A NOTE ON COMMUTATIVE BAER RINGS

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In this note we study commutative Baer rings, uniting the abstract algebraic approach with the approach of [3] using minimal prime ideals. Some new characterisations of this class of rings are obtained, relations between the minimal prime ideals of a commutative Baer ring B and its algebra E_B of idempotents are considered, and some results concerning the direct decomposition of commutative Baer rings are given. We then study Baer ideals, and finally state without proof a new construction of the Baer extension of a commutative semiprime ring.

It is a pleasure to acknowledge valuable discussions with Michael Evans on many aspects of this topic, and with Don Taylor on the material in section 6.

1

Throughout our notation and terminology will be the same as in [3], apart from some slight changes. Proofs of most of the basic results concerning minimal prime ideals are given in [3] and we will use these without comment.

2

In this section A denotes a commutative ring with identity, and we will give some necessary and sufficient conditions for A to be a commutative Baer ring. $I(A)$ denotes the lattice of all ideals of A.

THEOREM 2.1. *For a commutative semiprime ring A satisfying the following condition*: $C(*)$: *for every $a \in A$ there exists $a' \in A$ such that $(a)^{**} = (a')^*$ the following are pairwise equivalent*:

(i) $(a)^* + (a)^{**} = A$ *for every* $a \in A$.

(ii) $(ab)^* = (a)^* + (b)^*$ *for every a and $b \in A$.*

(iii) $\{(a)^{**} : a \in A\}$ *is a Boolean sublattice of* $I(A)$.

(iv) *For every a and $b \in A$ such that $ab = 0$ there exists $e = e^2$ such that $ae = a$ and $b(1 - e) = b$.*

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PROOF. (i) ⇒ (ii). If $(a)^*$ and $(b)^*$ are direct summands of A then there are idempotents e and f such that $(a)^* = (e)$, $(b)^* = (f)$. From this we have

$$(ab)^* = (ab)^{***} = ((a)^{**} \cap (b)^{**})^* = ((1-e) \cap (1-f))^* = (e+f-ef)$$
$$= (e) + (f) = (a)^* + (b)^*.$$

(ii) ⇒ (iii) Using condition $C(*)$ we see that

$$(a)^{**} + (b)^{**} = (a')^* + (b')^* = (a'b')^* = ((a'b')')^{**};$$

also $(a)^{**} \cap (b)^{**} = (ab)^{**}$ is always valid.

(iii) ⇒ (iv) and (iv) ⇒ (i) are easy to prove and we omit the details.

Our next result shows that under a suitable extra hypothesis, a converse to a result of Kist can be obtained.

THEOREM 2.2. *Let A be a commutative semiprime ring satisfying the following condition $GC(*)$: for every $a \in A$ there exists $\{a_1, \cdots, a_m\} \subseteq A$ such that $(a)^{**} = \bigcap_{i=1}^{m} (a_i)^*$. Then A is a commutative Baer ring iff for every pair of distinct minimal prime ideals M and N of A we have $M + N = A$.*

PROOF. The direct part of the theorem is Theorem 9.5 of [3]. For the converse suppose that A satisfies $GC(*)$ but there exists $a \in A$ with $(a)^* + (a)^{**} \neq A$. Then we can find a (proper) prime ideal P of A with $(a)^* + (a)^{**} \subseteq P \subseteq A$. By our assumption P contains a unique minimal prime ideal and so the localisation at P, the ring A_P, contains a unique minimal prime, i.e. is an integral domain. If $x \mapsto x/1$ is the canonical map from A into A_P then $a/1 \neq 0/1$ since $P \supseteq (a)^*$. Also if $a_i/1 = 0/1$ for every i we would have $a_i s_i = 0$ for some $s_i \notin P$ $(1 \leqq i \leqq m)$ whence $s = s_1 \cdot s_2 \cdot \cdots \cdot s_m \notin P$ satisfies

$$s \in \bigcap_{i=1}^{m} (a_i)^* = (a)^{**} \subseteq P,$$

an impossibility. So $a_j/1 \neq 0/1$ for some j. But now we obtain the final contradiction, for $a/1 \cdot a_j/1 = 0/1$ which is impossible in the integral domain A_I. Hence $(a)^* + (a)^{**} = A$ for every $a \in A$, and the result is proved

REMARK. The extra hypothesis $GC(*)$ is necessary. For if X is a compact F-space which is not basically disconnected, the ring $C(X)$ is not Baer, but distinct minimal primes are comaximal. For details see [2].

3

We now exhibit some connections between the minimal primes of a commutative Baer ring B and the Boolean lattice E_B of idempotents of B. The following lemmas are easily established.

