16

Orthogonal invariance of the CAR on Fock spaces

In this chapter we continue our study of the orthogonal invariance of the CAR. This invariance was already investigated in Chap. 14. However, whereas in Chap. 14 we used the representation-independent framework of CAR algebras, in this chapter we will consider Fock CAR representations in any dimensions. Therefore, to some extent, this chapter can be viewed as a continuation of Chap. 13 about Fock CAR representations.

Note also that this chapter is parallel to Chap. 11 about the symplectic invariance of the CCR on a Fock space.

16.1 Orthogonal group on a Kähler space

The framework of this section, as well as of most other sections of this chapter, is the same as that of Chap. 13 about the Fock representation of the CAR.

In particular, we assume that $(\mathcal{Y}, \nu)$ is a real Hilbert space with a Kähler anti-involution j. If r is a densely defined operator on $L(\mathcal{Y})$, then $r^{\#}$ denotes its adjoint for the scalar product ν. We also use the holomorphic space $\mathcal{Z} := \frac{\mathbb{1}-\mathrm{ij}}{2}\mathbb{C}\mathcal{Y}$ and the identification $\mathbb{C}\mathcal{Y} = \mathcal{Z} \oplus \overline{\mathcal{Z}}$.

In this section we study the orthogonal group and Lie algebra on a real Hilbert space equipped with a Kähler structure.

This section is parallel to Sect. 11.1 about the symplectic group on a Kähler space.

16.1.1 Basic properties

Recall that $O(\mathcal{Y})$ denotes the group of orthogonal transformations on $\mathcal{Y}$. Elements of $O(\mathcal{Y})$ are automatically bounded with a bounded inverse. Clearly, $r \in O(\mathcal{Y})$ iff

$$(a)\ r^{\#} r = \mathbb{1}, \qquad (b)\ r r^{\#} = \mathbb{1}.$$

In the context of real Hilbert spaces we adopt the following definition for the corresponding Lie algebra:

Definition 16.1 *$o(\mathcal{Y})$ denotes the Lie algebra of $a \in B(\mathcal{Y})$ satisfying $a^{\#} + a = 0$, that is, $o(\mathcal{Y}) = B_{\mathrm{a}}(\mathcal{Y})$.*

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Recall that every $r \in B(\mathcal{Y})$ extended to $\mathbb{C}\mathcal{Y} = \mathcal{Z} \oplus \overline{\mathcal{Z}}$ can be written as

$$r_{\mathbb{C}} = \begin{bmatrix} p & q \\ \overline{q} & \overline{p} \end{bmatrix}. \tag{16.1}$$

Proposition 16.2 *$r \in O(\mathcal{Y})$ iff $p \in B(\mathcal{Z})$, $q \in B(\overline{\mathcal{Z}}, \mathcal{Z})$ and the following conditions hold:*

Conditions implied by (a): $\quad p^*p + q^{\#}\overline{q} = \mathbb{1}, \quad p^*q + q^{\#}\overline{p} = 0;$

Conditions implied by (b): $\quad pp^* + qq^* = \mathbb{1}, \quad pq^{\#} + qp^{\#} = 0.$

Proposition 16.3 *$a \in o(\mathcal{Y})$ iff its extension to $\mathbb{C}\mathcal{Y}$ equals*

$$a_{\mathbb{C}} = \mathrm{i}\begin{bmatrix} h & g \\ -\overline{g} & -\overline{h} \end{bmatrix}, \tag{16.2}$$

with $h \in B_{\mathrm{h}}(\mathcal{Z})$, and $g \in B_{\mathrm{a}}(\overline{\mathcal{Z}}, \mathcal{Z})$ (h is self-adjoint and g is anti-symmetric).

16.1.2 j*-non-degenerate orthogonal maps*

The theory of orthogonal operators on a Kähler space is more complicated than that of symplectic operators on a Kähler space. For a symplectic transformation r, the operator p was automatically invertible, which greatly simplified the analysis. The analogous statement is not always true for a general orthogonal operator. Nevertheless, a large class of orthogonal transformations can be analyzed in a way parallel to symplectic transformations. These transformations, which we will call j-non-degenerate, will be studied in this subsection.

Proposition 16.4 *Let $r \in O(\mathcal{Y})$. Then the following conditions are equivalent:*

(1) $\mathrm{Ker}(r\mathrm{j} + \mathrm{j}r) = \{0\}$.
(2) $\mathrm{Ker}(r^{\#}\mathrm{j} + \mathrm{j}r^{\#}) = \{0\}$.
(3) $\mathrm{Ker}\, p = \{0\}$.
(4) $\mathrm{Ker}\, p^* = \{0\}$.

Proof (1)⇔(2), because

$$r^{\#}\mathrm{j} + \mathrm{j}r^{\#} = r^{\#}(\mathrm{j}r + r\mathrm{j})r^{\#}.$$

(1)⇔(3), because

$$r_{\mathbb{C}}\mathrm{j}_{\mathbb{C}} + \mathrm{j}_{\mathbb{C}}r_{\mathbb{C}} = \begin{bmatrix} p & q \\ \overline{q} & \overline{p} \end{bmatrix}\begin{bmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{bmatrix} + \begin{bmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{bmatrix}\begin{bmatrix} p & q \\ \overline{q} & \overline{p} \end{bmatrix} = 2\mathrm{i}\begin{bmatrix} p & 0 \\ 0 & -\overline{p} \end{bmatrix}.$$

Similarly we see that (2)⇔(4). □

Definition 16.5 *$r \in O(\mathcal{Y})$ is said to be* j-non-degenerate *if the equivalent conditions of Prop. 16.4 are satisfied.*

Recall that if h is a possibly unbounded operator with $\operatorname{Ker} h = \{0\}$, then we can define h^{-1} with domain $\operatorname{Dom} h^{-1} := \operatorname{Ran} h$. The operator h^{-1} is closed iff h is.

Recall also that $Cl_{\mathrm{a}}(\overline{\mathcal{Z}}, \mathcal{Z})$ denotes the set of closed densely defined operators c from $\overline{\mathcal{Z}}$ to $\mathcal{Z}$ satisfying $c^{\#} = -c$.

Let us now describe a convenient factorization of a j-non-degenerate orthogonal map. Note that if r is j-non-degenerate, then $(\operatorname{Ran} p)^{\mathrm{cl}} = (\operatorname{Ker} p^*)^{\perp} = \mathcal{Z}$. Therefore, the following operators are densely defined:

$$d := q\overline{p}^{-1}, \qquad \operatorname{Dom} d := \operatorname{Ran} \overline{p}; \tag{16.3}$$

$$c := -q^{\#}(p^{\#})^{-1}, \quad \operatorname{Dom} c := \operatorname{Ran} p^{\#}. \tag{16.4}$$

Proposition 16.6 (1) *c and d are closable. Let us denote their closures by the same symbols. Then $c, d \in Cl_{\mathrm{a}}(\overline{\mathcal{Z}}, \mathcal{Z})$.*

(2) *We have the following equivalent characterizations of c, d:*

$$d = -p^{*-1}q^{\#}, \quad \operatorname{Dom} d = \{\overline{z} \in \overline{\mathcal{Z}} \ : \ q^{\#}\overline{z} \in \operatorname{Ran} p^*\}; \tag{16.5}$$

$$c = p^{-1}q, \qquad \operatorname{Dom} c = \{\overline{z} \in \overline{\mathcal{Z}} \ : \ q\overline{z} \in \operatorname{Ran} p\}. \tag{16.6}$$

(3) *We have the following factorization, which holds as an operator identity:*

$$r_{\mathbb{C}} = \begin{bmatrix} 1\!\!1 & d \\ 0 & 1\!\!1 \end{bmatrix} \begin{bmatrix} (p^*)^{-1} & 0 \\ 0 & \overline{p} \end{bmatrix} \begin{bmatrix} 1\!\!1 & 0 \\ \overline{c} & 1\!\!1 \end{bmatrix}. \tag{16.7}$$

(4) *The following operator identities are true:*

$$\begin{aligned} (r_{\mathbb{C}}\mathrm{j}_{\mathbb{C}}r_{\mathbb{C}}^* - \mathrm{j}_{\mathbb{C}})(r_{\mathbb{C}}\mathrm{j}_{\mathbb{C}}r_{\mathbb{C}}^* + \mathrm{j}_{\mathbb{C}})^{-1} &= \begin{bmatrix} 0 & d \\ \overline{d} & 0 \end{bmatrix}, \\ (\mathrm{j}_{\mathbb{C}} - r_{\mathbb{C}}^*\mathrm{j}_{\mathbb{C}}r_{\mathbb{C}})(r_{\mathbb{C}}^*\mathrm{j}_{\mathbb{C}}r_{\mathbb{C}} + \mathrm{j}_{\mathbb{C}})^{-1} &= \begin{bmatrix} 0 & c \\ \overline{c} & 0 \end{bmatrix}. \end{aligned} \tag{16.8}$$

(Note that if r is j-non-degenerate, then $r\mathrm{j}r^* + \mathrm{j}$ and $r^*\mathrm{j}r + \mathrm{j}$ are injective with a dense range. Hence, in the identities (16.8) the meaning of the l.h.s. is described in (2.2) and (2.3).)

(5) *The following quadratic form identities are true:*

$$1\!\!1 + c^{\#}\overline{c} = p^{*-1}p^{-1}, \quad 1\!\!1 + d^*d = \overline{p}^{*-1}\overline{p}^{-1}.$$

Proof Consider $d = q\overline{p}^{-1}$. We have the identity

$$q^{\#}\overline{p} = -p^*q. \tag{16.9}$$

Therefore, $\operatorname{Ran} \overline{p}$ is contained in

$$\{z \in \mathcal{Z} \ : \ q^{\#}z \in \operatorname{Dom} p^{*-1} = \operatorname{Ran} p^*\}. \tag{16.10}$$

But $\operatorname{Ran} \overline{p}$ is dense. Thus (16.10) is dense. By Prop. 2.35 applied to the bounded operator q and the closed operator $\overline{p}^{-1}$,

$$(q\overline{p}^{-1})^{\#} = p^{*-1}q^{\#}. \tag{16.11}$$

But the identity (16.9) implies

$$p^{*-1}q^{\#}\,\overline{p} = -q,$$

and hence, on $\mathrm{Ran}\,\overline{p}$,

$$p^{*-1}q^{\#} = -q\overline{p}^{-1} = (q\overline{p}^{-1})^{\#},$$

by (16.11). Therefore, $d \subset -d^{\#}$, and hence d are closable. This easily implies (1) and (2).

We have

$$r_{\mathbb{C}}\mathrm{j}_{\mathbb{C}} - \mathrm{j}_{\mathbb{C}}r_{\mathbb{C}} = 2\mathrm{i}\begin{bmatrix} 0 & -q \\ -\overline{q} & 0 \end{bmatrix}, \; r_{\mathbb{C}}\mathrm{j}_{\mathbb{C}} + \mathrm{j}_{\mathbb{C}}r_{\mathbb{C}} = 2\mathrm{i}\begin{bmatrix} p & 0 \\ 0 & -\overline{p} \end{bmatrix}.$$

Hence,

$$\begin{bmatrix} 0 & d \\ \overline{d} & 0 \end{bmatrix} = (r_{\mathbb{C}}\mathrm{j}_{\mathbb{C}} - \mathrm{j}_{\mathbb{C}}r_{\mathbb{C}})(r_{\mathbb{C}}\mathrm{j}_{\mathbb{C}} + \mathrm{j}_{\mathbb{C}}r_{\mathbb{C}})^{-1} = (r_{\mathbb{C}}\mathrm{j}_{\mathbb{C}}r_{\mathbb{C}}^{*} - \mathrm{j}_{\mathbb{C}})(r_{\mathbb{C}}\mathrm{j}_{\mathbb{C}}r_{\mathbb{C}}^{*} + \mathrm{j}_{\mathbb{C}})^{-1}.$$

This proves the first identity of (4). □

Next we give a criterion for the j-non-degeneracy.

Lemma 16.7 *Assume that* $\|r - \mathbb{1}\| < 1$. *Then* r *is* j*-non-degenerate.*

Proof Let $y \in \mathcal{Y}$ such that $y \neq 0$ and $(r\mathrm{j} + \mathrm{j}r)y = 0$. Then,

$$2\mathrm{j}y = (\mathbb{1} - r)\mathrm{j}y + \mathrm{j}(\mathbb{1} - r)y.$$

Hence,

$$2\|y\| \leq 2\|\mathbb{1} - r\|\|y\|.$$

Therefore, $1 \leq \|\mathbb{1} - r\|$. □

16.1.3 j-*self-adjoint maps*

To some extent, this subsection can be viewed as parallel to Subsect. 11.1.4 about positive symplectic transformations.

Definition 16.8 *An operator* $r \in Cl(\mathcal{Y})$ *satisfying* $\mathrm{j}r = r^{\#}\mathrm{j}$ *is called* j-self-adjoint. *We say that it is* j-positive *if, in addition,* $\mathrm{j}r\mathrm{j}^{-1} + r^{\#} \geq 0$.

If the extension of r to $\mathbb{C}\mathcal{Y}$ is given by (16.1), then $r \in B(\mathcal{Y})$ is j-self-adjoint iff $q^{\#} = -q$, $p = p^{*}$. It is j-positive iff in addition $p \geq 0$.

Let $r \in B(\mathcal{Y})$ be j-self-adjoint. It belongs to $O(\mathcal{Y})$ iff

$$p^2 - q\overline{q} = \mathbb{1}, \quad pq - q\overline{p} = 0.$$

We now examine the form of the decomposition (16.7). Let $r \in O(\mathcal{Y})$ be j-non-degenerate. It is j-self-adjoint iff $c = d$, where $c, d \in Cl_{\mathrm{a}}(\overline{\mathcal{Z}}, \mathcal{Z})$ were defined in (16.3) and (16.4). j-non-degenerate j-positive elements of $O(\mathcal{Y})$ can be fully characterized by c:

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Proposition 16.9 *Let $r \in O(\mathcal{Y})$ be* j*-non-degenerate and* j*-positive. Let $c \in Cl_{\mathrm{a}}(\overline{\mathcal{Z}}, \mathcal{Z})$ be defined as in (16.4). Then one has*

$$r_{\mathbb{C}} = \begin{bmatrix} (\mathbb{1}+cc^*)^{-\frac{1}{2}} & (\mathbb{1}+cc^*)^{-\frac{1}{2}}c \\ -c^*(\mathbb{1}+cc^*)^{-\frac{1}{2}} & (\mathbb{1}+c^*c)^{-\frac{1}{2}} \end{bmatrix} \tag{16.12}$$
$$= \begin{bmatrix} \mathbb{1} & c \\ 0 & \mathbb{1} \end{bmatrix} \begin{bmatrix} (\mathbb{1}+cc^*)^{\frac{1}{2}} & 0 \\ 0 & (\mathbb{1}+c^*c)^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \mathbb{1} & 0 \\ -c^* & \mathbb{1} \end{bmatrix},$$

$$(r_{\mathbb{C}}^2 - \mathbb{1}_{\mathbb{C}})(r_{\mathbb{C}}^2 + \mathbb{1}_{\mathbb{C}})^{-1} = \begin{bmatrix} 0 & c \\ \overline{c} & 0 \end{bmatrix}. \tag{16.13}$$

Conversely let $c \in Cl_{\mathrm{a}}(\overline{\mathcal{Z}}, \mathcal{Z})$. Then r, defined by (16.12), belongs to $O(\mathcal{Y})$, is j*-non-degenerate and is* j*-positive.*

Proof Let $r \in O(\mathcal{Y})$ be j-non-degenerate and j-self-adjoint. We have $d = c = q\overline{p}^{-1}$ and $\mathbb{1} + c^{\#}\overline{c} = p^{-2}$. Using the positivity of p we obtain

$$p = (\mathbb{1} + c^{\#}\overline{c})^{-\frac{1}{2}}.$$

Now

$$q = c\overline{p} = c(\mathbb{1} + c^*c)^{-\frac{1}{2}} = (\mathbb{1} + cc^*)^{-\frac{1}{2}}c.$$

We then apply the decomposition (16.7) and formula (16.1) to get the first statement of the proposition. □

