ of infinite subsets of $G$, there exist an $x$ in $X$, $y$ in $Y$ and a positive integer $n = n(x,y)$ satisfying $[x,_n y] = 1$, then $G$ is nilpotent. The main purpose of this note is to improve this last result. We shall prove:

**THEOREM 1.** *Let $G$ be a finitely generated soluble group. Then the following properties are equivalent:*

(i) *$G$ is nilpotent-by-finite.*
(ii) *For every pair $X, Y$ of infinite subsets of $G$, there exist an $x$ in $X$, $y$ in $Y$ and two positive integers $m = m(x,y)$, $n = n(x,y)$ satisfying $[x,_n y^m] = 1$.*

From a result of Lennox [**4**], a finitely generated soluble group all of whose two-generator subgroups are nilpotent-by-finite, is itself nilpotent-by-finite. As an immediate consequence of Theorem 1, we have the following generalisation of Lennos's result:

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**COROLLARY 1.** *A finitely generated soluble group $G$ is nilpotent-by-finite if, and only if, for every pair $X$, $Y$ of infinite subsets of $G$, there exist an $x$ in $X$ and $y$ in $Y$ generating a nilpotent-by-finite group.*

As a consequence of Theorem 1, we shall prove the result:

**COROLLARY 2.** *Let $G$ be a finitely generated soluble group such that for every pair $X$, $Y$ of infinite subsets of $G$, there exist an $x$ in $X$, $y$ in $Y$ and a positive integer $m = m(x, y)$ satisfying $[x, y^m] = 1$. Then $G$ is Abelian-by-finite.*

Corollary 2 leads us to consider the following question: if the integers $n(x, y)$ of Theorem 1 are bounded by an integer $k$, then is $G$ a finite extension of a nilpotent group of class at most an integer depending only on $k$? We are unable to answer this in the general case. However, we shall prove:

**THEOREM 2.** *Let $G$ be a finitely generated metabelian group satisfying the condition* (ii) *of Theorem 1, and suppose that the integers $n(x, y)$ are bounded by a positive integer $k$. Then there is a function $c(k)$ of $k$ only, such that $G$ is a finite extension of a nilpotent group of class at most $c(k)$.*

Note that these results are not true for arbitrary groups. Indeed, Golod [**3**] showed that for each integer $d > 1$ and each prime $p$, there are infinte $d$-generator groups all of whose $(d-1)$-generator subgroups are finite $p$-groups. For $d = 3$, we obtain groups which satisfy the combinatorial conditions of the theorems and the corollaries, but which are not nilpotent-by-finite.

Now we turn our attention to the integers $m(x, y)$. We shall prove:

**THEOREM 3.** *Let $m$ be a positive integer and let $G$ be an infinite finitely generated soluble group. Then the following properties are equivalent:*

(i) *$G$ is nilpotent-by-(finite of exponent dividing $m$).*

(ii) *For every pair $X$, $Y$ of infinite subsets of $G$, there exist an $x$ in $X$, $y$ in $Y$ and a positive integer $n = n(x, y)$ such that $[x, {}_n y^m] = 1$.*

If we take $m = 1$, then we find again [**2,** Theorem 2].

Our notation and terminology are the usual ones, and can be found in [**10**]. In particular, $[x, {}_n y]$ is defined for each integer $n \geqslant 0$ by $[x, {}_0 y] = x$ and $[x, {}_{n+1} y] = [[x, {}_n y], y]$. We shall denote by $\Omega^*$ the class of groups satisfying the condition (ii) of Theorem 1.

## 2. Some preliminary Lemmas

**LEMMA 1.** *Let $G$ be a finitely generated metabelian group in the class $\Omega^*$. Then $G$ is nilpotent-by-finite.*

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PROOF: Let $G$ be a finitely generated metabelian group in the class $\Omega^*$. Suppose that $G$ is not nilpotent-by-finite. Since $\Omega^*$ is a quotient closed class of groups, and since finitely generated nilpotent-by-finite groups are finitely presented, it follows, by [**10**, Lemma 6.17], that we may assume that every proper homomorphic image of $G$ is nilpotent-by-finite. Since $G$ is metabelian, its Hirsch-Plotkin radical $H$ is non trivial; hence, $G/H$ is nilpotent-by-finite. It follows that $G$ contains a normal subgroup $K$ of finite index such that $K/H$ is nilpotent. If $K/H$ is infinite, then it contains an element $yH$ of infinite order [**10**, Theorem 2.24]. Thus, for any integer $k$, $y^k \notin H$; furthermore, for any $x \in G$, the subsets $\{y^i x : i \text{ positive integer}\}$ and $\{y^i : i \text{ positive integer}\}$ are infinite. Hence, there exist positive integers $r$, $k$, $m = m(x,y)$ and $n = n(x,y)$ such that $[y^r x, {}_n y^{km}] = 1$; so we get that $[x, {}_n y^{km}] = 1$. Since $G$ is a finitely generated metabelian group, it is eremitic [**5**, Theorem B]. This means that there is a positive integer $d$, depending only on $G$, such that $[a, b^d] = 1$ whenever $[a, b^i] = 1$, for any $a$, $b$ in $G$ and any positive integer $i$. Therefore, we deduce that $[[x, {}_{n-1} y^{km}],\ y^d] = 1$. The group $G$ being metabelian, it is easy to see that $[a,\ b,\ c] = [a,\ c,\ b]$ for all elements $a, b, c$ of $G$ such that $bc = cb$. Thus, we get that $[[x,\ y^d], {}_{n-1} y^{km}] = 1$; and by induction on $n$, we obtain that $[x, {}_n y^d] = 1$. Therefore, $y^d$ is a left Engel element of $G$. Since $G$ is metabelian, the set of left Engel elements of $G$ coincides with its Hirsch-Plotkin radical [**10**, Theorem 7.34]. So $y^d \in H$, and this contradicts the choice of $y$. It follows that $K/H$ is finite, so $G/H$ is finite. Since $G$ is finitely generated, $H$ is also finitely generated. Hence, $H$ is nilpotent; and $G$ is, therefore, nilpotent-by-finite, a contradiction which completes the proof. ∎