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LEMMA 3.1. *In any commutative Baer ring B, if $ab = 0$ then* $(a + b)^* = a^*b^*$ *and* $(a + b)^{**} = a^{**} + b^{**}$.

LEMMA 3.2. *In any commutative ring A, if $a \in A$ can be written as $a = \sum_{i=1}^m a_i e_i$ where $e_i^2 = e_i$ $(1 \leqq i \leqq m)$ then there are coefficients b_i and orthogonal idempotents f_i $(1 \leqq i \leqq m)$ with $f_i \leqq e_i$ $(1 \leqq i \leqq m)$ and* $a = \sum_{i=1}^m b_i f_i$.

THEOREM 3.3. *Let A be a commutative ring with identity. If M is a prime ideal of A then $M \cap E_A$ is a prime ideal of E_A. Conversely, if P is a prime ideal of E_A, and A is a commutative Baer ring, then the ideal (P) of A generated by P is a minimal prime ideal of A.*

PROOF. The first statement is straightforward. We will show that (P) is a minimal prime of A if P is a prime of E_A. Suppose $ab \in (P)$, i.e. $ab = \sum_{i=1}^m x_i e_i$ for $x_i \in A$, and $e_i \in P$ with $1 \leqq i \leqq m$. Then by 3.2 we also have $ab = \sum_{i=1}^m y_i f_i$ where $y_i \in A$ with $1 \leqq i \leqq m$ and the $\{f_i\}$ is a set of orthogonal idempotents, also in P. By a generalized form of 3.1 we have

$$a^{**}b^{**} = (ab)^{**} = \sum_{i=1}^{m} (y_i f_i)^{**} = \sum_{i=1}^{m} y_i^{**} f_i \in P$$

whence $a^{**} \in P$ or $b^{**} \in P$. This immediately implies that $a \in (P)$ or $b \in (P)$, and so (P) is prime. Finally (P) is minimal prime since $a = \sum_{i=1}^m a_i e_i \in (P)$ for $\{e_i\} \subseteq P$ implies that

$$g = (1 - e_1)(1 - e_2) \cdots (1 - e_m) \notin P$$

and so $g \notin (P)$, and $ag = 0$. The proof is complete.

This result enables us to obtain another characterisation of commutative Baer rings.

THEOREM 3.4. *Let A be a commutative semiprime ring satisfying GC(*) of 2.2. Then A is a commutative Baer ring iff for any minimal prime ideal M of A we have $M = (M \cap E_A)$.*

PROOF. It is straightforward to prove using 3.3 that if A is a commutative Baer ring then $M = (M \cap E_A)$ for M minimal prime. For the converse, suppose M and N are distinct minimal primes of A. By hypothesis we must have $M \cap E_A$ and $N \cap E_A$ distinct prime, and hence maximal, ideals of E_A. Thus $M \cap E_A$ and $N \cap E_A$ together generate E_A and so $M + N = A$. The theorem now follows from 2.2.

4

It is easy to show that (arbitrary) direct products of integral domains are commutative Baer rings. We now consider possible converse results.

LEMMA 4.1. *Let B a commutative Baer ring and e an atom of the Boolean algebra E_B of idempotents. Then $(e)_B$ is an integral domain.*

PROOF. Suppose $ab = 0$ for a and b in $(e)_B$. Then $a^{**}b^{**} = 0$ and, since e is an atom, $a^{**} = 0$ or e and $b^{**} = 0$ or e. Clearly either a^{**} or b^{**} must be 0 whence either a or b is zero.

THEOREM 4.2. *Let B be a commutative Baer ring. Then B is Noetherian iff B is a finite direct product of Noetherian integral domains. If B is finite then it is a finite product of finite fields.*

PROOF. Suppose B is a Noetherian commutative Baer ring. Then B possesses only finitely many minimal prime ideals and so we deduce by 3.4 that E_B is finite, hence atomic. If the atoms of E_B are $e_1, \cdots, e_m$, then it is easy to check that the map $a \to \langle ae_i \rangle_{i=1}^{n}$ for $a \in B$ defines as isomorphism of B onto the product $\times_{i=1}^{m} (e_i)_B$ of domains, each of which is clearly Noetherian. The converse and the final remark are both clear.

Next we ask when a commutative Baer ring B is a product of possibly infinitely many integral domains. To do this we use the partial order $\leqq$ on any commutative semiprime ring A which extends that on E_A, given by $a \leqq b$ if $ab = a^2$. For details and some results which we shall use, see [1]. A subset $S \subseteq A$ is said to be *orthogonal* if $st = 0$ for $s, t \in S$ with $s \neq t$. The ring A is said to be *orthogonally complete* if every orthogonal subset has a join relative to the partial order $\leqq$.