Proposition 16.10 *Let $a \in Cl(\mathcal{Y})$ be anti-self-adjoint and* j*-self-adjoint. Then there exists $g \in Cl_{\mathrm{a}}(\overline{\mathcal{Z}}, \mathcal{Z})$ such that*

$$a_{\mathbb{C}} = \mathrm{i}\begin{bmatrix} 0 & g \\ g^* & 0 \end{bmatrix}.$$

Moreover, e^a *belongs to $O(\mathcal{Y})$, is* j*-self-adjoint and*

$$\mathrm{e}^{a_{\mathbb{C}}} = \begin{bmatrix} \cos\sqrt{gg^*} & \mathrm{i}\frac{\sin\sqrt{gg^*}}{\sqrt{gg^*}}g \\ \mathrm{i}g^*\frac{\sin\sqrt{gg^*}}{\sqrt{gg^*}} & \cos\sqrt{g^*g} \end{bmatrix}, \tag{16.14}$$

$$c = \mathrm{i}\frac{\tan\sqrt{gg^*}}{\sqrt{gg^*}}g.$$

We have a complete description of j-non-degenerate j-self-adjoint elements of $O(\mathcal{Y})$:

Theorem 16.11 *Let $r \in O(\mathcal{Y})$ be* j*-non-degenerate and* j*-self-adjoint. Then $r = mr_0m^*$, where*

$$m_{\mathbb{C}} := \frac{1}{\sqrt{2}}\begin{bmatrix} \mathbb{1} & \mathrm{i}(\mathbb{1}-p^2)^{-\frac{1}{2}}q \\ -\mathrm{i}(\mathbb{1}-\overline{p}^2)^{-\frac{1}{2}}\overline{q} & \mathbb{1} \end{bmatrix},$$
$$r_{0\mathbb{C}} := \begin{bmatrix} p + \mathrm{i}(\mathbb{1}-p^2)^{\frac{1}{2}} & 0 \\ 0 & \overline{p} - \mathrm{i}(\mathbb{1}-\overline{p}^2)^{\frac{1}{2}} \end{bmatrix}.$$

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The transformation $m_{\mathbb{C}}$ *is unitary, hence* $r_{\mathbb{C}}$ *is unitarily equivalent to the diagonal operator* $r_{0\mathbb{C}}$*. Consequently*

$$\operatorname{spec} r = \operatorname{spec}\,(p + \mathrm{i}(\mathbb{1} - p^2)^{\frac12}) \cup \overline{\operatorname{spec}\,(p + \mathrm{i}(\mathbb{1} - p^2)^{\frac12})}.$$

In particular, r *is* j*-positive iff*

$$\operatorname{spec} r \subset \left\{ \mathrm{e}^{\mathrm{i}\phi} \;:\; \phi \in [-\pi/2, \pi/2] \right\}. \tag{16.15}$$

Proposition 16.12 *Let* $k \in O(\mathcal{Y})$ *be* j*-self-adjoint and such that* $\operatorname{Ker}(k + \mathbb{1}) = \{0\}$*. Let* $k^t \in O(\mathcal{Y})$ *be defined as in Subsect. 2.3.2 for* $t \in \mathbb{R}$*. Then* k^t *is also* j*-self-adjoint and* $(k^t)^{\#} = k^{-t}$*. Moreover, if* $|t| \le \frac12$*, then* k^t *is* j*-non-degenerate and* j*-positive.*

Proof We work on $\mathbb{C}\mathcal{Y}$ equipped with its unitary structure and consider $k_{\mathbb{C}}$. Clearly, $k_{\mathbb{C}}\mathrm{j}_{\mathbb{C}} = \mathrm{j}_{\mathbb{C}}k_{\mathbb{C}}^*$, so the identity

$$F(k_{\mathbb{C}})\mathrm{j}_{\mathbb{C}} = \mathrm{j}_{\mathbb{C}}F(k_{\mathbb{C}}^*) \tag{16.16}$$

holds for polynomials and extends by the usual argument to bounded Borel functions on $\operatorname{spec} k_{\mathbb{C}}$. Taking $F(z) = z^t$, we obtain by restriction to $\mathcal{Y}$ that $k^t\mathrm{j} = \mathrm{j}(k^t)^*$, so that k^t is j-self-adjoint.

Clearly, k^t is j-non-degenerate, and also j-positive for $|t| \le \frac12$, by criterion (16.15). □

16.1.4 j*-polar decomposition*

The following theorem gives a canonical decomposition of every j-non-degenerate orthogonal operator into a product of a unitary operator and a j-positive j-non-degenerate operator. This can be treated as a fermionic analog of the polar decomposition of symplectic transformations discussed in Subsect. 11.1.5.

Theorem 16.13 *Let* $r \in O(\mathcal{Y})$ *be* j*-non-degenerate. Set* $k := -\mathrm{j}r^{\#}\mathrm{j}r$*. Then*

(1) $k \in O(\mathcal{Y})$ *is* j*-self-adjoint;*
(2) $\operatorname{Ker}(k + \mathbb{1}) = \{0\}$*;*
(3) $k^{\frac12}$ *is* j*-positive and* j*-non-degenerate;*
(4) *For* $w := rk^{-\frac12} \in U(\mathcal{Y}^{\mathbb{C}})$ *we have*

$$r = wk^{\frac12}; \tag{16.17}$$

(5) *If in addition* r *is* j*-self-adjoint, then* $w = w^*$*,* $w^2 = \mathbb{1}$ *and* $r = k^{\frac12}w = wk^{\frac12}$*.*

Proof (1) follows from

$$\mathrm{j}k = r^{\#}\mathrm{j}r = k^{\#}\mathrm{j}.$$

(2) is a consequence of

$$\operatorname{Ker}(k + \mathbb{1}) = -\mathrm{j}r^{\#}\operatorname{Ker}(r\mathrm{j} + \mathrm{j}r) = \{0\}.$$

(3) follows from Prop. 16.12.

Let us prove (4). Clearly, $w \in O(\mathcal{Y})$. Moreover,

$$\begin{aligned} \mathrm{j}w &= \mathrm{j}rk^{-\frac{1}{2}} = \mathrm{j}rk^{-1}k^{\frac{1}{2}} \\ &= r\mathrm{j}k^{\frac{1}{2}} = rk^{-\frac{1}{2}}\mathrm{j} = w\mathrm{j}. \end{aligned}$$

So $w \in U(\mathcal{Y}^{\mathbb{C}})$. □

Definition 16.14 *We call (16.17) the* j-polar decomposition of r.

16.1.5 Conjugations on Kähler spaces

Conjugations on a unitary space were defined in Subsect. 1.2.10. We recall that they are anti-unitary involutions.

Conjugations on a Kähler space were defined in Subsect. 1.3.10. We recall that κ is a conjugation of the Kähler space $\mathcal{Y}$ if $\kappa \in O(\mathcal{Y})$, $\kappa^2 = \mathbb{1}$ and $\kappa\mathrm{j} = -\mathrm{j}\kappa$. Note that κ is self-adjoint, as well as anti-symplectic and infinitesimally symplectic.

Clearly, κ is a conjugation on a Kähler space $\mathcal{Y}$ iff it is a conjugation on the corresponding unitary space $\mathcal{Y}^{\mathbb{C}}$. It can be written as

$$\kappa_{\mathbb{C}} = \begin{bmatrix} 0 & t \\ \bar{t} & 0 \end{bmatrix},$$

where $t \in L(\overline{\mathcal{Z}}, \mathcal{Z})$, $t\bar{t} = \mathbb{1}_{\mathcal{Z}}$ and $t^{\#} = t$. If we set $uz := t\bar{z}$, then u is a conjugation of the Hilbert space $\mathcal{Z}$, which means an anti-unitary operator satisfying $u^2 = \mathbb{1}$.

Conversely, any conjugation on $\mathcal{Z}$ determines a conjugation on $\mathcal{Y}$.

Note also that if j is a Kähler anti-involution, then so is $-$j. If $\kappa \in O(\mathcal{Y})$ is a conjugation, then we have $\kappa\mathrm{j}\kappa^{\#} = -\mathrm{j}$.

16.1.6 Partial conjugations on Kähler spaces

Definition 16.15 *If $\mathcal{W}$ is a unitary space, we will say that $\kappa \in L(\mathcal{W}_{\mathbb{R}})$ is a* partial conjugation *if there exists a decomposition of $\mathcal{W}$ into an orthogonal direct sum of (complex) subspaces $\mathcal{W} = \mathcal{W}_{\mathrm{reg}} \oplus \mathcal{W}_{\mathrm{sg}}$ such that κ preserves this decomposition, is the identity on $\mathcal{W}_{\mathrm{reg}}$ and is a conjugation on $\mathcal{W}_{\mathrm{sg}}$.*

Definition 16.16 *If $(\mathcal{Y}, \nu, \mathrm{j})$ is a Kähler space, we say that $\kappa \in L(\mathcal{Y})$ is a* partial conjugation *if there exists an orthogonal decomposition $\mathcal{Y} = \mathcal{Y}_{\mathrm{reg}} \oplus \mathcal{Y}_{\mathrm{sg}}$ such that κ and* j *preserve this decomposition, κ is the identity on $\mathcal{Y}_{\mathrm{reg}}$ and a conjugation on $\mathcal{Y}_{\mathrm{sg}}$.*

Clearly, κ is a partial conjugation on a Kähler space $\mathcal{Y}$ iff it is a partial conjugation of the unitary space $\mathcal{Y}^{\mathbb{C}}$.

Let κ be a partial conjugation of $\mathcal{Y}$. If $\mathbb{1}_{\mathrm{reg}}$ and $\mathbb{1}_{\mathrm{sg}}$ are the orthogonal projections onto $\mathcal{Y}_{\mathrm{reg}}$ and $\mathcal{Y}_{\mathrm{sg}}$, then

$$\kappa\mathrm{j}\kappa^{\#} = \mathrm{j}\mathbb{1}_{\mathrm{reg}} - \mathrm{j}\mathbb{1}_{\mathrm{sg}}.$$

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Writing $\mathcal{Z} = \mathcal{Z}_{\mathrm{reg}} \oplus \mathcal{Z}_{\mathrm{sg}}$, we have

$$\kappa_{\mathbb{C}} = \begin{bmatrix} \mathbb{1} & 0 & 0 & 0 \\ 0 & \mathbb{1} & 0 & 0 \\ 0 & 0 & 0 & t \\ 0 & 0 & \overline{t} & 0 \end{bmatrix} \tag{16.18}$$

for $t \in L(\overline{\mathcal{Z}}_{\mathrm{sg}}, \mathcal{Z}_{\mathrm{sg}})$, $\overline{t}t = \mathbb{1}_{\overline{\mathcal{Z}}_{\mathrm{sg}}}$, $t^{\#} = t$.

16.1.7 Decomposition of orthogonal operators

Definition 16.17 *Let $r \in O(\mathcal{Y})$. We define the* regular *and* singular initial *and* final subspaces for r *by*

$$\begin{aligned} \mathcal{Y}_{-\mathrm{sg}} &:= \mathrm{Ker}(r\mathrm{j} + \mathrm{j}r), & \mathcal{Y}_{-\mathrm{reg}} &:= \mathcal{Y}_{-\mathrm{sg}}^{\perp}, \\ \mathcal{Y}_{+\mathrm{sg}} &:= \mathrm{Ker}(r^{\#}\mathrm{j} + \mathrm{j}r^{\#}), & \mathcal{Y}_{+\mathrm{reg}} &:= \mathcal{Y}_{+\mathrm{sg}}^{\perp}. \end{aligned}$$

We also introduce the corresponding holomorphic subspaces

$$\mathcal{Z}_{\pm\mathrm{sg}} := \mathbb{C}\mathcal{Y}_{\pm\mathrm{sg}} \cap \mathcal{Z}, \quad \mathcal{Z}_{\pm\mathrm{reg}} := \mathbb{C}\mathcal{Y}_{\pm\mathrm{reg}} \cap \mathcal{Z}.$$

We easily check that r maps $\mathcal{Y}_{-\mathrm{sg}}$ onto $\mathcal{Y}_{+\mathrm{sg}}$ and $\mathcal{Y}_{-\mathrm{reg}}$ onto $\mathcal{Y}_{+\mathrm{reg}}$. j preserves $\mathcal{Y}_{\pm\mathrm{sg}}$ and $\mathcal{Y}_{\pm\mathrm{reg}}$, and hence we have the decompositions

$$\mathbb{C}\mathcal{Y} = \mathcal{Z}_{-\mathrm{reg}} \oplus \overline{\mathcal{Z}}_{-\mathrm{reg}} \oplus \mathcal{Z}_{-\mathrm{sg}} \oplus \overline{\mathcal{Z}}_{-\mathrm{sg}}, \tag{16.19}$$

$$\mathbb{C}\mathcal{Y} = \mathcal{Z}_{+\mathrm{reg}} \oplus \overline{\mathcal{Z}}_{+\mathrm{reg}} \oplus \mathcal{Z}_{+\mathrm{sg}} \oplus \overline{\mathcal{Z}}_{+\mathrm{sg}}. \tag{16.20}$$

Note that $\mathrm{Ker}\, p = \mathcal{Z}_{-\mathrm{sg}}$ and $\mathrm{Ker}\, p^{*} = \mathcal{Z}_{+\mathrm{sg}}$. We can write $r_{\mathbb{C}}$ as a matrix from (16.19) to (16.20) as follows:

$$r_{\mathbb{C}} = \begin{bmatrix} p_{\mathrm{reg}} & q_{\mathrm{reg}} & 0 & 0 \\ \overline{q}_{\mathrm{reg}} & \overline{p}_{\mathrm{reg}} & 0 & 0 \\ 0 & 0 & 0 & q_{\mathrm{sg}} \\ 0 & 0 & \overline{q}_{\mathrm{sg}} & 0 \end{bmatrix}.$$

Clearly,

$$\mathrm{Ker}\, p_{\mathrm{reg}}^{*} = \mathrm{Ker}\, p_{\mathrm{reg}} = \{0\}, \quad q_{\mathrm{sg}}^{*} q_{\mathrm{sg}} = \mathbb{1}_{\overline{\mathcal{Z}}_{-\mathrm{sg}}}. \tag{16.21}$$

Proposition 16.18 *Let $r \in O(\mathcal{Y})$. Then there exists a decomposition $r = \kappa r_0$ such that $r_0 \in O(\mathcal{Y})$ is* j*-non-degenerate and κ is a partial conjugation.*

Proof Let κ be any partial conjugation such that $\kappa \mathrm{j} \kappa^{\#} = \mathrm{j}\mathbb{1}_{-\mathrm{reg}} - \mathrm{j}\mathbb{1}_{-\mathrm{sg}}$, so that in the matrix notation using (16.20)

$$\kappa_{\mathbb{C}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & t \\ 0 & 0 & \overline{t} & 0 \end{bmatrix}.$$

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We can set $r_0 = \kappa r$, which as a matrix from (16.19) to (16.20) has the form

$$r_{0\mathbb{C}} = \begin{bmatrix} p_{\text{reg}} & q_{\text{reg}} & 0 & 0 \\ \overline{q}_{\text{reg}} & \overline{p}_{\text{reg}} & 0 & 0 \\ 0 & 0 & t\overline{q}_{\text{sg}} & 0 \\ 0 & 0 & 0 & \overline{t}q_{\text{sg}} \end{bmatrix}. \tag{16.22}$$

Clearly, $r_0 \in O(\mathcal{Y})$, since $r, \kappa \in O(\mathcal{Y})$. (16.21) implies that r_0 is j-non-degenerate. □

Proposition 16.19 *Let $r \in O(\mathcal{Y})$ be* j*-self-adjoint. Then $\mathcal{Y}_{+\text{reg}} = \mathcal{Y}_{-\text{reg}} =: \mathcal{Y}_{\text{reg}}$ and $\mathcal{Y}_{+\text{sg}} = \mathcal{Y}_{-\text{sg}} =: \mathcal{Y}_{\text{sg}}$. We have the orthogonal decomposition $\mathcal{Y} = \mathcal{Y}_{\text{reg}} \oplus \mathcal{Y}_{\text{sg}}$ preserved by* j *and r. Let $\mathrm{j} = \mathrm{j}_{\text{reg}} \oplus \mathrm{j}_{\text{sg}}$ and $r = r_{\text{reg}} \oplus r_{\text{sg}}$. Then r_{reg} is j_{reg}-non-degenerate, and j_{reg}-self-adjoint on $\mathcal{Y}_{\text{reg}}$ and $r_{\text{sg}}\mathrm{j}_{\text{sg}}$ is a conjugation on $\mathcal{Y}_{\text{sg}}$.*

16.1.8 Restricted orthogonal group

The following subsection is parallel to Subsect. 11.1.6 about the restricted symplectic group. Recall that $B^2(\mathcal{Y})$ denotes the set of Hilbert–Schmidt operators on $\mathcal{Y}$.