We shall use the following lemma which is due to Lennox [**6**].

LEMMA 2. *Let $G$ be a finitely generated soluble group and $A$ a normal Abelian subgroup such that $G/A$ is polycyclic and $\langle a, g\rangle$ is polycyclic whenever $a \in A$ and $g \in G$. Then $G$ is polycyclic.*

LEMMA 3. *Let $G$ be a finitely generated soluble group in $\Omega^*$. Then $G$ is polycyclic.*

PROOF: Since polycyclic groups are finitely presented, and since $\Omega^*$ is a quotient closed class of groups, by [**10**, Lemma 6.17], we may assume that every proper homomorphic image of $G$ is polycyclic, but $G$ itself is not polycyclic. Since $G$ is soluble, it has a non trivial normal Abelian subgroup $A$; so $G/A$ is polycyclic. Let $g \in G$ and $a \in A$; $\langle a, g\rangle$ is, therefore, a finitely generated metabelian group in the class $\Omega^*$. It follows, from Lemma 1, that $\langle a,\ g\rangle$ is nilpotent-by-finite. Thus, $\langle a,\ g\rangle$ is polycyclic. From Lemma 2, we can deduce that $G$ is polycyclic, which is a contradiction.

## 3. Proofs of the results

PROOF OF THEOREM 1: Clearly we have only to show that (ii) implies (i). Suppose

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that $G$ is a finitely generated soluble group in the class $\Omega^*$; from Lemma 3, $G$ is polycyclic. The group $G$ contains, therefore, a normal subgroup $H$ of finite index, whose derived subgroup $H'$ is nilpotent [**11,** 15.1.6]. Since $G$ is polycyclic, it satisfies the maximal condition on normal subgroups; and since $\Omega^*$ is a quotient closed class, we may, therefore, assume that $G$ is not nilpotent-by-finite, but that every proper homomorphic image of $G$ is nilpotent-by-finite. If $H^{(2)}$, the third term of the derived series of $H$, is non trivial, then $G/H^{(2)}$ is nilpotent-by-finite. Hence, $G$ contains a normal subgroup $K$ of finite index such that $K/H^{(2)}$ is nilpotent. Now $K/H^{(2)}$ and $H'/H^{(2)}$ are two normal nilpotent subgroups of $G/H^{(2)}$, so their product $KH'/H^{(2)}$ is nilpotent [**10,** Theorem 2.18]. Also $H'$ and $KH'/H^{(2)}$ are nilpotent; by a result of Hall [**10,** Theorem 2.27], $KH'$, and so $K$, is nilpotent. Thus, $G$ is nilpotent-by-finite, which is a contradiction. So $H^{(2)} = 1$ and $H$ is, therefore, a finitely generated metabelian group. It follows, from Lemma 1, that $H$ is nilpotent-by-finite. So $G$ is nilpotent-by-finite, a contradiction which completes the proof. ∎

Proof of Corollary 2: Let $G$ be a finitely generated soluble group such that, for every pair $X$, $Y$ of infinite subsets of $G$, there exist an $x$ in $X$, $y$ in $Y$ and a positive integer $m = m(x,y)$ satisfying $[x, y^m] = 1$. Clearly, we may assume that $G$ is infinite. It follows, from Theorem 1, that $G$ is nilpotent-by-finite. Thus, $G$ has an infinite finitely generated nilpotent subgroup of finite index so, without loss of generality, we may suppose $G$ is finitely generated and nilpotent. Since finitely generated nilpotent groups are (torsion-free)-by-finite [**11,** 5.4.15(i)], we may assume also that $G$ is torsion-free. The group $G$, being nilpotent and finitely generated, contains a maximal normal Abelian subgroup $A$. We know that $C_G(A) = A$ [**11,** 5.2.3]. Let $a$ be a non trivial element of $A$, and let $g \in G$; since $G$ is torsion-free, the subsets $\{a^i : i \text{ integer}\}$ and $\{a^i g : i \text{ integer}\}$ are infinite. There exist, therefore, integers $i, j$ and $m = m(a,g)$ such that $[a^i, (a^j g)^m] = 1$. Since $A$ is a normal Abelian subgroup of $G$, we get that $[a, (a^j g)^m]^i = 1$. Thus, we obtain that $[a, (a^j g)^m] = 1$, because $G$ is torsion-free; hence, it is easy to deduce that $[a, g^m] = 1$. The group $G$, being nilpotent and finitely generated, is eremitic [**5,** Theorem B]. There is, therefore, a positive integer $d$, depending only on $G$, such that $[a, g^d] = 1$; so $g^d \in C_G(A)$. Now $A = C_G(A)$, thus $g^d \in A$. It follows, that $G/A$ is a periodic group. Therefore, $G/A$ being a periodic finitely generated nilpotent group, is finite. Hence, $G$ is Abelian-by-finite, as required. ∎

