ISOMORPHISMS OF MULTIPLIER ALGEBRAS

G. I. GAUDRY[1]

1. Introduction. Suppose that G_1 and G_2 are two locally compact Hausdorff groups with identity elements e and e' and with respective left Haar measures dx and dy. Let $1 \leqq p \leqq \infty$, and $L^p(G_i)$ be the usual Lebesgue space over G_i formed relative to left Haar measure on G_i. We denote by $M(G_i)$ the space of Radon measures, and by $M_{\text{bd}}(G_i)$ the space of bounded Radon measures on G_i. If $a \in G_i$, we write ϵ_a for the Dirac measure at the point a. $C_c(G_i)$ will denote the space of continuous, complex-valued functions on G_i with compact supports, whilst $C_c^+(G_i)$ will denote that subset of $C_c(G_i)$ consisting of those functions which are real-valued and non-negative.

Several interesting "isomorphism theorems" have already been proved for various convolution algebras over groups G_i. First, Kawada (**6**) proved that if there exists a bipositive isomorphism of $L^1(G_1)$ onto $L^1(G_2)$, then G_1 and G_2 are isomorphic (as topological groups). If T is an injection of an algebra $A(G_1)$ into an algebra $B(G_2)$, we say that T is bipositive if, when $f \in A(G_1)$, $Tf \geqq 0$ in $B(G_2)$ if and only if $f \geqq 0$ in $A(G_1)$. Wendel (**11**) established isomorphy of the groups from the hypothesis that there exists a norm non-increasing isomorphism of $L^1(G_1)$ onto $L^1(G_2)$. A later result established by Johnson (**5**) and independently by Strichartz (**10**) applies to the case where the algebras $M_{\text{bd}}(G_i)$ $(i = 1, 2)$ are isometrically isomorphic. We shall need this result later, thus we state it as Theorem A.

THEOREM A. *Let G_1 and G_2 be two locally compact Hausdorff groups. The algebras $M_{\text{bd}}(G_i)$ are isometrically isomorphic if and only if G_1 and G_2 are isomorphic topological groups.*

Edwards (**2**) considered the situation where the groups G_i are compact and there exists a bipositive isomorphism of $L^p(G_1)$ onto $L^p(G_2)$ $(1 \leqq p < \infty)$, and showed that under these conditions, the groups are isomorphic. (He also considered in the same paper the cases of bipositive and of isometric isomorphisms of $C_c(G_1)$ onto $C_c(G_2)$ (with the sup-norm topology) in the case where the groups are locally compact.) In (**2**), Edwards asked whether the compact groups G_1 and G_2 are necessarily isomorphic if there exists a norm-preserving isomorphism of $L^p(G_1)$ onto $L^p(G_2)$ $(1 \leqq p < \infty)$. Recently, an affirmative answer to this question was given by Strichartz (**9**), and inde-

Received January 13, 1967.

[1]This work was done while the author held an Overseas Studentship from the Commonwealth Scientific and Industrial Research Organisation (Australia) at the University of Paris.

https://doi.org/10.4153/CJM-1968-110-2 Published online by Cambridge University Press

pendently by Parrott (**8**), in the case where $p \neq 2$. The answer is easily seen to be in the negative when $p = 2$. The crucial step in solving the problem consists in giving the "appropriate" characterization of the norm-preserving right multipliers. (Cf. the corresponding step in Wendel (**11**).) By a right multiplier m of $L^p(G)$ we mean a continuous endomorphism of $L^p(G)$ which commutes with the operations of right translation:

$$m(\rho_a f) = \rho_a m(f) \qquad (a \in G), \tag{1.1}$$

where

$$\rho_a f(x) = f(xa^{-1}). \tag{1.2}$$

The characterization theorem runs as follows.

THEOREM B (Parrott (**8**, Theorem 1) and Strichartz (**9**, Theorem 1)). *If $1 \leqq p < \infty$, $p \neq 2$, the norm-preserving right multipliers of $L^p(G)$ are precisely those operators m of the form*

$$mf = \lambda \tau_a f \qquad (f \in L^p(G)), \tag{1.3}$$

where λ is a scalar, $|\lambda| = 1$, and τ_a is the operator of left translation by amount a $(\tau_a f(x) = f(a^{-1}x))$.