Proposition 16.20 *Let $r \in O(\mathcal{Y})$. Let $p, q, Z_{\pm\text{sg}}, \mathcal{Z}_{\pm\text{reg}}$ be defined as above. The following conditions are equivalent:*

(1) $\mathrm{j} - r^{-1}\mathrm{j}r \in B^2(\mathcal{Y})$, (2) $r\mathrm{j} - \mathrm{j}r \in B^2(\mathcal{Y})$.

(3) $\mathrm{Tr}(q^*q) < \infty$, (4) $\mathrm{Tr}(p^*p - \mathbb{1}) < \infty$, (5) $\mathrm{Tr}(pp^* - \mathbb{1}) < \infty$.

(6) $\dim \mathcal{Z}_{+\text{sg}} < \infty$ *and* $d \in B^2(\overline{\mathcal{Z}}_{+\text{reg}}, \mathcal{Z}_{+\text{reg}})$.

(7) $\dim \mathcal{Z}_{-\text{sg}} < \infty$ *and* $c \in B^2(\overline{\mathcal{Z}}_{-\text{reg}}, \mathcal{Z}_{-\text{reg}})$.

If the above conditions are true, then $\dim \mathcal{Y}_{-\text{sg}} = \dim \mathcal{Y}_{+\text{sg}} < \infty$.

Proof The proof of the equivalence of the first five conditions is identical to the proof in Prop. 11.12. Assume now that condition (3) (and hence (4), (5)) holds. Since $\mathcal{Z}_{-\text{sg}} = \mathrm{Ker}\, p$, $\mathcal{Z}_{+\text{sg}} = \mathrm{Ker}\, p^*$, these spaces are finite-dimensional, and $p : \mathcal{Z}_{-\text{reg}} \to \mathcal{Z}_{+\text{reg}}$, $p^* : \mathcal{Z}_{+\text{reg}} \to \mathcal{Z}_{-\text{reg}}$ are invertible with bounded inverses. It follows then from (3) that $d = q\overline{p}^{-1} \in B^2(\overline{\mathcal{Z}}_{+\text{reg}}, \mathcal{Z}_{+\text{reg}})$ and $c = q^{\#}(p^{\#})^{-1} \in B^2(\overline{\mathcal{Z}}_{-\text{reg}}, \mathcal{Z}_{-\text{reg}})$, so (3) ⇒ (6), (7). To prove that (6), (7) ⇒ (3), we argue similarly, using the identities $\mathbb{1} + c^{\#}\overline{c} = (pp^*)^{-1}$, $\mathbb{1} + d^*d = (\overline{pp^*})^{-1}$. □

Definition 16.21 *Let $O_{\mathrm{j}}(\mathcal{Y})$ be the set of $r \in O(\mathcal{Y})$ satisfying the conditions of Prop. 16.20. $O_{\mathrm{j}}(\mathcal{Y})$ is called the* restricted orthogonal group *and is equipped with the metric*

$$d_{\mathrm{j}}(r_1, r_2) := \|p_1 - p_2\| + \|q_1 - q_2\|_2.$$

Equivalent metrics are $\|[\mathrm{j}, r_1 - r_2]_+\| + \|[\mathrm{j}, r_1 - r_2]\|_2$ *and* $\|r_1 - r_2\| + \|[\mathrm{j}, r_1 - r_2]\|_2$.

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Noting that since $\mathcal{Y}_{-\mathrm{sg}}$ *is* j*-invariant its dimension as a real vector space is even, we define*

$$SO_{\mathrm{j}}(\mathcal{Y}) := \Big\{ r \in O_{\mathrm{j}}(\mathcal{Y}) \ : \ \frac{1}{2}\dim \mathcal{Y}_{-\mathrm{sg}} \ \textit{is even} \Big\}.$$

We set $\det r = 1$ *if* $r \in SO_{\mathrm{j}}(\mathcal{Y})$ *and* $\det r = -1$ *if* $r \in O_{\mathrm{j}}(\mathcal{Y}) \backslash SO_{\mathrm{j}}(\mathcal{Y})$.

We say that $a \in o_{\mathrm{j}}(\mathcal{Y})$ *if* $a \in o(\mathcal{Y})$ *and* $[a, \mathrm{j}] \in B^2(\mathcal{Y})$*, or equivalently if* $g \in B^2(\overline{\mathcal{Z}}, \mathcal{Z})$*, where we use the decomposition (16.2).*

Recall that the groups $O_2(\mathcal{Y})$ and $SO_2(\mathcal{Y})$ and the Lie algebra $o_2(\mathcal{Y})$ were defined in Subsect. 14.1.2.

Proposition 16.22 (1) $O_{\mathrm{j}}(\mathcal{Y})$ *and* $SO_{\mathrm{j}}(\mathcal{Y})$ *are topological groups containing* $O_2(\mathcal{Y})$ *and* $SO_2(\mathcal{Y})$*, and we have an exact sequence*

$$1 \to SO_{\mathrm{j}}(\mathcal{Y}) \to O_{\mathrm{j}}(\mathcal{Y}) \to \mathbb{Z}_2 \to 1.$$

(2) $o_{\mathrm{j}}(\mathcal{Y})$ *is a Lie algebra containing* $o_2(\mathcal{Y})$.
(3) *If* $a \in o_{\mathrm{j}}(\mathcal{Y})$*, then* $\mathrm{e}^a \in SO_{\mathrm{j}}(\mathcal{Y})$.

In the following lemma, which we will prove before we prove the above proposition, we use the concept of the regularized determinant, defined for $b \in B^2(\mathcal{Y})$ as $\det_2(\mathbb{1} + b) := \det\big((\mathbb{1} + b)\mathrm{e}^{-b}\big)$; see (2.4).

Lemma 16.23 *Let* $r \in O_{\mathrm{j}}(\mathcal{Y})$*. Then* r *is* j*-non-degenerate iff* $\det_2\big(\mathbb{1} + b(r)\big) \neq 0$ *for* $b(r) := \frac{1}{2}\mathrm{j}r^{\#}[\mathrm{j}, r]$.

Proof We have

$$r\mathrm{j} + \mathrm{j}r = 2r\mathrm{j}\big(\mathbb{1} + b(r)\big).$$

This implies that $\mathrm{Ker}(r\mathrm{j} + \mathrm{j}r) = \mathrm{Ker}\big(\mathbb{1} + b(r)\big)$ Then we use Prop. 2.49. □

Proof of Prop. 16.22. The fact that $O_{\mathrm{j}}(\mathcal{Y})$ is a topological group, and $o_{\mathrm{j}}(\mathcal{Y})$ is a Lie algebra, follows by the same arguments as in Prop. 11.14. To show the remaining facts, we will use Thm. 16.43, to be proven later on.

Since $SO_{\mathrm{j}}(\mathcal{Y})$ is also the set of $r \in O_{\mathrm{j}}(\mathcal{Y})$ implementable by even unitaries, we see that $SO_{\mathrm{j}}(\mathcal{Y})$ is a subgroup of $O_{\mathrm{j}}(\mathcal{Y})$, using Thm. 16.43 (2)(ii) and (2)(iv).

Let us now prove that $SO_{\mathrm{j}}(\mathcal{Y})$ is closed. Let $r_n \in SO_{\mathrm{j}}(\mathcal{Y})$ converge to r, and let U_{r_n} be the corresponding Bogoliubov implementers. By Thm. 16.43 (2)(v), there exist μ_n, $|\mu_n| = 1$ such that $\mu_n U_{r_n} \to U_r$. U_{r_n} are even, and so is U_r. Hence $r \in SO_{\mathrm{j}}(\mathcal{Y})$.

Let us now prove (3). Set $f(t) = \det_2\big(\mathbb{1} + b(\mathrm{e}^{ta})\big)$, where $b(\mathrm{e}^{ta})$ is given by Lemma 16.23. The map $t \mapsto f(t)$ is real analytic, and $f(0) \neq 0$, hence $f(t)$ is not identically zero. So we can find a sequence $t_n \underset{n\to\infty}{\to} 1$ such that $f(t_n) \neq 0$. By Lemma 16.23, $\mathrm{e}^{t_n a}$ are j-non-degenerate, hence they belong to $SO_{\mathrm{j}}(\mathcal{Y})$.

But $o_{\rm j}(\mathcal{Y}) \ni a \mapsto {\rm e}^a \in O_{\rm j}(\mathcal{Y})$ is continuous and $SO_{\rm j}(\mathcal{Y})$ is closed. Hence, ${\rm e}^a = \lim_{n\to\infty} {\rm e}^{t_n a}$ belongs to $SO_{\rm j}(\mathcal{Y})$. □

16.1.9 Anomaly-free orthogonal group

This subsection is parallel to Subsect. 11.1.7 about the anomaly-free symplectic group. Recall that the groups $O_1(\mathcal{Y})$ and $SO_1(\mathcal{Y})$ and the Lie algebra $o_1(\mathcal{Y})$ were defined in Subsect. 14.1.2. Recall also that $B^1(\mathcal{Y})$ denotes the set of trace-class operators on $\mathcal{Y}$.

Definition 16.24 *Let $O_{\rm j,af}(\mathcal{Y})$ be the set of $r \in O_{\rm j}(\mathcal{Y})$ such that $2{\rm j} - ({\rm j}r + r{\rm j}) \in B^1(\mathcal{Y})$, or equivalently $p - \mathbb{1}_{\mathcal{Z}} \in B^1(\mathcal{Z})$. $O_{\rm j,af}(\mathcal{Y})$ will be called the* anomaly-free orthogonal group *and will be equipped with the metric*

$$d_{\rm j,af}(r_1, r_2) := \|p_1 - p_2\|_1 + \|q_1 - q_2\|_2.$$

An equivalent metric is $\|[{\rm j}, r_1 - r_2]_+\|_1 + \|[{\rm j}, r_1 - r_2]\|_2$.

We set $SO_{\rm j,af}(\mathcal{Y}) := O_{\rm j,af}(\mathcal{Y}) \cap SO_{\rm j}(\mathcal{Y})$.

We say that $a \in o_{\rm j,af}(\mathcal{Y})$ if $a \in o_{\rm j}(\mathcal{Y})$ and $a{\rm j} + {\rm j}a \in B^1(\mathcal{Y})$, or equivalently $h \in B^1(\mathcal{Y})$, where we use the decomposition (16.2).

Proposition 16.25 (1) *$O_{\rm j,af}(\mathcal{Y})$ and $SO_{\rm j,af}(\mathcal{Y})$ are topological groups containing $O_1(\mathcal{Y})$ and $SO_1(\mathcal{Y})$ respectively, and we have an exact sequence*

$$1 \to SO_{\rm j,af}(\mathcal{Y}) \to O_{\rm j,af}(\mathcal{Y}) \to \mathbb{Z}_2 \to 1.$$

(2) *$o_{\rm j,af}(\mathcal{Y})$ is a Lie algebra containing $o_1(\mathcal{Y})$.*
(3) *If $a \in o_{\rm j,af}(\mathcal{Y})$, then ${\rm e}^a \in SO_{\rm j,af}(\mathcal{Y})$.*

Proof The proof is completely analogous to that of Prop. 16.22. □

Proposition 16.26 (1) *Let $r \in O(\mathcal{Y})$ be* j*-positive. Then $r \in O_{\rm j}(\mathcal{Y})$ iff $r \in O_{\rm j,af}(\mathcal{Y})$.*
(2) *Let $a \in o(\mathcal{Y})$ be* j*-self-adjoint. Then $a \in o_{\rm j}(\mathcal{Y})$ iff $a \in o_{\rm j,af}(\mathcal{Y})$.*

Proof (1) We know that $r \in O_{\rm j}(\mathcal{Y})$ iff $c \in B^2(\overline{\mathcal{Z}}_{\rm reg}, \mathcal{Z}_{\rm reg})$ and $\dim \mathcal{Y}_{\rm sg} < \infty$. But then (16.12) implies $r \in O_{\rm j,af}(\mathcal{Y})$.

(2) By the decomposition (16.2), $a \in o_{\rm j}(\mathcal{Y})$ iff $h = 0$ and $g \in B^2_{\rm a}(\overline{\mathcal{Z}}, \mathcal{Z})$. □

We will also need the following lemma:

Lemma 16.27 *Let $r \in O_{\rm j,af}(\mathcal{Y})$, $\epsilon > 0$. There exists a decomposition $r = ts$ such that $\mathbb{1} - s$ is finite rank and $t \in O_{\rm j,af}(\mathcal{Y})$, $\|\mathbb{1} - t\| \le \epsilon$.*

Proof $\mathbb{1} - r$ is compact. Hence, there exists an o.n. basis $(e_1, e_2, \dots)$ in $\mathbb{C}\mathcal{Y}$ such that

$$r_{\mathbb{C}} = \sum_j \lambda_j |e_j)(e_j|$$

with $|\lambda_j| = 1$ and $\lambda_j \to 1$. Then we set

$$t_{\mathbb{C}} := \sum_{|\lambda_j - 1| \le \epsilon} \lambda_j |e_j)(e_j| + \sum_{|\lambda_j - 1| > \epsilon} |e_j)(e_j|, \tag{16.23}$$

$$s_{\mathbb{C}} := \sum_{|\lambda_j - 1| \le \epsilon} |e_j)(e_j| + \sum_{|\lambda_j - 1| > \epsilon} \lambda_j |e_j)(e_j|. \tag{16.24}$$

(We easily see that the r.h.s. of (16.23) and (16.24) restrict to operators on $\mathcal{Y}$.) □

16.1.10 Pairs of Kähler structures on real Hilbert spaces

This subsection is parallel to Subsect. 11.1.8 about pairs of Kähler structures in a symplectic space.

Recall that, as usual in this chapter, $(\mathcal{Y}, \nu)$ is a real Hilbert space. Let us first describe the action of the orthogonal group on Kähler anti-involutions.

Proposition 16.28 *Let $r \in O(\mathcal{Y})$ and let* j *be a Kähler anti-involution. Then*

(1) $\mathrm{j}_1 = r^{-1}\mathrm{j}r$ *is a Kähler anti-involution;*
(2) $r \in U(\mathcal{Y}^{\mathbb{C}})$ *iff* $\mathrm{j}_1 = \mathrm{j}$*;*
(3) r *is* j*-non-degenerate iff* $\mathrm{Ker}(\mathrm{j} + \mathrm{j}_1) = \{0\}$*.*

In the following theorem, for two Kähler anti-involutions j and j_1 we try to construct $r \in O(\mathcal{Y})$ such that

$$r^{-1}\mathrm{j}r = \mathrm{j}_1. \tag{16.25}$$

Note that this problem is more complicated for $O(\mathcal{Y})$ than for $Sp(\mathcal{Y})$ (see Subsect. 11.1.8).

Theorem 16.29 (1) *Let* j, j_1 *be Kähler anti-involutions on a real Hilbert space* $\mathcal{Y}$*. Then* $k := -\mathrm{j}\mathrm{j}_1$ *is a* j*-self-adjoint orthogonal transformation.*
(2) *Let* $k \in O(\mathcal{Y})$ *be* j*-self-adjoint for a Kähler anti-involution* j*. Then* $\mathrm{j}_1 := \mathrm{j}k$ *is a Kähler anti-involution.*
(3) *In what follows we assume that* $\mathrm{j}, \mathrm{j}_1, k$ *are as above. Then* $\mathrm{Ker}(\mathrm{j} + \mathrm{j}_1) = \mathrm{Ker}(k + \mathbb{1})$ *is invariant under* j *and* j_1*, and so is its orthogonal complement.*
(4) *There exists* $r \in O(\mathcal{Y})$ *satisfying (16.25) iff* $\mathrm{Ker}(\mathrm{j} + \mathrm{j}_1)$ *is even- or infinite-dimensional.*
(5) *If there exists a* j*-positive* $r \in O(\mathcal{Y})$ *satisfying (16.25), then* $\mathrm{Ker}(\mathrm{j} + \mathrm{j}_1) = \{0\}$*.*
(6) *Assume that* $\mathrm{Ker}(\mathrm{j} + \mathrm{j}_1) = \{0\}$*. Then* $r := k^{\frac{1}{2}}$ *defined in Thm. 16.13 is the unique* j*-positive element of* $O(\mathcal{Y})$ *satisfying (16.25).*
(7) *There exists* $c \in B_{\mathrm{a}}(\overline{\mathcal{Z}}, \mathcal{Z})$ *such that*

$$\left(\frac{k - \mathbb{1}}{k + \mathbb{1}}\right)_{\mathbb{C}} = \begin{bmatrix} 0 & c \\ \overline{c} & 0 \end{bmatrix}. \tag{16.26}$$

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(8) *We have*

$$r_{\mathbb{C}} = \begin{bmatrix} (\mathbb{1}+cc^*)^{-\frac12} & (\mathbb{1}+cc^*)^{-\frac12}c \\ -c^*(\mathbb{1}+cc^*)^{-\frac12} & (\mathbb{1}+c^*c)^{-\frac12} \end{bmatrix}, \tag{16.27}$$

$$k_{\mathbb{C}} == \begin{bmatrix} (\mathbb{1}-cc^*)(\mathbb{1}+cc^*)^{-1} & 2(\mathbb{1}+cc^*)^{-1}c \\ -2c^*(\mathbb{1}+cc^*)^{-1} & (\mathbb{1}-c^*c)(\mathbb{1}+c^*c)^{-1} \end{bmatrix}, \tag{16.28}$$

$$\mathrm{j}_{1\mathbb{C}} = \mathrm{i}\begin{bmatrix} (\mathbb{1}-cc^*)(\mathbb{1}+cc^*)^{-1} & 2(\mathbb{1}+cc^*)^{-1}c \\ 2c^*(\mathbb{1}+cc^*)^{-1} & (c^*c-\mathbb{1})(\mathbb{1}+c^*c)^{-1} \end{bmatrix}. \tag{16.29}$$

Proof (1)–(3) are straightforward.