Proof of Theorem 2: Let $G$ be a finitely generated metabelian group in the class $\Omega^*$, such that the integers $n(x,y)$ are bounded by a positive integer $k$. Clearly, we may assume that $G$ is infinite. It follows, from Theorem 1, that $G$ is nilpotent-by-finite. Hence, $G$ contains a normal nilpotent subgroup $H$ of finite index. Since finitely generated nilpotent groups are (torsion-free)-by-finite [**11,** 5.4.15 (i)], there is

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no loss of generality if we assume that $H$ is torsion-free. Since $G$ is infinite, $H$ is an infinite finitely generated nilpotent group. Hence, $\zeta(H)$, the centre of $H$, is infinite [**10,** Theorem 2.24]. Thus, for any $x, y$ in $H$, the subsets $x\zeta(H)$ and $y\zeta(H)$ are infinite. There exist, therefore, $a, b$ in $\zeta(H)$ and integers $n = n(x,y)$, $m = m(x,y)$ such that $[xa, {}_n(yb)^m] = 1$; so $[x, {}_n y^m] = 1$. Now $n \leqslant k$, so $[x, {}_k y^m] = 1$. Since $G$ is a finitely generated metabelian group, it is eremitic [**5,** Theorem B]. We proceed then as in Lemma 1; there is, therefore, a positive integer $d$, depending only on $H$, such that for any $x$, $y$ in $H$, we have $[x, {}_k y^d] = 1$. So $y^d$ is a left $k$-Engel element of $H$. Since $H$ is a finitely generated nilpotent group, then, according to a result of Mal'cev [**8**], the set $\{h^d : h \in H\}$ contains a normal subgroup $K$ of $H$, of finite index in $H$. Since for any $x$, $y$ in $H$ we have $[x^d, {}_k y^d] = 1$ then $K$ is a $k$-Engel group. By a result of Zelmanov [**12**], there is an integer $c = c(k)$, depending only on $k$, such that $K$ is nilpotent of class at most $c(k)$. Hence, $H$, and therefore $G$, is a finite extension of a nilpotent group of class at most $c(k)$ as required. ∎

Proof of Theorem 3: Clearly, every nilpotent-by-(finite of exponent dividing $m$) group satisfies the condition (ii). Now suppose that $G$ is an infinite finitely generated soluble group in the class $\Omega^*$, such that the integer $m$ is the same for any pair of infinite subsets $X$, $Y$ of $G$. We have to show that $G$ is an extension of a nilpotent group by a finite group of exponent dividing $m$. Since $G$ is a finitely generated soluble group, $G/G^m$ is a finite group of exponent dividing $m$ [**11,** 5.4.11]. It suffices, therefore, to show that $G^m$ is nilpotent; and from a result of Robinson and Wehrfritz [**11,** 15.5.3], it suffices to show that any finite homomorphic image of $G^m$ is nilpotent. Let $N$ be a normal subgroup of $G^m$, of finite index. Since $G/G^m$ is finite, $N$ is of finite index in $G$. Hence, there is a $G$-admissible subgroup $M$ of $N$, of finite index in $G$. So, if $T$ is a left transversal of $M$ in $G$, then $T$ is finite; and since $G$ is infinite, $M$ is also infinite. Thus, for any $x$, $y$ in $T$, the subsets $xM$ and $yM$ are infinite. There exist, therefore, $a$, $b$ in $M$ and an integer $n = n(x, y, M)$, such that $[xa, {}_n(yb)^m] = 1$; so $[x, {}_n y^m] \in M$. Since $T$ is finite, it follows that there is a positive integer $n$, depending only on $M$, such that for any $x$, $y$ in $T$, we have $[x, {}_n y^m] \in M$. This means that $G/M$ satisfies the identity $[x, {}_n y^m] = 1$, and from the corollary of [**1**], $(G/M)^m$ is, therefore, nilpotent. Now $(G/M)^m = G^m/M$, so $G^m/M$ is nilpotent. Hence, $G^m/N$, as a homomorphic image of a nilpotent group, is nilpotent. ∎

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