THEOREM 4.3. *Let the ring A be isomorphic to a direct product of integral domains. Then A is a commutative Baer ring satisfying*:

(i) *E_A is an atomic Boolean ring;*

(ii) *A is orthogonally complete.*

Conversely, let A be a commutative Baer ring satisfying (i) *and* (ii) *above. Then A is isomorphic to a direct product of integral domains.*

PROOF. Let $\phi: A \to \times_{i \in I} D_i$ be an isomorphism of A onto a product of domains, such that $a\phi = \langle a_i \rangle_{i \in I}$. Then by results in [1] we see that A is a commutative Baer ring, and the idempotents of A are the inverse images of element $\langle e_i \rangle_{i \in I}$ where $e_i = 0_i$ with $i \in I_1$ and $e_i = 1_i$ with $i \in I \setminus I_1$, for I_1 any subset of I. Thus E_A is seen to be isomorphic to the Boolean ring of subsets of I, which is atomic. The fact that A is orthogonally complete is proved exactly as in [1] p. 506 and so we omit the details.

Now for the converse. Index the atoms of E_A by the set I and for the atom e_i write $D_i = (e_i)_A$, a domain by 4.1. We define a map

$$\psi: A \to \times_{i \in I} (e_i) \text{ by } a\psi = \langle ae_i \rangle_{i \in I}.$$

It is easy to see that the elements ae_i are all zero if $a = a^{**} = 0$, and so ψ is seen to be a ring monomorphism. Suppose $\langle a_i \rangle \in \times_{i \in I}(e_i)$, then the set $\{a_i : i \in I\}$ is orthogonal in A and hence has a supremum a (relative to $\leqq$). Then by a result in [1] we have

$$ae_j = \left(\bigvee_i a_i \right) e_j = \bigvee_i a_i e_j = a_j$$

and so $a\psi = \langle a_i \rangle_{i \in I}$ proving that ψ is onto, and completing our proof.

REMARK. In a first draft of this work an equivalent theorem was proved which did not use the partial order; after seeing [1] the above neater formulation was obtained.

5

Next, we consider Baer-ideals, the kernels of the morphisms of a commutative Baer ring when the latter is considered as an abstract algebra.

THEOREM 5.1. *Let B be a commutative Baer ring. Then the following are pairwise equivalent, for an ideal J of B.*

(i) *J is a Baer-ideal, i.e., $a - b \in J$ implies $a^* - b^* \in J$.*
(ii) *If $a \in J$ and $a^* = b^*$ then $b \in J$.*
(iii) *$a \in J$ iff $a^{**} \in J$.*
(iv) *$J = \cap \{M \in \mathcal{M}_B : M \supseteq J\}$.*

Here $\mathcal{M}_B$ is the set of all minimal prime ideals of B.

PROOF. (*i*) $\Rightarrow$ (*ii*) Suppose J is a Baer-ideal, and $a \in J$ and $b^* = a^*$. Then $a^* - 1 = b^* - 1 \in J$ and so $b^{**} \in J$ whence $b = bb^{**} \in J$, proving (ii).

(*ii*) $\Rightarrow$ (*iii*) If $a \in J$ then $(a^{**})^* = a^*$ implies $a^{**} \in J$. Also $a^{**} \in J$ implies $a = a^{**}a \in J$.

(*iii*) $\Rightarrow$ (*iv*) We note firstly that J is a radical ideal. For if $a^n \in J$ then $(a^n)^{**} = a^{**} \in J$ and so $a = aa^{**} \in J$. This means that J is the intersection of all the prime ideals of B minimal with respect to containing J, and we now show that each such ideal is actually a minimal prime ideal of B. Let P be a minimal prime belonging to J; there exists for every $a \in P$ an element $x \in P$ such that $ax \in J$. Clearly $(ax)^{**} \subseteq J \subseteq P$ and so $(ax)^* \nsubseteq P$ since B is a Baer ring. Thus there is $t \in (ax)^* \setminus P$ i.e. $t \notin P$ such that $axt = 0$. But $xt \notin P$ and hence P is characterised as a minimal prime ideal of B, and J is the intersection of minimal primes as asserted.

(*iv*) $\Rightarrow$ (*i*) We omit the easy proof that any minimal prime ideal, and hence any intersection of minimal primes, is a Baer idael.

COROLLARY 5.2. *For any $K \subset B$, K^* is a Baer-ideal.*

PROOF. $K^* = \cap\{M \in \mathscr{M}_B : M \nsupseteq K\}$ by a result in [3].