Set $b = \frac{k-\mathbb{1}}{k+\mathbb{1}}$. We check that $\mathrm{j}b = -b\mathrm{j}$. This implies (16.26). Then using (16.13), we see that $r_{\mathbb{C}}$ equals (16.12), which is repeated as (16.27).

By the properties of the Cayley transform we have $k = \frac{\mathbb{1}+b}{\mathbb{1}-b}$, which yields (16.28). Alternatively, we can use $k = r^2$. (16.29) follows from $\mathrm{j}_1 = \mathrm{j}k$. □

Theorem 16.30 *Let $\mathcal{Z}$ and $\mathcal{Z}_1$ be the holomorphic subspaces of $\mathbb{C}\mathcal{Y}$ for the Kähler anti-involutions j and j_1. Suppose that $\mathrm{Ker}(\mathrm{j}+\mathrm{j}_1) = \{0\}$. Then*

$$\big\{(z, -\overline{c}z) \ : \ z \in \mathrm{Dom}\,\overline{c}\big\} \textit{ is dense in } \mathcal{Z}_1,$$

$$\big\{(-c\overline{z}, \overline{z}) \ : \ \overline{z} \in \mathrm{Dom}\,c\big\} \textit{ is dense in } \overline{\mathcal{Z}_1}.$$

Proof Every vector of $\mathcal{Z}_1$ is of the form $(\mathbb{1}-\mathrm{i}\mathrm{j}_1)y_1$ for $y_1 \in \mathcal{Y}$. Since $\mathrm{Ker}(\mathrm{j}+\mathrm{j}_1) = \{0\}$, $\mathrm{Ran}\,(k+\mathbb{1})$ is dense in $\mathcal{Y}$, hence the vectors of the form $(\mathbb{1}-\mathrm{i}\mathrm{j}_1)(\mathbb{1}+k)^{-1}y$, for $y \in \mathrm{Ran}\,(k+\mathbb{1})$ are dense in $\mathcal{Z}_1$. As in the proof of Prop. 11.21, we get that

$$(\mathbb{1}-\mathrm{i}\mathrm{j}_1)(\mathbb{1}+k)^{-1}y = z - \overline{c}z,$$

for $z = \mathbb{1}_{\mathcal{Z}}y \in \mathrm{Dom}\,\overline{c}$. □

Proposition 16.31 *Let j, j_1, k be as in Thm. 16.29. Set $\mathcal{Y}_{\mathrm{sg}} := \mathrm{Ker}(\mathrm{j}+\mathrm{j}_1)$ and $\mathcal{Y}_{\mathrm{reg}} := \mathcal{Y}_{\mathrm{sg}}^{\perp}$. Note that $\mathcal{Y}_{\mathrm{reg}}$ and $\mathcal{Y}_{\mathrm{sg}}$ are preserved by j and j_1. Let $\mathcal{Z}_{\mathrm{reg}}$ and $\mathcal{Z}_{\mathrm{sg}}$ be the corresponding holomorphic spaces. Recall also that one can define*

$$c := (k_{\mathbb{C}} - \mathbb{1})(\mathbb{1}+k_{\mathbb{C}})^{-1}\big|_{\overline{\mathcal{Z}}_{\mathrm{reg}}} \in Cl_{\mathrm{a}}(\overline{\mathcal{Z}}_{\mathrm{reg}}, \mathcal{Z}_{\mathrm{reg}}). \tag{16.30}$$

Then the following conditions are equivalent:

(1) $\mathrm{j} - \mathrm{j}_1 \in B^2(\mathcal{Y})$.
(2) $\mathbb{1} - k \in B^2(\mathcal{Y})$.
(3) $c \in B^2(\overline{\mathcal{Z}}_{\mathrm{reg}}, \mathcal{Z}_{\mathrm{reg}})$ *and* $\dim \mathcal{Z}_{\mathrm{sg}}$ *is finite.*
(4) *There exists a* j*-positive* $r \in O_{\mathrm{j,af}}(\mathcal{Y})$ *such that* $\mathrm{j}_1 = r\mathrm{j}r^{\#}$.
(5) *There exists* $r \in O_{\mathrm{j}}(\mathcal{Y})$ *such that* $\mathrm{j}_1 = r\mathrm{j}r^{\#}$.

Proof The identity $-\mathrm{j}(\mathrm{j}-\mathrm{j}_1) = \mathbb{1} - k$ and $\mathrm{j} \in O(\mathcal{Y})$ imply the equivalence of (1) and (2).

(2)⇒(3). Since $\mathcal{Y}_{\rm sg} = \mathrm{Ker}(\mathbb{1}+k)$ and $\mathbb{1}-k$ is compact, $\mathcal{Y}_{\rm sg}$ and hence $\mathcal{Z}_{\rm sg}$ are finite-dimensional. Moreover k preserves $\mathcal{Y}_{\rm reg}$ and $(\mathbb{1}+k)^{-1}\big|_{\mathcal{Y}_{\rm reg}}$ is bounded. Using (16.26), we obtain that $c \in B^2(\overline{\mathcal{Z}}_{\rm reg}, \mathcal{Z}_{\rm reg})$.

(3)⇒(2). By (16.26), we see that $(\mathbb{1}-k)\big|_{\mathcal{Y}_{\rm reg}} \in B^2(\mathcal{Y}_{\rm reg})$. $\mathcal{Y}_{\rm sg} = \mathrm{Ker}(\mathbb{1}+k)$ is finite-dimensional, hence we get that $\mathbb{1}-k \in B^2(\mathcal{Y})$.

(4)⇒(5)⇒(1) is obvious. (3)⇒(4) follows by setting $r := r_{\rm reg} \oplus r_{\rm sg}$, where $r_{\rm reg} \in O(\mathcal{Y}_{\rm reg})$ is defined as in Thm. 16.29 (5), and $r_{\rm sg}$ is any conjugation on $\mathcal{Y}_{\rm sg}$. □

16.2 Fermionic quadratic Hamiltonians on Fock spaces

As elsewhere in this chapter, $\mathcal{Z}$ is a Hilbert space and $\mathcal{Y} = \mathrm{Re}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$ is the corresponding Kähler space with the dual $\mathcal{Y}^\# = \mathrm{Re}(\overline{\mathcal{Z}} \oplus \mathcal{Z})$. We consider the Fock representation over $\mathcal{Y}$ in $\Gamma_{\rm a}(\mathcal{Z})$.

We study quadratic Hamiltonians on a fermionic Fock space. This section is parallel to Sect. 11.2 about quadratic Hamiltonians on a bosonic Fock space. It is also a continuation of Sect. 14.2, where quadratic fermionic Hamiltonians were studied in an algebraic setting.

16.2.1 Quadratic anti-commuting polynomials and their quantization

Let $h \in B^{\rm fd}(\mathcal{Z})$ (h is finite rank). It corresponds to the anti-symmetric polynomial

$$\mathcal{Y}^\# \times \mathcal{Y}^\# \ni \big((\overline{z}_1, z_1), (\overline{z}_2, z_2)\big) \mapsto \frac{1}{2}\big(\overline{z}_1 \cdot h z_2 - z_1 \cdot h^\# \overline{z}_2\big). \tag{16.31}$$

Its Wick, anti-symmetric and anti-Wick quantizations are

$$\mathrm{d}\Gamma(h), \qquad \mathrm{d}\Gamma(h) - \frac{\mathrm{Tr}\, h}{2}\mathbb{1}, \qquad \mathrm{d}\Gamma(h) - (\mathrm{Tr}\, h)\mathbb{1}.$$

Note that the anti-Wick and anti-symmetric quantizations can be extended to the case $h \in B^1(\mathcal{Z})$ (h is trace-class). The Wick quantization of (16.31) is well defined for much more general h.

Suppose that $g \in \overset{\rm al}{\Gamma}{}^2_{\rm a}(\mathcal{Z}) \simeq B^{\rm fd}_{\rm a}(\overline{\mathcal{Z}}, \mathcal{Z})$ (g is anti-symmetric finite rank). Consider the polynomial

$$\mathcal{Y}^\# \times \mathcal{Y}^\# \ni \big((\overline{z}_1, z_1), (\overline{z}_2, z_2)\big) \mapsto (z_1 \otimes_{\rm a} z_2 | g) = \overline{z}_1 \cdot g \overline{z}_2. \tag{16.32}$$

The Wick, anti-symmetric and anti-Wick quantizations of (16.32) are the "two-particle creation operator" $a^*(g)$ defined in Subsect. 3.4.4. According to the notation of Def. 13.27, this can be written as $\mathrm{Op}^{a^*,a}\big(|g)\big)$. It can be defined as a bounded operator also if $g \in \Gamma^2_{\rm a}(\mathcal{Z}) \simeq B^2_{\rm a}(\overline{\mathcal{Z}}, \mathcal{Z})$. It will act on $\Psi_n \in \Gamma^n_{\rm a}(\mathcal{Z})$ as

$$a^*(g)\Psi_n := \sqrt{(n+2)(n+1)}\, g \otimes_{\rm a} \Psi_n. \tag{16.33}$$

(On the right of (16.33) we interpret g as an element of $\Gamma^2_{\rm a}(\mathcal{Z})$.)

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The polynomial complex conjugate to (16.32) times -1 is

$$\mathcal{Y}^{\#} \times \mathcal{Y}^{\#} \ni \big((\overline{z}_1, z_1), (\overline{z}_2, z_2)\big) \mapsto (g|z_2 \otimes_{\mathrm{a}} z_1) = z_1 \cdot g^* z_2. \tag{16.34}$$

(-1 comes from the operator Λ; see (3.29).) The Wick, anti-symmetric and anti-Wick quantizations of (16.34) are the "two particle annihilation operator" $a^*(g)^* = a(g)$ defined in Subsect. 3.4.4. According to the notation of Def. 13.27, this can be written as $\mathrm{Op}^{a^*,a}\big(|g)\big)$.

A general element of $\mathbb{C}\mathrm{Pol}^2_{\mathrm{a}}(\mathcal{Y}^{\#})$ is

$$\big((\overline{z}_1, z_1), (\overline{z}_2, z_2)\big) \mapsto \overline{z}_1 \cdot h z_2 - z_1 \cdot h^{\#} \overline{z}_2 + \overline{z}_1 \cdot g_1 \overline{z}_2 - z_1 \cdot \overline{g}_2 z_2, \tag{16.35}$$

where $h \in B^{\mathrm{fd}}(\mathcal{Z})$, $g_1, g_2 \in B^{\mathrm{fd}}_{\mathrm{a}}(\overline{\mathcal{Z}}, \mathcal{Z})$. We can write (16.35) as

$$(\overline{z}_1, z_1) \cdot \zeta(\overline{z}_2, z_2), \quad \zeta_{\mathbb{C}} = \begin{bmatrix} g_1 & h \\ -h^{\#} & -\overline{g}_2 \end{bmatrix}.$$

(Recall that we use elements of $L_{\mathrm{a}}(\mathcal{Y}^{\#}, \mathcal{Y})$ for symbols of fermionic quadratic Hamiltonians, as in Subsect. 14.2.3.)

The quantizations of ζ are

$$\mathrm{Op}^{a^*,a}(\zeta) = 2\mathrm{d}\Gamma(h) + a^*(g_1) + a(g_2), \tag{16.36}$$

$$\mathrm{Op}(\zeta) = 2\mathrm{d}\Gamma(h) - (\mathrm{Tr}\, h)\mathbb{1} + a^*(g_1) + a(g_2), \tag{16.37}$$

$$\mathrm{Op}^{a,a^*}(\zeta) = 2\mathrm{d}\Gamma(h) - (2\mathrm{Tr}\, h)\mathbb{1} + a^*(g_1) + a(g_2).$$

Note that

$$\mathrm{Op}(\zeta) = \frac{1}{2}\left(\mathrm{Op}^{a^*,a}(\zeta) + \mathrm{Op}^{a,a^*}(\zeta)\right).$$

In particular, we can extend the definition of $\mathrm{Op}(\zeta)$ and $\mathrm{Op}^{a,a^*}(\zeta)$ to the case when $g_1, g_2 \in B^2(\overline{\mathcal{Z}}, \mathcal{Z})$ and $h \in B^1(\mathcal{Z})$. $\mathrm{Op}^{a^*,a}(\zeta)$ is defined under much more general conditions. All these quantizations are self-adjoint iff $h = h^*$ and $g_1 = g_2$.

16.2.2 Fermionic Schwinger term

Recall from Thm. 14.13 that the anti-symmetric quantization restricted to quadratic symbols yields an isomorphism of Lie algebra $o_1(\mathcal{Y})$ into quadratic Hamiltonians in $\mathrm{CAR}^{C^*}(\mathcal{Y})$. This is no longer true in the case of the Wick quantization, where the so-called Schwinger term appears. This is described in the following proposition:

Proposition 16.32 *Let $\zeta, \zeta_i \in B(\mathcal{Y}^{\#}, \mathcal{Y})$, $i = 1, 2$. Then,*

$$\mathrm{Op}(\zeta) = \mathrm{Op}^{a^*,a}(\zeta) + \frac{\mathrm{i}}{2}(\mathrm{Tr}\zeta\nu\mathrm{j})\, \mathbb{1}, \tag{16.38}$$

$$\big[\mathrm{Op}^{a^*,a}(\zeta_1), \mathrm{Op}^{a^*a}(\zeta_2)\big] = 4\mathrm{Op}^{a^*,a}(\zeta_1\nu\zeta_2 - \zeta_2\nu\zeta_1) + \mathrm{i}2(\mathrm{Tr}[\zeta_1\nu, \zeta_2\nu]\mathrm{j})\, \mathbb{1}.$$

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Proof We have $\nu_{\mathbb{C}} = \frac{1}{2}\begin{bmatrix} 0 & 1\!\!1 \\ 1\!\!1 & 0 \end{bmatrix} \in B(\mathcal{Z} \oplus \overline{\mathcal{Z}}, \overline{\mathcal{Z}} \oplus \mathcal{Z})$. Therefore,

$$\zeta_{\mathbb{C}}\nu_{\mathbb{C}} = \frac{1}{2}\begin{bmatrix} h & g \\ -\overline{g} & -h^{\#} \end{bmatrix}, \quad \zeta_{\mathbb{C}}\nu_{\mathbb{C}}\mathrm{j}_{\mathbb{C}} = \frac{1}{2}\begin{bmatrix} \mathrm{i}h & -\mathrm{i}g \\ -\mathrm{i}\overline{g} & \mathrm{i}h^{\#} \end{bmatrix}. \tag{16.39}$$

Hence, $-\mathrm{Tr}\, h = \frac{\mathrm{i}}{2}\mathrm{Tr}\zeta\nu\mathrm{j}$, which implies (16.38).

Now, to compute the Schwinger term we note that by (14.9)

$$\left[\mathrm{Op}^{a^*,a}(\zeta_1), \mathrm{Op}^{a^*a}(\zeta_2)\right] = 4\mathrm{Op}(\zeta_1\nu\zeta_2 - \zeta_2\nu\zeta_1).$$

Then we apply (16.38). □

16.2.3 Infimum of quadratic fermionic Hamiltonians

For simplicity, in this subsection we assume that $\mathcal{Z}$ is a finite-dimensional Hilbert space.