Denoting the set of all right multipliers of $L^p(G)$ by $m_p(G)$ $(1 \leqq p < \infty)$, we observe that $m_p(G)$ is a Banach algebra under the operator norm and the usual operations of addition, multiplication by scalars, and (composition) multiplication of operators. (In the case where G is Abelian, multiplication of multipliers may be regarded as convolution of the corresponding pseudo-measures. (Cf. Gaudry (**3**, Theorem 5.1).) In this paper, we consider the situation where there exists an isomorphism of the algebras $m_p(G_i)$ $(i = 1, 2)$ which is either (i) isometric, or (ii) bipositive, and show that in the first case, we are able to deduce that the groups G_i are isomorphic if $p \neq 2$, and that in the second case, the same conclusion is reached without any restriction on p in the range $1 \leqq p < \infty$.[2]

2. The isometric case.

THEOREM 1. *Suppose that G_1 and G_2 are locally compact Hausdorff groups, that $1 \leqq p < \infty$, $p \neq 2$, and that there exists an isometric isomorphism T of $m_p(G_1)$ onto $m_p(G_2)$. Then G_1 and G_2 are isomorphic topological groups.*

Proof. Case 1: $p = 1$. In this case, $m_1(G_i)$ is isometrically isomorphic to $M_{\text{bd}}(G_i)$ (see Wendel (**11**)); therefore the result follows immediately from Theorem A.

[2]I am indebted to Robert Strichartz for first posing part (i) of the problem, and for some later discussions on it, and for bringing to my attention the paper of Kunze and Stein.

https://doi.org/10.4153/CJM-1968-110-2 Published online by Cambridge University Press

Case 2: $1 < p < \infty$, $p \neq 2$. For each $a \in G_1$, consider the right multiplier τ_a defined as the operator of translation on the left by amount a:

$$\tau_a f(x) = f(a^{-1}x). \tag{2.1}$$

This multiplier is norm-preserving and has norm-preserving inverse $\tau_{a^{-1}}$. Thus $T\tau_a$ has norm one, and so does its inverse. It follows that $T\tau_a$ is in fact norm-preserving. It follows from Theorem B that $T\tau_a$ has the form

$$T\tau_a = \lambda(a)\tau_{a'}, \tag{2.2}$$

where $|\lambda(a)| = 1$ and $a' \in G_2$. It is easy to see from the defining relation (2.1) and the fact that T is an isomorphism that a' is uniquely determined by a and that the mapping ϕ: $a \rightarrow a'$ is an *algebraic* isomorphism of G_1 onto G_2. All that remains to prove is that the mapping ϕ is continuous. (A repetition of the argument in the case where T^{-1} replaces T and ϕ^{-1} replaces ϕ will then yield the result that ϕ^{-1} is continuous.) To show that ϕ is continuous, it will suffice to show that if (a_i) is a net in G_1 and if $a_i \rightarrow e$, then $\phi(a_i) \rightarrow e'$ in G_2.

Suppose the contrary, that $\phi(a_i)$ does not tend to e' in G_2. Then there exists an open neighbourhood V of e' and a subnet of $(\phi(a_i))$ whose elements remain outside of V for all sufficiently large indices. We shall suppose, without loss of generality, that $\phi(a_i) \in \complement V$ (the complement of V) for all i. Consider then the net $(T\tau_{a_i})$ of continuous endomorphisms of $L^p(G_2)$: this net is bounded in norm, and therefore, since $1 < p < \infty$, and the bounded subsets of L^p are weakly relatively compact when $1 < p < \infty$, the net $(T\tau_{a_i})$ has a limiting point for the weak operator topology; suppose that m is any such weak operator topology limiting point and suppose, without loss of generality, that $T\tau_{a_i} \rightarrow m$ in the weak operator topology. It is easy to see that $m \in m_p(G_2)$. Then for any $f \in L^p(G_2)$, $g \in L^{p'}(G_2)$,