THEOREM 5.3 *Let B be a commutative Baer ring. Then the lattice $I_b(B)$ of all Baer-ideals of B is isomorphic to the lattice $I(E_B)$ of all ideals of the Boolean ring E_B.*

PROOF. The map $J \mapsto J \cap E_B$ of $I_b(B)$ into $I(E_B)$ is mono by 5.1 (iii) above. It is certainly onto all prime ideals of E_B by 3.4 and so, using 5.1 (iv) it is onto $I(E_B)$. Since the map is clearly order-preserving the theorem is proved.

COROLLARY 5.4. *The lattice $I_b(B)$ of all Baer-ideals of B is a complete, relatively pseudo-complemented (and hence distributive) lattice. $I_b(B)$ is a Stone lattice if E_B is a complete Boolean algebra.*

PROOF. These facts follow from known results concerning the lattice of ideals of a Boolean algebra and 5.3.

6

To close this note we mention an alternative construction of the Baer extension of a commutative semiprime ring. The main virtue of our approach is that it enables (*i*) a functorial characterisation of the map $A \to B(A)$ and (*ii*) a characterisation of the map $A \to B(A)$ in terms of the dual semilattice of [3], to be given. We do not give any details as the proofs are all very similar to ones appearing in another context [4]. Let us call a ring morphism $\phi: A \to A'$ *R-compatible* if $(a)^* = (b)^*$ in A implies $(a\phi)^* = (b\phi)^*$ in A'. It is easy to see that if A and A' are commutative Baer rings then ϕ is R-compatible if the kernel $\ker \phi$ of ϕ is a Baer-ideal of A.

THEOREM 6.1. *Let A be a commutative semiprime ring. Then there is a commutative Baer ring $B(A)$ and an R-compatible ring monomorphism $\beta: A \to B(A)$ with the following property: for every R-compatible ring morphism $\phi: A \to B$ of A into a commutative Baer ring B, there is a unique Baer morphism $\phi: B(A) \to B$ such that $\beta \circ \phi = \phi$. The pair $(\beta, B(A))$ is unique.*

COROLLARY 6.2. *Let* E *be a commutative Baer ring. Then there is a Baer monomorphism β from B onto a Baer subring of a direct product of integral domains.*

For the next corollary we need some notation. If $\mathscr{M}_A$ is the set of minimal prime ideals of A, the dual semilattice is

$$\mu_A = \{\mathscr{M}_A(a) : a \in A\}$$

where $\mathscr{M}_A(a) = \{M \in \mathscr{M}_A : a \notin M\}$. We write $\bar{\mu}_A$ for the Boolean lattice generated by μ_A. If $\phi: A \to A'$ is R-compatible, we have an induced map

$$\phi^*: \mu_A \to \mu_{A'} \text{ given by } \mathscr{M}_A(a)\phi^* = \mathscr{M}_{A'}(a\phi),$$

and this map extends to $\bar{\mu}_A$.

COROLLARY 6.3. *The pair* $(\beta, B(A))$ *satisfies the following conditions:*

(i) $\beta: A \to B(A)$ *is an R-compatible ring monomorphism of A into a commutative Baer ring;*

(ii) *The induced map* $\beta^*: \bar{\mu}_A \to \mu_{B(A)}$ *is a Boolean isomorphism;*

(iii) *For every* $s \in B(A)$ *there are elements* $a_1, \cdots, a_m$ *of A and idempotents* $e_1, \cdots, e_m$ *of* $B(A)$ *with* $e_ie_j = 0$ $(i \neq j)$ *and* $\sum_{i=1}^m e_i = 1$ *such that* $s = \sum_{i=1}^m (a_i\beta)e_i$.

Thus our Baer extension satisfies conditions given in Kist [3]. We now give our final result.

THEOREM 6.4. *Let* $(k, K(A))$ *be an extension of the commutative semiprime ring A satisfying conditions (i), (ii), (iii) of* 6.3. *Then there are Baer isomorphisms* $\bar{k}: B(A) \to K(A)$ *and* $\bar{\beta}: K(A) \to B(A)$ *such that* $\bar{k} \circ \bar{\beta} = \iota_{B(A)}, \bar{\beta} \circ \bar{k} = \iota_{K(A)}$.

References

[1] A. Abian, 'Direct Product Decomposition of Commutative Semisimple Rings', *Proc. Amer. Math. Soc.* 24 (1970), 502–507.

[2] L. Gillman and M, Jerison, *Rings of Continuous Functions* (Van Nostrand 1960.)

[3] J. Kist, 'Minimal Prime Ideals in Commutative Semigroups', *Proc. Lond. Math. Soc. Ser.*3, 13 (1963), 31–50.

[4] T. P. Speed, 'A Note on Stone Lattices', *Can. Math. Bull.* 14 (1971) 81–86.

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