Theorem 16.33 *Let $h \in B_{\mathrm{h}}(\mathcal{Z})$, $g \in B^2_{\mathrm{a}}(\overline{\mathcal{Z}}, \mathcal{Z})$. Let*

$$\zeta_{\mathbb{C}} = \begin{bmatrix} g & h \\ -h^{\#} & -\overline{g} \end{bmatrix}. \tag{16.40}$$

Then,

$$\inf \mathrm{Op}^{a^*,a}(\zeta) = \frac{1}{2}\mathrm{Tr}\left(- \begin{bmatrix} h^2 + gg^* & hg - gh^{\#} \\ g^*h - h^{\#}g^* & \overline{h}^2 + g^*g \end{bmatrix}^{\frac{1}{2}} + \begin{bmatrix} h & 0 \\ 0 & h^{\#} \end{bmatrix}\right).$$

Proof Clearly, $\zeta\nu$ is self-adjoint and

$$(\zeta_{\mathbb{C}}\nu_{\mathbb{C}})^2 = \frac{1}{4}\begin{bmatrix} h^2 + gg^* & hg - gh^{\#} \\ g^*h - h^{\#}g^* & h^{\#\,2} + g^*g \end{bmatrix}.$$

Thus, by Thm. 14.13,

$$\begin{aligned} \inf \mathrm{Op}^{a^*,a}(\zeta) - \mathrm{Tr}\, h &= \inf \mathrm{Op}(\zeta) \\ &= -\mathrm{Tr}|\zeta\nu| = -\frac{1}{2}\mathrm{Tr}\begin{bmatrix} h^2 + gg^* & hg - gh^{\#} \\ g^*h - h^{\#}g^* & h^{\#\,2} + g^*g \end{bmatrix}^{\frac{1}{2}}. \end{aligned}$$ □

16.2.4 Two-particle creation and annihilation operators

In this subsection we allow the dimension of $\mathcal{Z}$ to be infinite. We study two-particle creation and annihilation operators. Recall that they are defined for $c \in \Gamma^2_{\mathrm{a}}(\mathcal{Z}) \simeq B^2_{\mathrm{a}}(\overline{\mathcal{Z}}, \mathcal{Z})$.

Proposition 16.34 *Let $c \in \Gamma^2_{\mathrm{a}}(\mathcal{Z})$. Then $a(c)$, $a^*(c)$ are bounded operators with*

$$\|a(c)\| = \|a^*(c)\| = \|c\|_2; \tag{16.41}$$

$$\mathrm{e}^{-\frac{1}{2}a^*(c)}a(z)\mathrm{e}^{\frac{1}{2}a^*(c)} = a(z) - a^*(c\overline{z}), \qquad z \in \mathcal{Z}; \tag{16.42}$$

$$\mathrm{e}^{\frac{1}{2}a(c)}a^*(z)\mathrm{e}^{-\frac{1}{2}a(c)} = a^*(z) - a(c\overline{z}), \qquad z \in \mathcal{Z}. \tag{16.43}$$

Proof Since c is anti-symmetric Hilbert–Schmidt, by Corollary 2.88 there exists an o.n. family

$$(w_{1,+}, w_{1,-}, w_{2,+}, w_{2,-} \dots) \tag{16.44}$$

and positive numbers $(\lambda_1, \lambda_2, \dots)$ such that

$$c = \sum_{j=1}^{\infty} \frac{\lambda_j}{2} \big(|w_{j,+}\rangle\langle w_{j,-}| - |w_{j,-}\rangle\langle w_{j,+}| \big).$$

Then,

$$a^*(c) = \sum_{j=1}^{\infty} \lambda_j a^*(w_{j,+}) a^*(w_{j,-}).$$

Using the Jordan–Wigner representation compatible with the o.n. family (16.44), we easily obtain

$$\|a^*(c)\|^2 = \sum_{j=1}^{\infty} \lambda_j^2. \qquad \square$$

16.2.5 Fermionic Gaussian vectors

Let $c \in \Gamma_{\mathrm{a}}^2(\mathcal{Z}) \simeq B_{\mathrm{a}}^2(\overline{\mathcal{Z}}, \mathcal{Z})$. Then c^*c is trace-class, so $\det(\mathbb{1} + c^*c)$ is well defined.

Definition 16.35 *The* fermionic Gaussian vector associated with c *is defined as*

$$\Omega_c := \det(\mathbb{1} + c^*c)^{-\frac14} \mathrm{e}^{-\frac12 a^*(c)} \Omega.$$

Theorem 16.36 (1) *If $c \in B_{\mathrm{a}}^2(\overline{\mathcal{Z}}, \mathcal{Z})$, then Ω_c is a normalized vector in $\Gamma_{\mathrm{a}}(\mathcal{Z})$ satisfying*

$$\big(a(z) - a^*(c\overline{z})\big)\Psi = 0, \quad z \in \mathcal{Z}, \quad (\Omega_c|\Omega_c) > 0.$$

(2) *Let $c \in Cl_{\mathrm{a}}(\overline{\mathcal{Z}}, \mathcal{Z})$. Assume that there exists a non-zero $\Psi \in \Gamma_{\mathrm{a}}(\mathcal{Z})$ satisfying*

$$\big(a(z) - a^*(c\overline{z})\big)\Psi = 0, \quad \overline{z} \in \mathrm{Dom}\, c.$$

Then $c \in B_{\mathrm{a}}^2(\overline{\mathcal{Z}}, \mathcal{Z})$. Moreover Ψ is proportional to Ω_c.

(3) *Let $c_1, c_2 \in B_{\mathrm{a}}^2(\overline{\mathcal{Z}}, \mathcal{Z})$. Then*

$$(\Omega_{c_1}|\Omega_{c_2}) = \det(\mathbb{1} + c_1^*c_1)^{-\frac14} \det(\mathbb{1} + c_2^*c_2)^{-\frac14} \mathrm{Pf} \begin{bmatrix} \overline{c}_1 & -\mathbb{1} \\ \mathbb{1} & c_2 \end{bmatrix}.$$

To make the above theorem complete we need to define the Pfaffian of certain infinite-dimensional operators, which is provided by the following proposition:

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Proposition 16.37 *Let* $c_i \in B^2_{\mathrm{a}}(\overline{\mathcal{Z}}, \mathcal{Z})$, $i = 1, 2$. *Set* $\zeta \in B_{\mathrm{a}}(\mathcal{Z} \oplus \overline{\mathcal{Z}}, \overline{\mathcal{Z}} \oplus \mathcal{Z})$ *equal to*

$$\zeta = \begin{bmatrix} \overline{c}_1 & -1\!\!1_{\overline{\mathcal{Z}}} \\ 1\!\!1_{\mathcal{Z}} & c_2 \end{bmatrix}.$$

Let π_n *be an increasing family of finite rank projections on* $\mathcal{Z}$ *with* $\mathrm{s} - \lim\limits_{n\to\infty} \pi_n = 1\!\!1$. *Set* $\mathcal{Z}_n = \pi_n \mathcal{Z}$, *and* $\zeta_n = (\overline{\pi}_n \oplus \pi_n)\zeta(\pi_n \oplus \overline{\pi}_n)$. *Then*

$$\lim_{n\to\infty} \mathrm{Pf}\zeta_n =: \mathrm{Pf}\zeta$$

exists, where, for each n $\mathrm{Pf}\zeta_n$, *is computed w.r.t. the Liouville form on* $\overline{\mathcal{Z}}_n \oplus \mathcal{Z}_n$. *Moreover*

$$(\mathrm{Pf}\zeta)^2 = \det(1\!\!1 - \overline{c}_1 c_2).$$

Proof of Thm. 16.36 and Prop. 16.37. Let Ψ be as in (2). Arguing as in the proof of Thm. 11.28, we obtain, for $\overline{z}_i \in \mathrm{Dom}\, c$ and $\lambda := (\Omega|\Psi)$,

$$\begin{aligned} &\big(a^*(z_{2m+1}) \cdots a^*(z_1)\Omega|\Psi\big) = 0, \\ &\quad \big(a^*(z_{2m}) \cdots a^*(z_1)\Omega|\Psi\big) = \lambda \sum_{\sigma\in\mathrm{Pair}_{2m}} \mathrm{sgn}(\sigma) \prod_{j=1}^{m} (z_{\sigma(2j)}|c\overline{z}_{\sigma(2j+1)}). \end{aligned}$$

Therefore, $\lambda = 0$ implies $\Psi = 0$. Hence, $\lambda \neq 0$. In particular, for $\overline{z}_1, \overline{z}_2 \in \mathrm{Dom}\; c$ this gives the following formula for the two-particle component of Ψ:

$$\sqrt{2}(z_2 \otimes_{\mathrm{a}} z_1|\Psi_2) = \lambda(z_1|c\overline{z}_2). \tag{16.45}$$

As in Thm. 11.28, this implies that $c \in B^2_{\mathrm{a}}(\overline{\mathcal{Z}}, \mathcal{Z})$ and $\Psi_2 = -\frac{\lambda}{\sqrt{2}}c$.

We have

$$(z_1 \otimes_{\mathrm{a}} \cdots \otimes_{\mathrm{a}} z_{2m}|c^{\otimes_{\mathrm{a}} m}) = \frac{m!2^m}{2m!} \sum_{\sigma\in\mathrm{Pair}_m} \prod_{i=0}^{m-1} \mathrm{sgn}(\sigma)\,(z_{\sigma(2i+1)}|c\overline{z}_{\sigma(2i+2)}),$$

which implies that

$$\begin{aligned} \Psi_{2m} &= \lambda(-1)^m \frac{\sqrt{(2m)!}}{2^m m!} c^{\otimes_{\mathrm{a}} m} \;=\; \lambda(-1)^m \frac{1}{2^m m!} \big(a^*(c)\big)^m \Omega, \\ \Psi_{2m+1} &= 0, \end{aligned}$$

i.e.

$$\Psi = \lambda \mathrm{e}^{-\frac{1}{2}a^*(c)}\Omega.$$

Let us now compute $\|\Psi\|^2$. Without loss of generality we can assume $\lambda = 1$. Since c is compact, we can by Corollary 2.88 find an o.n. basis $\{z_{i,+}, z_{i,-}\}_{i\in I}$ of $(\mathrm{Ran}\, c)^{\perp}$, such that $c\overline{z_{i,-}} = \lambda_i z_{i,+}$, $c\overline{z_{i,+}} = -\lambda_i z_{i,+}$. Thus $c^* c\overline{z}_{i,\pm} = \lambda_i^2 \overline{z}_{i,\pm}$. Using the corresponding basis in $\Gamma_{\mathrm{a}}(\mathcal{Z})$, we obtain

$$\|\Psi\|^2 = \prod_{i\in I}(1 + \lambda_i^2) = \det(1\!\!1 + c^* c)^{\frac{1}{2}}. \tag{16.46}$$

This shows that the vector Ω_c is normalized.

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It remains to show (3). Let us first assume that $\mathcal{Z}$ is finite-dimensional. In the fermionic complex-wave representation, $\mathrm{e}^{-\frac{1}{2}a^*(c)}\Omega$ equals $\mathrm{e}^{-\frac{1}{2}\overline{z}\cdot c\overline{z}}$.

$$
\begin{aligned}
\big(\mathrm{e}^{-\frac{1}{2}a^*(c_1)}\Omega|\mathrm{e}^{-\frac{1}{2}a^*(c_2)}\Omega\big) &= \int \mathrm{e}^{z\cdot\overline{z}}\mathrm{e}^{-\frac{1}{2}z\cdot\overline{c_1}z}\mathrm{e}^{-\frac{1}{2}\overline{z}\cdot c_2\overline{z}}\mathrm{d}z\mathrm{d}\overline{z}\\
&= \int \exp\frac{1}{2}[\overline{z},z]\cdot\begin{bmatrix}-\overline{c}_1 & -\mathbb{1}\\ \mathbb{1} & -c_2\end{bmatrix}\begin{bmatrix}z\\ \overline{z}\end{bmatrix}\mathrm{d}z\mathrm{d}\overline{z}\\
&= \mathrm{Pf}\begin{bmatrix}-\overline{c}_1 & -\mathbb{1}\\ \mathbb{1} & -c_2\end{bmatrix} = \det(\mathbb{1}-\overline{c}_1c_2)^{\frac{1}{2}},
\end{aligned}
$$

using the formulas in Subsect. 1.1.2.

Let us now consider the general case. We first claim that the map

$$B^2_{\mathrm{a}}(\overline{\mathcal{Z}},\mathcal{Z}) \ni c \mapsto \Omega_c \in \Gamma_{\mathrm{a}}(\mathcal{Z}) \tag{16.47}$$

is continuous for the Hilbert–Schmidt norm. Recall from Prop. 16.34 that

$$\|a^*(c)\| = \|c\|_2. \tag{16.48}$$

Note now that if a_1, a_2 are two bounded operators then

$$\|\mathrm{e}^{a_1}-\mathrm{e}^{a_2}\| \leq \|a_1-a_2\|\frac{\mathrm{e}^{\|a_1\|}-\mathrm{e}^{\|a_2\|}}{\|a_1\|-\|a_2\|}.$$

Using (16.48), for $a_i = a^*(c_i)$ with $\|c_i\|_2 \leq C$ this yields,

$$\|\mathrm{e}^{-\frac{1}{2}a^*(c_1)}-\mathrm{e}^{-\frac{1}{2}a^*(c_2)}\| \leq C'\|c_1-c_2\|_2.$$

Since $c\mapsto\det(\mathbb{1}+c^*c)^{-\frac{1}{4}}$ is continuous for the Hilbert–Schmidt norm, this proves (16.47).

We can now complete the proof of (3) in the general case. Let us choose an increasing sequence of finite rank projections π_n and set $c_{i,n} = \pi_n c_i \overline{\pi}_n$, $i = 1, 2$. We have $c_{i,n} \to c_i$ in the Hilbert–Schmidt norm. Hence, by (16.47), $\Omega_{c_{i,n}} \to \Omega_{c_i}$, and thus

$$(\Omega_{c_1}|\Omega_{c_2}) = \lim_{n\to\infty}(\Omega_{c_{1,n}}|\Omega_{c_{2,n}}),$$

which proves (3) in the general case. □

16.3 Fermionic Bogoliubov transformations on Fock spaces

We keep the same framework and notation as in the rest of the chapter. That is, $\mathcal{Z}$ is a Hilbert space, $\mathcal{Y} := \mathrm{Re}(\mathcal{Z}\oplus\overline{\mathcal{Z}})$ is the corresponding complete Kähler space, equipped with ν, j. We also consider the Fock CAR representation

$$\mathcal{Y} \ni y \mapsto \phi(y) \in B_{\mathrm{h}}\big(\Gamma_{\mathrm{a}}(\mathcal{Z})\big).$$

We are going to study the implementation of orthogonal transformations on a fermionic Fock space. The central result of the section is the Shale–Stinespring

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theorem, which says that an orthogonal transformation is implementable iff it belongs to the restricted orthogonal group. The unitary operators implementing the corresponding Bogoliubov automorphisms form a group, denoted $Pin_{\mathrm{j}}^{\mathrm{c}}(\mathcal{Y})$, which is one of the generalizations of the Pin^{c} group from the finite-dimensional case and contains the group $Pin_2^{\mathrm{c}}(\mathcal{Y})$, which is a subgroup of the unitary part of $\mathrm{CAR}^{W^*}(\mathcal{Y})$.

We will also describe the group $Pin_{\mathrm{j,af}}(\mathcal{Y})$, which is one of the generalizations of the Pin group from the finite-dimensional case. It contains the group $Pin_2(\mathcal{Y})$ as a proper subgroup.

Clearly, both $Pin_{\mathrm{j}}^{\mathrm{c}}(\mathcal{Y})$ and $Pin_{\mathrm{j,af}}(\mathcal{Y})$ depend on the Kähler structure of $\mathcal{Y}$.

This section is parallel to Sect. 11.3, where Bogoliubov transformations on bosonic Fock spaces were studied. It can be viewed as a continuation of Sect. 14.3, which described the implementability of Bogoliubov transformations in the C^*- and W^*-CAR algebras.

16.3.1 Extending parity and complex conjugation

Clearly, we can isometrically embed $\mathrm{CAR}^{C^*}(\mathcal{Y})$ in $B\big(\Gamma_{\mathrm{a}}(\mathcal{Y})\big)$. The parity automorphism α defined on $\mathrm{CAR}^{C^*}(\mathcal{Y})$ extends to a weakly continuous involution on the whole $B\big(\Gamma_{\mathrm{a}}(\mathcal{Z})\big)$ by setting

$$\alpha(A) := IAI. \tag{16.49}$$

Thus we can speak about even and odd operators on $B\big(\Gamma_{\mathrm{a}}(\mathcal{Z})\big)$.