$$\int T\tau_{a_i}(f)g\,dy \rightarrow \int m(f)g\,dy. \tag{2.3}$$

Suppose now that $h \in C_c(G_1)$ and that we consider the element m_h of $m_p(G_1)$ defined by convolution on the left by h. It is evident that $\tau_{a_i}h \rightarrow h$ in $M_{\text{bd}}(G_1)$; since the topology of M_{bd} is stronger than that induced by m_p, it follows that $m_{\tau_{a_i}h} \rightarrow m_h$ in $m_p(G_1)$. Since T is continuous, $T(m_{\tau_{a_i}h}) \rightarrow T(m_h)$ in $m_p(G_2)$. But $m_{\tau_{a_i}h} = \tau_{a_i} \cdot m_h$; thus, since T is an isomorphism,

$$(T\tau_{a_i}) \cdot (Tm_h) \rightarrow Tm_h$$

in $m_p(G_2)$, *a fortiori* in the weak operator topology. Hence, if $f, g \in C_c(G_2)$,

$$\int (T\tau_{a_i}) \cdot (Tm_h)(f)g\,dy \rightarrow \int (Tm_h)(f)g\,dy. \tag{2.4}$$

But $(Tm_h)(f) \in L^p(G_2)$; therefore it follows from (2.3) that

$$\int (T\tau_{a_i}) \cdot (Tm_h)(f)g\,dy \rightarrow \int m \cdot (Tm_h)(f)\, g\,dy.$$

It follows, on using (2.4), that

(2.5) $$m\cdot(Tm_h) = Tm_h \qquad (h \in C_c(G_1)).$$

Applying T^{-1} to (2.5), we deduce that

(2.6) $$(T^{-1}m)\cdot m_h = m_h$$

for all $h \in C_c(G_1)$. Therefore $(T^{-1}m - I)(h * k) = 0$ for all $h, k \in C_c(G_1)$, where I is the identity endomorphism of $L^p(G_1)$. But the set of elements $h * k$, where h and k range over $C_c(G_1)$, is dense in $L^p(G_1)$, since $p < \infty$. Hence $T^{-1}m = I$, and therefore $m = I'$, the identity endomorphism of $L^p(G_2)$. Thus we have proved that

(2.7) $$T\tau_{a_i} = \lambda(a_i)\tau_{a_i'} \to \tau_{e'} = I'$$

in the weak operator topology. We now show that this implies that the net (a_i') has at least one limiting point in G_2.

For any compact non-negligible subset K of G_2, denote by χ_K the characteristic function of K. Then $\chi_K \in L^p(G_2) \cap L^{p'}(G_2)$ and by (2.7),

$$\int (\lambda(a_i)\tau_{a_i'}\chi_K - \chi_K)\chi_K\, dy \to 0.$$

It follows that there exists an index i_0 and a compact subset K_0 of G_2 such that $a_i' \in K_0$ for all $i \geqq i_0$. The net (a_i') $(i \geqq i_0)$ therefore has a limiting point in K_0, say a'. Since $a_i' \in \complement V$, and V is open, it follows that $a' \neq e'$. To save renaming, suppose, without loss of generality, that $a_i' \to a'$ in G_2. Since $a' \neq e'$, we may choose a relatively compact neighbourhood U of e' such that $a'U \cap U = \emptyset$. Now choose a symmetric neighbourhood W of e' such that $W^2 \subset U$. Then, since $a_i' \to a'$, it follows that for all sufficiently large indices i, $a_i'W \cap W = \emptyset$. Denote by χ_W the characteristic function of W. By (2.7),

$$\int (\lambda(a_i)\tau_{a_i'}\chi_W - \chi_W)\chi_W\, dy \to 0.$$

But for all sufficiently large i,

$$\int (\lambda(a_i)\tau_{a_i'}\chi_W - \chi_W)\chi_W\, dy = -\int \chi_W\, dy$$

since for such indices, $a_i'W \cap W = \emptyset$. Since W is a neighbourhood of e', $\int \chi_W\, dy \neq 0$. Thus, we have reached a contradiction. The mapping $a \to a'$ is therefore continuous, and the proof is complete.