Unfortunately, there seems to be no analog of (16.49) for the complex conjugation $A \mapsto \mathrm{c}(A)$ on the Fock space, as seen from the following proposition:

Proposition 16.38 *Let $\mathcal{Y}$ be infinite-dimensional. Then $\mathrm{Cliff}^{C^*}(\mathcal{Y})$ is weakly dense in $B\big(\Gamma_{\mathrm{a}}(\mathcal{Z})\big)$. Hence, the anti-linear automorphism $A \to \mathrm{c}(A)$ cannot be extended from $\mathrm{CAR}^{C^*}(\mathcal{Y})$ to a strongly continuous automorphism of $B\big(\Gamma_{\mathrm{a}}(\mathcal{Z})\big)$.*

Proof It is sufficient to assume that $\mathcal{Z}$ has an o.n. basis $(e_1, e_2, \dots)$. Let $\theta \in \mathbb{R}$. Let $u_n \in U(\mathcal{Z})$ be defined by

$$u_n e_j := \begin{cases} \mathrm{e}^{\frac{\mathrm{i}2}{n}\theta} e_j, & j = 1, \dots, n; \\ 0, & j = n+1, \dots. \end{cases}$$

One finds that if $e \in \mathcal{Z}$ is a normalized vector, then

$$2\mathrm{i}a^*(e)a(e) = \phi(\mathrm{i}e, -\mathrm{i}\overline{e})\phi(e, \overline{e}) + \mathrm{i}.$$

Hence,

$$\Gamma(u_n) = \exp\Big(\sum_{j=1}^{n} \frac{\theta}{n} 2\mathrm{i}a^*(e_j)a(e_j)\Big) = \mathrm{e}^{\mathrm{i}\theta} \exp\Big(\sum_{j=1}^{n} \frac{\theta}{n} \phi(\mathrm{i}e_j, -\mathrm{i}\overline{e_j})\phi(e_j, \overline{e_j})\Big).$$

Therefore, $U_n := \Gamma(u_n)\mathrm{e}^{-\mathrm{i}\theta} \in \mathrm{Cliff}^{C^*}(\mathcal{Y})$. Clearly,

$$\mathrm{s} - \lim_{n\to\infty} U_n = \mathrm{e}^{\mathrm{i}\theta}\mathbb{1}.$$

Consequently, $\mathrm{e}^{\mathrm{i}\theta}\mathbb{1}$ belongs to the strong closure of $\mathrm{Cliff}^{C^*}(\mathcal{Y})$. Hence, the strong closure of $\mathrm{Cliff}^{C^*}(\mathcal{Y})$ contains $\mathrm{CAR}^{C^*}(\mathcal{Y})$. But $\mathrm{CAR}^{C^*}(\mathcal{Y})$ is strongly dense in $B\big(\Gamma_{\mathrm{a}}(\mathcal{Z})\big)$. □

16.3.2 Group $Pin_{\mathrm{j}}^{\mathrm{c}}(\mathcal{Y})$

Definition 16.39 *We define $Pin_{\mathrm{j}}^{\mathrm{c}}(\mathcal{Y})$ to be the set of $U \in U\big(\Gamma_{\mathrm{a}}(\mathcal{Z})\big)$ such that*

$$\big\{U\phi(y)U^* \ : \ y \in \mathcal{Y}\big\} = \big\{\phi(y) \ : \ y \in \mathcal{Y}\big\}.$$

We set

$$Spin_{\mathrm{j}}^{\mathrm{c}}(\mathcal{Y}) := \big\{U \in Pin_{\mathrm{j}}^{\mathrm{c}}(\mathcal{Y}) \ : \ \alpha(U) = U\big\}.$$

We equip $Pin_{\mathrm{j}}^{\mathrm{c}}(\mathcal{Y})$ with the strong operator topology.

It is obvious that $Pin_{\mathrm{j}}^{\mathrm{c}}(\mathcal{Y})$ is a topological group and $Spin_{\mathrm{j}}^{\mathrm{c}}(\mathcal{Y})$ is its closed subgroup.

The following definitions are parallel to definitions of Sect. 14.3.

Definition 16.40 *Let $A \in B\big(\Gamma_{\mathrm{a}}(\mathcal{Z})\big)$ and $r \in O(\mathcal{Y})$.*

(1) *We say that A* intertwines *r if*

$$A\phi(y) = \phi(ry)A, \quad y \in \mathcal{Y}. \tag{16.50}$$

(2) *If in addition A is unitary then we also say that A* implements *r.*
(3) *If there exists $U \in U\big(\Gamma_{\mathrm{a}}(\mathcal{Z})\big)$ that implements r, then we say that r is* implementable in the Fock representation.

It is clear that the map $Pin_{\mathrm{j}}^{\mathrm{c}}(\mathcal{Y}) \to O(\mathcal{Y})$ defined by (16.50) is a group homomorphism. However, one prefers to use a different homomorphism, arising from the following definition:

Definition 16.41 *Let $r \in O(\mathcal{Y})$.*

(1) *We say that $A \in B\big(\Gamma_{\mathrm{a}}(\mathcal{Z})\big)$* α-intertwines *$r \in O(\mathcal{Y})$ if*

$$\alpha(A)\phi(y) = \phi(ry)A, \quad y \in \mathcal{Y}.$$

(2) *If in addition A is unitary then we also say that A* α-implements *r.*
(3) *If there exists $U \in U\big(\Gamma_{\mathrm{a}}(\mathcal{Z})\big)$ that α-implements r, then we say that r is* α-implementable in the Fock representation.

We will see in Thm. 16.43 that if r is α-implementable in the Fock representation, then necessarily $r \in O_{\mathrm{j}}(\mathcal{Y})$. Therefore, $\det r$ is well defined by Def. 16.21

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and we can introduce the notion of det-implementation, essentially equivalent to α-implementability.

Definition 16.42 *Let $r \in O_{\mathrm{j}}(\mathcal{Y})$.*

(1) *We say that $A \in B\big(\Gamma_{\mathrm{a}}(\mathcal{Z})\big)$* det-intertwines *$r$ if*

$$A\phi(y) = \det r\, \phi(ry)A, \quad y \in \mathcal{Y}. \tag{16.51}$$

(2) *If in addition A is unitary, then we also say that A* det-implements *r.*
(3) *If there exists $U \in U\big(\Gamma_{\mathrm{a}}(\mathcal{Z})\big)$ that det-implements r, then we say that r is* det-implementable in the Fock representation.

We will prove

Theorem 16.43 (The Shale–Stinespring theorem about Bogoliubov transformations)

(1) *Let $r \in O(\mathcal{Y})$. The following statements are equivalent:*
   (i) *r is α-implementable in the Fock representation.*
   (ii) *r is det-implementable.*
   (iii) *r is implementable in the Fock representation.*
   (iv) *$r \in O_{\mathrm{j}}(\mathcal{Y})$.*
(2) *Suppose now that $r \in O_{\mathrm{j}}(\mathcal{Y})$. Then the following is true:*
   (i) *There exists $U_r \in Pin_{\mathrm{j}}^{\mathrm{c}}(\mathcal{Y})$ such that the set of elements of $U\big(\Gamma_{\mathrm{a}}(\mathcal{Z})\big)$ α-implementing r consists of operators of the form μU_r with $|\mu| = 1$.*
   (ii) *U_r is even iff $r \in SO_{\mathrm{j}}(\mathcal{Y})$. Otherwise, it is odd. Hence, U_r α-implements r iff it det-implements r.*
   (iii) *The set of elements of $U\big(\Gamma_{\mathrm{a}}(\mathcal{Z})\big)$ implementing r consists of operators of the form μU_r with $|\mu| = 1$ if $\det r = 1$ and μU_{-r} with $|\mu| = 1$ if $\det r = -1$.*
   (iv) *If $r_1, r_2 \in O_{\mathrm{j}}(\mathcal{Y})$, then $U_{r_1}U_{r_2} = \mu U_{r_1 r_2}$ for some μ such that $|\mu| = 1$.*
   (v) *If $r_n \to r$ in $O_{\mathrm{j}}(\mathcal{Y})$, then there exist μ_n, $|\mu_n| = 1$, such that $\mu_n U_{r_n} \to U_r$ strongly.*
(3) *Most of the above statements can be summarized by the following commuting diagram of Lie groups and their continuous homomorphisms, where all vertical and horizontal sequences are exact:*

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \to & U(1) & \to & U(1) & \to & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & Spin_{\mathrm{j}}^{\mathrm{c}}(\mathcal{Y}) & \to & Pin_{\mathrm{j}}^{\mathrm{c}}(\mathcal{Y}) & \to & \mathbb{Z}_2 & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & SO_{\mathrm{j}}(\mathcal{Y}) & \to & O_{\mathrm{j}}(\mathcal{Y}) & \to & \mathbb{Z}_2 & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array} \tag{16.52}$$

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As a preparation for the proof of the above theorem we will first show the following lemma:

Lemma 16.44 *Let $r \in O(\mathcal{Y})$. Then the set of elements of $A \in B\big(\Gamma_{\mathrm{a}}(\mathcal{Z})\big)$ α-intertwining r is either empty or of the form $\{\mu U \ : \ \mu \in \mathbb{C}\}$, where U is unitary. Besides, U is even or odd.*

Proof Let $A = A_0 + A_1$ with A_0 even and A_1 odd. Then

$$(A_0 + A_1)\phi(y) = \phi(ry)(A_0 - A_1), \quad y \in \mathcal{Y}. \tag{16.53}$$

Comparing even and odd terms in (16.53), we obtain

$$A_0\phi(y) = \phi(ry)A_0, \quad A_1\phi(y) = -\phi(ry)A_1, \quad y \in \mathcal{Y}. \tag{16.54}$$

Hence, $A_0^*A_0$ and $A_1^*A_1$ commute with $\phi(y)$, $y \in \mathcal{Y}$. Clearly, they are even. Hence, by the irreducibility of the Fock CAR representation, they are proportional to identity. Hence, the operators A_i are proportional to a unitary operator.

(16.54) implies also that $A_1^*A_0$ anti-commutes with $\phi(y)$, $y \in \mathcal{Y}$. By Prop. 13.3, this implies that $A_1^*A_0$ is even. But $A_1^*A_0$ is odd. Hence, $A_1^*A_0 = 0$. Thus one A_i is zero. □

16.3.3 Implementation of partial conjugations

Let $\kappa \in O(\mathcal{Y})$ be an involution with $\mathrm{Ker}(\kappa + \mathbb{1})$ finite. Clearly, $\kappa \in O_1(\mathcal{Y})$. Hence, κ is det-implementable in $\mathrm{Cliff}^{\mathrm{alg}}(\mathcal{Y})$. In fact, if $(e_1, \dots, e_n)$ is an o.n. basis of $\mathrm{Ker}(\kappa + \mathbb{1})$, then

$$U_\kappa = \phi(e_1)\cdots\phi(e_n) \in \mathrm{Cliff}^{\mathrm{alg}}(\mathcal{Y})$$

det-implements κ.

Recall that $\mathrm{Cliff}^{\mathrm{alg}}(\mathcal{Y})$ can be treated as a sub-algebra of $B\big(\Gamma_{\mathrm{a}}(\mathcal{Z})\big)$. Hence, U_κ det-implements κ in the Fock representation.

In the case of the Fock CAR representation one can distinguish a class of orthogonal involutions with special properties – the so-called partial conjugations; see Def. 16.16. Assume now that κ is not only an orthogonal involution, but also a partial conjugation on the Kähler space $\mathcal{Y}$. Let $\mathcal{W} := \frac{\mathbb{1}-\mathrm{i}j}{2}\mathrm{Ker}(\kappa + \mathbb{1})$ be the holomorphic subspace associated with $\mathrm{Ker}(\kappa + \mathbb{1})$. It is easy to see that, setting

$$w_j := \frac{1}{2}(e_j - \mathrm{i}je_j), \ j = 1, \dots, n,$$

we obtain an o.n. basis of $\mathcal{W}$, and

$$\kappa_{\mathbb{C}} w_j = \overline{w}_j, \quad \kappa_{\mathbb{C}}\overline{w}_j = w_j.$$

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Note that in this case U_κ transforms the vacuum into the Slater determinant associated with the subspace $\mathcal{W}$:

$$U_\kappa \Omega = a^*(w_1) \cdots a^*(w_n)\Omega.$$

Moreover, we can easily put U_κ in the Wick-ordered form:

$$\begin{aligned} U_\kappa &= (a^*(w_1) + a(w_1)) \cdots (a^*(w_n) + a(w_n)) \\ &= \sum \mathrm{sgn}(i_1, \dots, i_k)\, a^*(w_{i_1}) \cdots a^*(w_{i_k}) a(w_{j_1}) \cdots a(w_{j_{n-k}}), \end{aligned}$$

where we sum over all $1 \le i_1 < \cdots < i_k \le n$, $1 \le j_1 < \cdots < j_{n-k} \le n$ with $\{i_1, \dots, i_k\} \cup \{j_1, \dots, j_{n-k}\} = \{1, \dots, n\}$, and $\mathrm{sgn}(i_1, \dots, i_k)$ is the sign of the permutation $(i_1, \dots, i_k, j_1, \dots, j_{n-k})$.

16.3.4 Implementation of j-non-degenerate transformations

For j-non-degenerate orthogonal transformations we can write down a formula for its Bogoliubov implementer that is parallel to that of the bosonic case (11.42).

Theorem 16.45 *Let $r \in O_{\mathrm{j}}(\mathcal{Y})$ be* j*-non-degenerate. Let p, c, d be defined as in Subsect. 16.1.1. Set*

$$U_r^{\mathrm{j}} = |\det pp^*|^{\frac{1}{4}} \mathrm{e}^{\frac{1}{2} a^*(d)} \Gamma\big((p^*)^{-1}\big) \mathrm{e}^{\frac{1}{2} a(c)}. \tag{16.55}$$

Then U_r^{j} is the unique unitary operator implementing r such that

$$(\Omega | U_r^{\mathrm{j}} \Omega) > 0. \tag{16.56}$$

We have $\alpha(U_r^{\mathrm{j}}) = U_r^{\mathrm{j}}$. Thus $U_r^{\mathrm{j}} \in Spin_{\mathrm{j}}^{\mathrm{c}}(\mathcal{Y})$.

Proof Let $z \in \mathcal{Z}$. Recall that

$$\Gamma(p) a^*(z) \Gamma(p^{-1}) = a^*(pz), \quad \Gamma(p) a(z) \Gamma(p^{-1}) = a\left(p^{*-1} z\right). \tag{16.57}$$

Using (16.57), (16.42) and (16.43), we obtain

$$\begin{aligned} U_r^{\mathrm{j}} a^*(z) &= \big(a^*(p^{*-1} z - dpc\overline{z}) - a(pc\overline{z})\big) U_r^{\mathrm{j}} \\ &= \big(a^*(pz) + a(q\overline{z})\big) U_r^{\mathrm{j}}, \\ U_r^{\mathrm{j}} a(z) &= \big(a(pz) - a^*(d\overline{pz})\big) U_r^{\mathrm{j}} \\ &= \big(a(pz) + a^*(q\overline{z})\big) U_r^{\mathrm{j}}. \end{aligned}$$

Thus U_r^{j} implements r.

By Lemma 16.44, U_r^{j} is proportional to a unitary operator. By Thm. 16.36 (1),

$$U_r^{\mathrm{j}} \Omega = \Omega_{-d}$$

is of norm 1. Hence, U_r^{j} is unitary. Finally, $(\Omega | U_r^{\mathrm{j}} \Omega) = |\det pp^*|^{\frac{1}{4}} > 0$. $\square$

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16.3.5 End of proof of the Shale–Stinespring theorem

In this subsection we finish the proof of the implementability of the restricted orthogonal group.

Lemma 16.46 *Let $\mathcal{W}$ be an infinite-dimensional subspace of $\mathcal{Z}$, and $\Psi \in \Gamma_{\mathrm{a}}(\mathcal{Z})$ such that*

$$a^*(w)\Psi = 0, \quad w \in \mathcal{W}.$$

Then $\Psi = 0$.

Proof We have

$$a(w)a^*(w) = -a^*(w)a(w) + \|w\|^2 \mathbb{1}, \quad w \in \mathcal{Z}.$$

Hence, for any projection π_n of dimension n with range contained in $\mathcal{W}$, we have

$$\mathrm{e}^{\mathrm{i}t\mathrm{d}\Gamma(\pi_n)}\Psi = \mathrm{e}^{\mathrm{i}tn}\Psi. \tag{16.58}$$

Now suppose that $\dim \mathcal{W} = \infty$. Then we can find a sequence of projections $\pi_n \leq \mathbb{1}_{\mathcal{W}}$ going strongly to an infinite-dimensional projection π. Then the l.h.s. of (16.58) converges to $\mathrm{e}^{\mathrm{i}t\mathrm{d}\Gamma(\pi)}\Psi$ and the r.h.s. has no limit if $\Psi \neq 0$, which is a contradiction. □

Proof of Thm. 16.43. Let $r \in O_{\mathrm{j}}(\mathcal{Y})$. By Prop. 16.20, r has a finite-dimensional singular space. By Prop. 16.18, it can be represented as a product of a j-non-degenerate transformation and a partial conjugation with a finite dimensional singular space. The former is implementable in $B\big(\Gamma_{\mathrm{a}}(\mathcal{Z})\big)$ by Thm. 16.45, and the latter by Subsect. 16.3.3. This proves that r is implementable in $B\big(\Gamma_{\mathrm{a}}(\mathcal{Z})\big)$.

Suppose that $r \in O(\mathcal{Y})$ is implemented by $U \in U\big(\Gamma_{\mathrm{a}}(\mathcal{Z})\big)$. Let $\Psi := U\Omega$. Note that, for any $z \in \mathcal{Z}$, $a(z)\Omega = 0$ and

$$Ua(z)U^* = a(pz) + a^*(q\overline{z}).$$

Therefore,

$$\big(a(pz) + a^*(q\overline{z})\big)\Psi = 0, \quad z \in \mathcal{Z}. \tag{16.59}$$

Assume first that r is j-non-degenerate. Then (16.59) implies

$$\big(a(z) + a^*(d\overline{z})\big)\Psi = 0, \quad z \in \operatorname{Ran} p.$$

Using the fact that $\operatorname{Ran}\overline{p}$ is dense and Thm. 16.36 (2), we obtain that $d \in B^2(\overline{\mathcal{Z}}, \mathcal{Z})$, and hence $r \in O_{\mathrm{j}}(\mathcal{Y})$.

Suppose now that $r \in O(\mathcal{Y})$ is arbitrary. (16.59) yields

$$a^*(z)\Psi = 0, \quad z \in q \operatorname{Ker}\overline{p}. \tag{16.60}$$

By Lemma 16.46, this implies that $q \operatorname{Ker}\overline{p}$ is finite-dimensional. But, for $z \in \operatorname{Ker}\overline{p}$, $\|qz\| = \|z\|$. Hence, $\dim q \operatorname{Ker}\overline{p} = \dim \operatorname{Ker}\overline{p}$. So $\operatorname{Ker} p$ is finite-dimensional.

By Prop. 16.18, we can find a partial conjugation κ such that $r\kappa$ is j-non-degenerate. The dimension of the singular space of κ is $\dim \operatorname{Ker} p$. By Subsect. 16.3.3, κ is implementable. Hence, $r\kappa$ is implementable. By what we have just proven, $r\kappa \in O_{\mathrm{j}}(\mathcal{Y})$. Clearly, $\kappa \in O_{\mathrm{j}}(\mathcal{Y})$. Hence, $r \in O_{\mathrm{j}}(\mathcal{Y})$.

We write r as κr_0, $U_r = U_\kappa U_{r_0}$, where κ is a partial conjugation and r_0 is j-non-degenerate. Then the Gaussian vector $U_{r_0}\Omega$ is an even vector, as seen in the proof of Thm. 16.36. From the form of U_κ given in Subsect. 16.3.3, we see that $U_r\Omega$ is even (resp. odd) if r is such. Hence, $\alpha(U_r) = U_r$ if $\det r = 1$, and $\alpha(U_r) = -U_r$ if $\det r = -1$.

Finally, let us prove (2)(v). Let $r_n \in O_{\mathrm{j}}(\mathcal{Y})$ such that $r_n \to r$. From Thm. 16.43, we know that $U_{r_n r^{-1}} = \pm U_{r_n} U_r^{-1}$. For n large enough, $r_n r^{-1}$ is close to $\mathbb{1}$ in the topology of $O_{\mathrm{j}}(\mathcal{Y})$, hence is j-non-degenerate and belongs to $SO_{\mathrm{j}}(\mathcal{Y})$. From the explicit form of U_r for j-non-degenerate r given in Thm. 16.45, we see that $U_{r_n r^{-1}} \to \mathbb{1}$ strongly, hence $U_{r_n} \to \pm U_r$ strongly. □

16.3.6 One-parameter groups of Bogoliubov transformations

Let $h \in B_{\mathrm{h}}(\mathcal{Z})$, $g \in B^2_{\mathrm{a}}(\overline{\mathcal{Z}}, \mathcal{Z})$. Let $\zeta = \begin{bmatrix} g & h \\ -h^\# & -\overline{g} \end{bmatrix}$. Recall that

$$\mathrm{Op}^{a^*,a}(\zeta) := 2\mathrm{d}\Gamma(h) + a^*(g) + a(g)$$

is a self-adjoint operator. If in addition $h \in B^1(\mathcal{Z})$, then we can use the anti-symmetric quantization to quantize ζ obtaining

$$\mathrm{Op}(\zeta) := 2\mathrm{d}\Gamma(h) + a^*(g) + a(g) - (\operatorname{Tr} h)\mathbb{1}.$$

Let $a \in o(\mathcal{Y})$ be given by

$$a_{\mathbb{C}} = \mathrm{i}\zeta_{\mathbb{C}}\nu_{\mathbb{C}} = \frac{\mathrm{i}}{2}\begin{bmatrix} h & g \\ -\overline{g} & -h^\# \end{bmatrix};$$

see (16.39). Let $r_t = \mathrm{e}^{ta}$ and

$$r_{t\mathbb{C}} = \begin{bmatrix} p_t & q_t \\ \overline{q}_t & \overline{p}_t \end{bmatrix}.$$

For $t \in \mathbb{R}$ such that r_t is non-degenerate, we set

$$d_t := q_t \overline{p}_t^{-1}, \quad c_t := -q_t^\# (p_t^\#)^{-1}.$$

The following formula gives the unitary group generated by $\mathrm{Op}^{a^*,a}(\zeta)$:

Theorem 16.47 (1) *Let $t \in \mathbb{R}$ be such that r_t is non-degenerate. Then $p_t\mathrm{e}^{-\mathrm{i}th} - \mathbb{1} \in B^1(\mathcal{Z})$, $d_t, c_t \in B^2(\overline{\mathcal{Z}}, \mathcal{Z})$, and*

$$\mathrm{e}^{\mathrm{i}t\mathrm{Op}^{a^*,a}(\zeta)} = \det\big(p_t\mathrm{e}^{-\mathrm{i}th}\big)^{\frac{1}{2}} \mathrm{e}^{\frac{1}{2}a^*(d_t)}\Gamma(p_t^{*-1})\mathrm{e}^{\frac{1}{2}a(c_t)}. \tag{16.61}$$

Besides, (16.61) implements r_t.

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(2) *If in addition $h \in B^1(\mathcal{Z})$, then*

$$\mathrm{e}^{\mathrm{i}t\mathrm{Op}(\zeta)} = \det p_t^{\frac{1}{2}} \mathrm{e}^{\frac{1}{2}a^*(d_t)} \Gamma(p_t^{*-1}) \mathrm{e}^{\frac{1}{2}a(c_t)}. \tag{16.62}$$

(In both (16.61) and (16.62) the branch of the square root is determined by continuity.)

16.3.7 Implementation of j-non-degenerate j-positive transformations

In this subsection we consider a j-non-degenerate j-positive orthogonal transformation r, considered in Subsect. 16.1.3. From formula (16.12) we see that there exists $c \in B_{\mathrm{a}}^2(\overline{\mathcal{Z}}, \mathcal{Z})$ and

$$r_{\mathbb{C}} = \begin{bmatrix} \mathbb{1} & c \\ 0 & \mathbb{1} \end{bmatrix} \begin{bmatrix} (\mathbb{1} + cc^*)^{\frac{1}{2}} & 0 \\ 0 & (\mathbb{1} + c^*c)^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \mathbb{1} & 0 \\ -c^* & \mathbb{1} \end{bmatrix}.$$

By Thm. 16.45, r is then implemented by

$$U_r^{\mathrm{j}} = \det(\mathbb{1} + cc^*)^{-1/4} \mathrm{e}^{\frac{1}{2}a^*(c)} \Gamma(\mathbb{1} + cc^*)^{\frac{1}{2}} \mathrm{e}^{\frac{1}{2}a(c)}. \tag{16.63}$$

We recall also that $a \in o(\mathcal{Y})$ is j-self-adjoint iff

$$a_{\mathbb{C}} = \mathrm{i} \begin{bmatrix} 0 & g \\ g^* & 0 \end{bmatrix}, \tag{16.64}$$

for $g \in Cl_{\mathrm{a}}(\overline{\mathcal{Z}}, \mathcal{Z})$. Clearly, $r = \mathrm{e}^a \in O_{\mathrm{j}}(\mathcal{Y})$ iff $a \in B^2(\mathcal{Y})$, i.e. $g \in B_{\mathrm{a}}^2(\overline{\mathcal{Z}}, \mathcal{Z})$. For such a, we obtain an implementable one-parameter group of j-non-degenerate j-positive orthogonal transformations $\mathbb{R} \ni t \mapsto \mathrm{e}^{ta} = r_t$. On the quantum level this corresponds to

$$\begin{aligned} U_{r_t}^{\mathrm{j}} &= \mathrm{e}^{\frac{\mathrm{i}}{2}\left(a^*(g) + a(g)\right)} \qquad (16.65) \\ &= \left(\det \cos(t\sqrt{gg^*})\right)^{\frac{1}{2}} \mathrm{e}^{\frac{\mathrm{i}t}{2}a^*\left(\frac{\tan\sqrt{gg^*}}{\sqrt{gg^*}}g\right)} \Gamma\left(\cos(t\sqrt{gg^*})\right)^{-1} \mathrm{e}^{-\frac{\mathrm{i}t}{2}a\left(\frac{\tan\sqrt{gg^*}}{\sqrt{gg^*}}g\right)}. \end{aligned}$$

Clearly, (16.65) is essentially a special case of (16.61).

16.3.8 Pin group in the Fock representation

Recall that in Subsect. 14.3.2 for an arbitrary Euclidean space $\mathcal{Y}$ we defined the group $Pin_1(\mathcal{Y})$ satisfying the exact sequence

$$1 \to Pin_1(\mathcal{Y}) \to O_1(\mathcal{Y}) \to \mathbb{Z}_2 \to 1. \tag{16.66}$$

We also defined the group $Pin_1^{\mathrm{c}}(\mathcal{Y})$, which satisfied

$$1 \to Pin_1^{\mathrm{c}}(\mathcal{Y}) \to O_1(\mathcal{Y}) \to U(1) \to 1. \tag{16.67}$$

We had the property

$$1 \to Pin_1(\mathcal{Y}) \to Pin_1^{\mathrm{c}}(\mathcal{Y}) \to U(1) \to 1. \tag{16.68}$$

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Recall that both $\mathrm{CAR}^{C^*}(\mathcal{Y})$ and $\mathrm{Cliff}^{C^*}(\mathcal{Y})$ can be embedded in $B\big(\Gamma_\mathrm{a}(\mathcal{Z})\big)$ (where $\mathcal{Y} = \mathrm{Re}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$). Hence, we can embed $Pin_1^\mathrm{c}(\mathcal{Y})$ and $Pin_1(\mathcal{Y})$ in $U\big(\Gamma_\mathrm{a}(\mathcal{Z})\big)$. It is natural to ask whether both these groups have natural extensions in the Fock representation.

The group $Pin_\mathrm{j}^\mathrm{c}(\mathcal{Y})$ defined in Def. 16.39 is, in some sense, the maximal extension of $Pin_1^\mathrm{c}(\mathcal{Y})$. In this subsection we will construct the group $Pin_{\mathrm{j,af}}(\mathcal{Y})$, which can be viewed as the maximal extension of $Pin_1(\mathcal{Y})$,

The analog of (16.68) will not however be true for $Pin_\mathrm{j}^\mathrm{c}(\mathcal{Y})$ and $Pin_{\mathrm{j,af}}(\mathcal{Y})$ if $\mathcal{Y}$ is infinite-dimensional. In fact, in this case the factor group $Pin_\mathrm{j}^\mathrm{c}(\mathcal{Y})/Pin_{\mathrm{j,af}}(\mathcal{Y})$ is much larger than $U(1)$. In quantum field theory this is responsible for the so-called *anomalies* – symmetries of the classical system that cannot be lifted to the quantum level.

The definition of the group $Pin_{\mathrm{j,af}}(\mathcal{Y})$ is somewhat complicated. We first define its j-non-degenerate elements. Then we use the representation-independent construction, which we discussed in the definition of $Pin_1(\mathcal{Y})$ in Subsect. 14.3.2, to handle j-degenerate elements.

Definition 16.48 *Let $r \in O_{\mathrm{j,af}}(\mathcal{Y})$ be* j*-non-degenerate and p, c, d be given by Sect. 16.1. We define the pair of operators*

$$U_r = \pm(\det p^*)^{\frac{1}{2}} \mathrm{e}^{\frac{1}{2} a^*(d)} \Gamma(p^{*-1}) \mathrm{e}^{\frac{1}{2} a(c)}. \tag{16.69}$$

$Pin_{\mathrm{j,af}}(\mathcal{Y})$ is defined as the set of operators $\pm U_t U_s$ in $U\big(\Gamma_\mathrm{a}(\mathcal{Z})\big)$, where $t \in O_{\mathrm{j,af}}(\mathcal{Y})$ is j*-non-degenerate, $s \in O_1(\mathcal{Y})$, $\pm U_t$ is defined as in (16.69) and U_s is defined as in Subsect. 14.3.2. We set $Spin_{\mathrm{j,af}}(\mathcal{Y}) := Pin_{\mathrm{j,af}}(\mathcal{Y}) \cap Spin_\mathrm{j}(\mathcal{Y})$.*

Theorem 16.49 *$Pin_{\mathrm{j,af}}(\mathcal{Y})$ is a subgroup of $Pin_\mathrm{j}^\mathrm{c}(\mathcal{Y})$. $Spin_{\mathrm{j,af}}(\mathcal{Y})$ is a subgroup of $Spin_\mathrm{j}^\mathrm{c}(\mathcal{Y})$. $Pin_\mathrm{j}^\mathrm{c}(\mathcal{Y}) \to O_\mathrm{j}(\mathcal{Y})$ restricts to a surjective homomorphism $Pin_{\mathrm{j,af}}(\mathcal{Y}) \to O_{\mathrm{j,af}}(\mathcal{Y})$. The pre-image of each $r \in O_{\mathrm{j,af}}(\mathcal{Y})$ consists of precisely two elements of $Pin_{\mathrm{j,af}}(\mathcal{Y})$ differing by the sign, which will be denoted by $\pm U_r$.*

The above statements can be summarized by the following commuting diagram of groups and their continuous homomorphisms, where all vertical and horizontal sequences are exact:

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \to & \mathbb{Z}_2 & \to & \mathbb{Z}_2 & \to & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & Spin_{\mathrm{j,af}}(\mathcal{Y}) & \to & Pin_{\mathrm{j,af}}(\mathcal{Y}) & \to & \mathbb{Z}_2 & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & SO_{\mathrm{j,af}}(\mathcal{Y}) & \to & O_{\mathrm{j,af}}(\mathcal{Y}) & \to & \mathbb{Z}_2 & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array} \tag{16.70}$$

Furthermore, if $r \in O_1(\mathcal{Y})$, then $\pm U_r$ defined in Subsect. 14.3.2 coincides with $\pm U_r$ defined in Thm. 16.49.

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The proof of the above theorem is divided into a sequence of steps.