Remarks. (i) It is easy to see that the result is, in general, false when $p = 2$. As a counterexample, take $G_1 = T$, the circle group, and $G_2 = T \times T$, the two-dimensional torus. Then $m_2(G_1) \cong l^\infty(Z)$, where Z is the additive group of the integers, and $m_2(G_2) \cong l^\infty(Z \times Z)$, the algebras l^∞ being taken here with pointwise operations and the usual sup-norm. Each of Z and $Z \times Z$ is countable: let ϕ be any one-to-one correspondence between $Z \times Z$ and Z. Then the mapping T_ϕ of $l^\infty(Z)$ onto $l^\infty(Z \times Z)$ defined by

$$T_\phi\psi(m, n) = \psi(\phi(m, n))$$

https://doi.org/10.4153/CJM-1968-110-2 Published online by Cambridge University Press

is an isometric isomorphism of $l^\infty(Z)$ onto $l^\infty(Z \times Z)$. It is clear, however, that Z and $Z \times Z$ are not algebraically isomorphic; therefore the groups G_1 and G_2 are certainly not isomorphic.

(ii) We do not know whether the assumption that T is isometric can be replaced by the assumption that it is norm non-increasing.

3. Bipositive isomorphisms. The principal result needed in order to establish our results is a special case of a theorem due to Brainerd and Edwards (**1**, Theorem 3.5).

THEOREM C. *Let G be a locally compact Hausdorff group, let $1 \leqq p < \infty$, and suppose that m is a positive right [left] multiplier of $L^p(G)$ (i.e., $mf \geqq 0$ if $f \geqq 0$). Then there exists a positive measure $\mu \in M(G)$ such that*

$$mf = \mu * f \qquad [f * \mu] \tag{3.1}$$

for all $f \in C_c(G)$.

Remarks. (i) In the case where G is Abelian, it is known that a positive multiplier of $L^p(G)$ is defined by convolution with a positive *bounded* measure. This follows, for example, from the fact that when G is Abelian, a multiplier is defined by convolution with a pseudomeasure, and that a positive pseudomeasure is a bounded measure (Gaudry (**4**, Theorem 1.4.9)).

When G is non-compact and non-Abelian, and $1 < p < \infty$, it is *not*, in general, true that a positive left multiplier of $L^p(G)$ is defined by convolution on the right by a bounded measure. A counterexample is provided by a result contained in a paper by Kunze and Stein (**7**). They considered, in detail, harmonic analysis on the real 2×2 unimodular group G; one of their results is the following very interesting theorem.

THEOREM D (Theorem 9 of (**7**)). *If $f \in L^2(G)$ and $g \in L^p(G)$, $1 \leqq p < 2$, and if $h = f * g$, then $h \in L^2(G)$, and*

$$||h||_2 \leqq A_p \, ||f||_2 \, ||g||_p, \tag{3.2}$$

where A_p does not depend on f or g. Hence the operator of convolution by a function $g \in L^p(G)$, $1 \leqq p < 2$, is a bounded operator on $L^2(G)$.

We need only consider the operator $m: f \to f * g$ on $L^2(G)$, where $g \geqq 0$, $g \in L^p(G)$, $1 < p < 2$, and $g \notin L^1(G)$. Then m is a positive left multiplier of $L^2(G)$ which is not defined by convolution on the right with a positive bounded measure.

(ii) It should also be noted that, *in general*, the positive multipliers of $L^p(G)$ $(1 < p < \infty)$ do not generate all the multipliers. For example, if G is an Abelian group, the positive multipliers "are" the bounded measures; and there are, for most Abelian groups, multipliers of L^p which are not bounded measures.

https://doi.org/10.4153/CJM-1968-110-2 Published online by Cambridge University Press

THEOREM 2. *Let G_1 and G_2 be locally compact Hausdorff groups and* $1 \leqq p < \infty$. *Suppose that there exists a bipositive isomorphism T of $m_p(G_1)$ onto $m_p(G_2)$. Then G_1 and G_2 are isomorphic topological groups.*

Proof. Case 1: $p = 1$. When $p = 1$, $m_p(G_i)$ is isomorphic to $M_{\text{bd}}(G_i)$, the isomorphism being bipositive; therefore T may be regarded as a bipositive isomorphism of $M_{\text{bd}}(G_1)$ onto $M_{\text{bd}}(G_2)$.

If $a \in G_1$, ϵ_a is a positive measure with inverse $\epsilon_{a^{-1}}$. $T\epsilon_a$ and $T\epsilon_{a^{-1}}$ are positive measures, say μ and ν, respectively. We show now that each of μ and ν has one-point support.

Note first that $\mu * \nu = T\epsilon_e = \epsilon_{e'}$, and that each of μ and ν is non-zero. Suppose, for example, that b_1 and b_2 are two distinct points of the support of μ and that c is a point of the support of ν. Choose a relatively compact neighbourhood U of e' such that $b_1UcU \cap b_2UcU = \emptyset$ and a function $\psi \in C_c(G_2)$ with $0 \leqq \psi \leqq 1$, $\psi(e') = 1$ and support $\psi \subset U$. Define $\mu_1 = (\tau_{b_1}\psi)\mu + (\tau_{b_2}\psi)\mu$ and $\nu_1 = (\tau_c\psi)\nu$. Then μ_1 and ν_1 are positive, non-zero measures with compact supports, and $\mu_1 \leqq \mu$, $\nu_1 \leqq \nu$. Therefore

$$\mu_1 * \nu_1 \leqq \mu * \nu = \epsilon_{e'}.$$

But $\mu_1 * \nu_1$ is a positive measure with at least two distinct points in its support, while $\epsilon_{e'}$ has one-point support. We have a contradiction. Hence, each of $T\epsilon_a$ and $T\epsilon_{a^{-1}}$ has one-point support, so that $T\epsilon_a = \lambda(a)\epsilon_{a'}$ and $T\epsilon_{a^{-1}} = \lambda(a^{-1})\epsilon_{b'}$, say, where $a', b' \in G_2$ and $\lambda(a) > 0$, $\lambda(a^{-1}) > 0$. It is easy to see that $b' = (a')^{-1}$ and that λ defines a homomorphism of G_1 into the positive real numbers. We now note that $\lambda(a) = 1$ for all $a \in G_1$. For, if $\lambda(a) > 1$ for some a, then, since λ is a homomorphism, we can find a sequence (a_n) of points of G_1 such that $\lambda(a_n) > n^3$. Then $m = \sum_1^\infty (1/n^2)\epsilon_{a_n}$ is a positive bounded measure with $m \geqq (1/n^2)\epsilon_{a_n}$ for each n; therefore

$$Tm \geqq (1/n^2)T\epsilon_{a_n} \geqq n\epsilon_{a_n'},$$

a contradiction if n is large enough.

The mapping $\phi: a \rightarrow a'$ is an algebraic isomorphism of G_1 onto G_2. It remains to show that it is continuous. First, we note the following simple fact: there exists a constant $c \geqq 0$ such that

$$||T\mu|| \leqq c\,||\mu|| \tag{3.3}$$

for all $\mu \in M_{\text{bd}}(G_1)$ with $\mu \geqq 0$. The proof of this is trivial, and uses much the same ideas as the proof given above that $\lambda = 1$.

Now examine the part of the proof of Theorem 1 applying to the continuity of the map ϕ. This can be adapted immediately to the present situation once the following comments are made.

(i) Since $T\epsilon_a = \epsilon_{a'}$ for $a \in G_1$, the measures $T\epsilon_a$ $(a \in G_1)$ are clearly bounded in norm.

https://doi.org/10.4153/CJM-1968-110-2 Published online by Cambridge University Press

(ii) The norm-bounded subsets of $M_{\mathrm{bd}}(G_i)$ are weakly relatively compact (for the weak topology $\sigma(M_{\mathrm{bd}}, C_0)$) so that when $a_i \to e$, the set of measures $(T\epsilon_{a_i})$ has a weak limiting point in $M_{\mathrm{bd}}(G_2)$, say β.

(iii) If $h \in C_c^+(G_1)$ and $a_i \to e$ in G_1, then $(\epsilon_{a_i} * h) \to h$ in $M_{\mathrm{bd}}(G_1)$ and $(\epsilon_{a_i} * h) \geqq 0$. Therefore by (3.3), $T(\epsilon_{a_i} * h) \to Th$ in $M_{\mathrm{bd}}(G_2)$.

(iv) In place of (2.6) we shall have

$$(T^{-1}\beta) * h = h \tag{3.4}$$

for all $h \in C_c^+(G_1)$. (3.4) then implies that $T^{-1}\beta = \epsilon_e$ so that $\beta = \epsilon_{e'}$.

Thus the proof of the continuity of ϕ is virtually the same as it was before.

Case 2: $1 < p < \infty$. In this case we begin with the multipliers τ_a and $\tau_{a^{-1}}$, where $a \in G_1$ and observe that $T\tau_a$ and $T\tau_{a^{-1}}$ are both positive multipliers. Appeal to Theorem C now yields the existence of positive measures μ and ν such that

$$(T\tau_a)(f) = \mu * f$$

and

$$(T\tau_{a^{-1}})(f) = \nu * f$$

for $f \in C_c(G_2)$. Then it is not difficult to establish, along the lines of the argument used in the proof of Case 1, that μ and ν are both Dirac measures and that $T\tau_a = \tau_{a'}$, $T\tau_{a^{-1}} = \tau_{b'}$, say, where $b' = (a')^{-1}$.

In order to complete the proof by showing that the mapping ϕ: $a \to a'$ is continuous, we note that we may mimic the argument used in Case 1 once we observe that we can establish, in place of (3.3), that

$$||Tm||_{m_p} \leqq c||m||_{m_p} \tag{3.5}$$

for all $m \in m_p(G_1)$ with $m \geqq 0$.

The proof of the theorem is therefore complete.

Note (*added in proof*). R. Rigelhof has recently proved that when $p = 1$, the conclusion of Theorem 1 remains valid if T is assumed to be norm non-increasing.

References

1. B. Brainerd and R. E. Edwards, *Linear operators which commute with translations, Part* I: *Representation theorems*, J. Austral. Math. Soc. *6* (1966), 289–327.
2. R. E. Edwards, *Bipositive and isometric isomorphisms of some convolution algebras*, Can. J. Math. *17* (1965), 839–846.
3. G. I. Gaudry, *Quasimeasures and operators commuting with convolution*, Pacific J. Math. *18* (1966), 461–476.
4. ——— *Quasimeasures and multiplier problems* (Thesis, Australian National University, Canberra, 1966).
5. B. E. Johnson, *Isometric isomorphism of measure algebras*, Proc. Amer. Math. Soc. *15* (1964), 186–188.

6. Y. Kawada, *On the group ring of a topological group*, Math. Japon. *1* (1948), 1–5.
7. R. A. Kunze and E. M. Stein, *Uniformly bounded representations and harmonic analysis of the* 2×2 *real unimodular group*, Amer. J. Math. *82* (1960), 1–62.
8. S. K. Parrott, *Isometric multipliers*, Pacific J. Math. *25* (1968), 159–166.
9. Robert S. Strichartz, *Isomorphisms of group algebras*, Proc. Amer. Math. Soc. *17* (1966), 858–862.
10. ——— *Isometric isomorphisms of measure algebras*, Pacific J. Math. *15* (1965), 315–317.
11. J. G. Wendel, *Left centralizers and isomorphisms of group algebras*, Pacific J. Math. *2* (1952), 251–261.

Institut Henri Poincaré (Université de Paris),
Paris, France;
University of Warwick,
Coventry, England

https://doi.org/10.4153/CJM-1968-110-2 Published online by Cambridge University Press