Lemma 16.50 *Suppose that $r \in O_{\mathrm{j,af}}$ is* j*-non-degenerate. Then U_r defined in Subsect. 14.3.2 coincides with $\pm U_r$ defined in (16.69). In particular, we have the following cases:*

(1) *If $w \in U_1(\mathcal{Y}^{\mathbb{C}})$, so that we can write $w_{\mathbb{C}} = \begin{bmatrix} u & 0 \\ 0 & \overline{u} \end{bmatrix}$ for $u \in U_1(\mathcal{Z})$, then*

$$U_w = \pm(\det u)^{\frac{1}{2}}\Gamma(u). \tag{16.71}$$

(2) *If r is* j*-non-degenerate and* j*-positive, so that we can write*
$r_{\mathbb{C}} = \begin{bmatrix} 1\!\!1 & c \\ 0 & 1\!\!1 \end{bmatrix}\begin{bmatrix} p^{-1} & 0 \\ 0 & \overline{p} \end{bmatrix}\begin{bmatrix} 1\!\!1 & 0 \\ -c^* & 1\!\!1 \end{bmatrix}$ *for $c \in B^2_{\mathrm{a}}(\overline{\mathcal{Z}}, \mathcal{Z})$, $p = (1\!\!1 + cc^*)^{-\frac{1}{2}}$, then*

$$\pm U_r = \pm(\det p)^{\frac{1}{2}}\mathrm{e}^{\frac{1}{2}a^*(c)}\Gamma(p^{-1})\mathrm{e}^{\frac{1}{2}a(c)} = \pm U_r^{\mathrm{j}}.$$

Proof Consider first (1). We can find $h \in B^1_{\mathrm{h}}(\mathcal{Z})$ such that $u = \mathrm{e}^{\mathrm{i}h}$. Then $w = \mathrm{e}^a$ with $a_{\mathbb{C}} = \mathrm{i}\begin{bmatrix} h & 0 \\ 0 & -h^{\#} \end{bmatrix}$. By Prop. 14.27, $U_w = \pm\mathrm{e}^{\frac{1}{4}\mathrm{Op}(a\nu^{-1})}$. But

$$\frac{1}{4}(a\nu^{-1})_{\mathbb{C}} = \frac{\mathrm{i}}{2}\begin{bmatrix} 0 & h \\ -h^{\#} & 0 \end{bmatrix}, \quad \mathrm{Op}(\frac{1}{4}a\nu^{-1}) = \mathrm{id}\Gamma(h) - \frac{\mathrm{i}}{2}\mathrm{Tr}\, h.$$

Thus

$$\pm U_w = \pm\mathrm{e}^{\mathrm{id}\Gamma(h) - \frac{\mathrm{i}}{2}\mathrm{Tr}\, h} = \pm(\det \mathrm{e}^{-\mathrm{i}h})^{\frac{1}{2}}\Gamma(\mathrm{e}^{\mathrm{i}h}) = \pm(\det u^*)^{\frac{1}{2}}\Gamma(u).$$

Let us prove (2). Let $r \in O_1(\mathcal{Y})$ be j-non-degenerate and j-positive. We can find $g \in B^1_{\mathrm{a}}(\overline{\mathcal{Z}}, \mathcal{Z})$ such that $a_{\mathbb{C}} = \mathrm{i}\begin{bmatrix} 0 & g \\ g^* & 0 \end{bmatrix}$ and $r = \mathrm{e}^a$. We have

$$\frac{1}{4}(a\nu^{-1})_{\mathbb{C}} = \frac{\mathrm{i}}{2}\begin{bmatrix} g & 0 \\ 0 & g^* \end{bmatrix}, \quad \mathrm{Op}(\frac{1}{4}a\nu^{-1}) = \frac{\mathrm{i}}{2}(a^*(g) + a(g)).$$

By Prop. 14.27, $\pm U_r^{\mathrm{j}} = \pm\mathrm{e}^{\mathrm{Op}(\frac{1}{4}a\nu^{-1})}$, and by (16.65),

$$\pm\mathrm{e}^{\mathrm{Op}(\frac{1}{4}a\nu^{-1})} = \pm(\det p)^{\frac{1}{2}}\mathrm{e}^{\frac{1}{2}a^*(c)}\Gamma(p^{-1})\mathrm{e}^{\frac{1}{2}a(c)} = \pm U_r^{\mathrm{j}}.$$

If r is an arbitrary j-non-degenerate element of $O_1(\mathcal{Y})$, by Thm. 16.13, we can write $r = wr_0$ with w unitary and r_0 j-non-degenerate and j-positive. By the proof of Thm. 16.13, $r_0, w \in O_1(\mathcal{Y})$. Then with

$$w_{\mathbb{C}} = \begin{bmatrix} u & 0 \\ 0 & \overline{u} \end{bmatrix}, \quad (r_0)_{\mathbb{C}} = \begin{bmatrix} p_0 & q_0 \\ \overline{q}_0 & \overline{p}_0 \end{bmatrix},$$

we have

$$r_{\mathbb{C}} = (wr_0)_{\mathbb{C}} = \begin{bmatrix} up_0 & uq_0 \\ \overline{uq_0} & \overline{up_0} \end{bmatrix}.$$

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Using (1) and (2), we obtain

$$\begin{aligned}\pm U_r \ = \ \pm U_w U_{r_0} &= \pm(\det u^*)^{\frac12}\Gamma(u)(\det p_0)^{\frac12}\mathrm{e}^{\frac12 a^*(c_0)}\Gamma(p_0^{-1})\mathrm{e}^{\frac12 a(c_0)}\\ &= \pm\big(\det(up_0)^*\big)^{\frac12}\mathrm{e}^{\frac12 a^*(uc_0)}\Gamma\big((up_0)^{*-1}\big)\mathrm{e}^{\frac12 a(c_0)}. \quad (16.72)\end{aligned}$$

But $up_0 = p$, $uc_0 = d$ and $c_0 = c$, hence (16.72) coincides with (16.69). □

Lemma 16.51 *Let $r \in O_{\mathrm{j,af}}(\mathcal{Y})$ be* j*-non-degenerate. Then (16.69) defines a pair of even unitary operators differing by a sign implementing r. $\pm U_r$ depends continuously on $r \in O_{\mathrm{j,af}}(\mathcal{Y})$, where $O_{\mathrm{j,af}}(\mathcal{Y})$ is equipped with its usual topology.*

Proof To see that $\pm U_r$ implements r, it is enough to note that it is proportional to U_r^{j} defined in (16.55). □

Lemma 16.52 *Let $t, ts \in O_{\mathrm{j,af}}(\mathcal{Y})$ be* j*-non-degenerate and $s \in O_1(\mathcal{Y})$. Then*

$$\pm U_t U_s = \pm U_{ts}, \tag{16.73}$$

where $\pm U_t$, $\pm U_{ts}$ are defined by (16.69) and $\pm U_s$ was defined in Subsect. 14.3.2.

Proof Let $t_n \in O_{\mathrm{j,af}}(\mathcal{Y})$ be a sequence convergent in the metric of $O_{\mathrm{j,af}}(\mathcal{Y})$ to t. Then, for any $s \in O_{\mathrm{j,af}}(\mathcal{Y})$, $t_n s \to ts$ in the same metric.

The set of j-non-degenerate elements is open in $O_{\mathrm{j,af}}(\mathcal{Y})$. Therefore, for sufficiently large indices, t_n and $t_n s_n$ are j-non-degenerate. Hence,

$$\pm U_{t_n} \to \pm U_t, \quad \pm U_{t_n s} \to \pm U_{ts}.$$

Therefore, $\pm U_{t_n} U_s \to \pm U_t U_s$.

Suppose in addition that $s \in O_1(\mathcal{Y})$. Since $O_1(\mathcal{Y})$ is dense in $O_{\mathrm{j,af}}(\mathcal{Y})$, we can demand that $t_n \in O_1(\mathcal{Y})$ Therefore, $\pm U_{t_n} U_s = \pm U_{t_n s}$. □

Lemma 16.53 *Let $t_1 s_2 = t_2 s_2$, where $t_i \in O_{\mathrm{j,af}}(\mathcal{Y})$ are* j*-non-degenerate and $s_i \in O_1(\mathcal{Y})$. Then*

$$\pm U_{t_1} U_{s_1} = \pm U_{t_2} U_{s_2}, \tag{16.74}$$

where $\pm U_{t_i}$ are defined by (16.69) and $\pm U_{s_i}$ were defined in Subsect. 14.3.2.

Proof We have $t_1(s_1 s_2^{\#}) = t_2$, and hence, by Lemma 16.52,

$$\pm U_{t_1} U_{s_1 s_2^{\#}} = \pm U_{t_2}.$$

But $\pm U_{s_1 s_2^{\#}} = \pm U_{s_1} U_{s_2}^*$, because $s_1, s_2 \in O_1(\mathcal{Y})$. □

Proof of Thm. 16.49. We know by Lemmas 16.27 and 16.7 that every $r \in O_{\mathrm{j,af}}(\mathcal{Y})$ can be written as $r = ts$, where $t \in O_{\mathrm{j,af}}(\mathcal{Y})$ is j-non-degenerate and $s \in O_1(\mathcal{Y})$. By Lemma 16.53,

$$\pm U_r := \pm U_t U_s,$$

where U_t is defined as in (16.69) and $\pm U_s$ were defined in Subsect. 14.3.3, does not depend on the decomposition.

For $s \in O_{\mathrm{j,af}}(\mathcal{Y})$, set

$$\mathcal{U}_s := \{r \in O_{\mathrm{j,af}}(\mathcal{Y}) \ : \ rs \text{ is j-non-degenerate}\}.$$

Clearly, $\mathcal{U}_s$ are open in $O_{\mathrm{j,af}}(\mathcal{Y})$. Besides, by Lemma 16.51,

$$\mathcal{U}_{1\!\!1} \ni r \mapsto \pm U_r \in Pin_{\mathrm{j,af}}(\mathcal{Y})/\{1\!\!1, -1\!\!1\} \tag{16.75}$$

is continuous. Using Def. 16.48 and the continuity of multiplication in $O_{\mathrm{j,af}}(\mathcal{Y})$, we see that, for $s \in O_1(\mathcal{Y})$,

$$\mathcal{U}_s \ni r \mapsto \pm U_r \in Pin_{\mathrm{j,af}}(\mathcal{Y})/\{1\!\!1, -1\!\!1\} \tag{16.76}$$

is also continuous. But $\mathcal{U}_s$ with $s \in O_1(\mathcal{Y})$ cover $O_{\mathrm{j,af}}(\mathcal{Y})$. Hence,

$$O_{\mathrm{j,af}} \ni r \mapsto \pm U_r \in Pin_{\mathrm{j,af}}(\mathcal{Y})/\{1\!\!1, -1\!\!1\} \tag{16.77}$$

is continuous.

We know that

$$\pm U_{r_1} U_{r_2} = \pm U_{r_1 r_2} \tag{16.78}$$

is true for $r_1, r_2 \in O_1(\mathcal{Y})$. But $O_1(\mathcal{Y})$ is dense in $O_{\mathrm{j,af}}(\mathcal{Y})$. Hence, (16.78) holds for $r_1, r_2 \in O_{\mathrm{j,af}}(\mathcal{Y})$. This proves that $Pin_{\mathrm{j,af}}(\mathcal{Y}) \to O_{\mathrm{j,af}}(\mathcal{Y})$ is a homomorphism. □

As an exercise, we give an alternative proof of the group property of $O_{\mathrm{j,af}}(\mathcal{Y})$ restricted to j-non-degenerate elements.

Lemma 16.54 *Let $r = r_1 r_2$ with $r_1, r_2 \in O_{\mathrm{j,af}}(\mathcal{Y})$. Assume that r, r_1, r_2 are j-non-degenerate. Then*

$$U_{r_1} U_{r_2} = \pm U_{r_1 r_2}. \tag{16.79}$$

Proof We know that

$$(\Omega | U_{r_1 r_2} \Omega) = \pm (\det p^*)^{\frac{1}{2}} = \pm \big(\det(p_1 p_2 + q_1 \overline{q}_2)^*\big)^{\frac{1}{2}}. \tag{16.80}$$

Moreover,

$$\begin{aligned}
(\Omega | U_{r_1} U_{r_2} \Omega) &= \pm \big(\mathrm{e}^{-\frac{1}{2} a^*(c_1)} \Omega | \mathrm{e}^{\frac{1}{2} a^*(d_2)} \Omega\big) (\det p_1^*)^{\frac{1}{2}} (\det p_2^*)^{\frac{1}{2}} \\
&= \pm \det(1\!\!1 + d_2 c_1^*)^{\frac{1}{2}} (\det p_1^*)^{\frac{1}{2}} (\det p_2^*)^{\frac{1}{2}} \\
&= \pm \big(\det(p_1 p_2 + q_1 \overline{q}_2)^*\big)^{\frac{1}{2}}.
\end{aligned}$$

Hence,

$$(\Omega | U_{r_1} U_{r_2} \Omega) = \pm (\Omega | U_{r_1 r_2} \Omega).$$

We know that $U_{r_1} U_{r_2}$ and $U_{r_1 r_2}$ implement $r_1 r_2$ and the representation is irreducible. Hence, (16.79) is true. □

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16.4 Fock sector of a CAR representation

The main result of this section is a necessary and sufficient criterion for two Fock CAR representations to be unitarily equivalent. This result goes under the name *Shale–Stinespring theorem*, and is closely related to Thm. 16.43 about the implementability of fermionic Bogoliubov transformations, which can be viewed as another version of the Shale–Stinespring theorem.

Another, closely related, subject of this chapter can be described as follows. We consider a Euclidean space $\mathcal{Y}$ and a CAR representation in a Hilbert space $\mathcal{H}$. We suppose that we are given a Kähler anti-involution j. We will describe how to find a subspace of $\mathcal{H}$ on which this representation is unitarily equivalent to the Fock CAR representation associated with j.

Throughout the section, $(\mathcal{Y}, \nu)$ is a real Hilbert space and j is a Kähler anti-involution on $\mathcal{Y}$.

We use the notation and results of Subsects. 1.3.6, 1.3.8 and 1.3.9. As usual, $\mathcal{Z}$, $\overline{\mathcal{Z}}$ are the holomorphic and anti-holomorphic subspaces of $\mathbb{C}\mathcal{Y}$. Recall that $\mathcal{Y}$ is identified with $\mathrm{Re}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$ by

$$\mathcal{Y} \ni y \mapsto \Big(\frac{1}{2}(y - \mathrm{i}\mathrm{j}y), \frac{1}{2}(y + \mathrm{i}\mathrm{j}y)\Big) \in \mathrm{Re}(\mathcal{Z} \oplus \overline{\mathcal{Z}}).$$

We equip $\mathcal{Z}$ with the unitary structure associated with 2ν and j.

16.4.1 Vacua of CAR representations

Let

$$\mathcal{Y} \ni y \mapsto \phi^{\pi}(y) \in B(\mathcal{H})$$

be a representation of CAR over $(\mathcal{Y}, \nu)$. Recall that by complex linearity we extend the definition of $\phi^{\pi}(y)$ to arguments in $\mathbb{C}\mathcal{Y} = \mathcal{Z} \oplus \overline{\mathcal{Z}}$. As in Subsect. 12.1.6, we introduce the creation, resp. annihilation operators $a^{\pi *}(z)$, resp. $a^{\pi}(z)$ by

$$a^{\pi *}(z) := \phi^{\pi}(z), \quad a^{\pi}(z) := \phi^{\pi}(\overline{z}), \quad z \in \mathcal{Z}. \tag{16.81}$$

As in the bosonic case, sometimes we will call them j-*creation*, resp. j-*annihilation operators*.

Definition 16.55 *We define the space of* j-vacua *as*

$$\mathcal{K}^{\pi} := \big\{\Psi \in \mathcal{H} \;:\; a^{\pi}(z)\Psi = 0, \quad z \in \mathcal{Z}\big\}.$$

Theorem 16.56 (1) $\mathcal{K}^{\pi}$ *is a closed subspace of* $\mathcal{H}$.
(2) *If* $\Phi, \Psi \in \mathcal{K}^{\pi}$, *then*

$$\big(\Phi|\phi^{\pi}(y_1)\phi^{\pi}(y_2)\Psi\big) = (\Phi|\Psi)(y_1 \cdot \nu y_2 - \mathrm{i} y_1 \cdot \nu \mathrm{j} y_2), \quad y_1, y_2 \in \mathcal{Y}.$$

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Proof We will suppress the superscript π to simplify notation.

$\mathcal{K}$ is closed as the intersection of kernels of bounded operators. To prove (2), we set $(z_i, \overline{z}_i) = y_i$, so that $\phi(y_i) = a^*(z_i) + a(z_i)$. Using the CAR, we obtain

$$\big(\Phi|\phi(y_1)\phi(y_2)\Psi\big) = (z_1|z_2)(\Phi|\Psi).$$

Since $(z_1|z_2) = y_1\cdot\nu y_2 - \mathrm{i}y_1\cdot\nu\mathrm{j}y_2$, we obtain (2). □

16.4.2 Fock CAR representations

Recall that in Sect. 3.4, for $z \in \mathcal{Z}$, we introduced creation, resp. annihilation operators $a^*(z)$, resp. $a(z)$ acting on the fermionic Fock space $\Gamma_{\mathrm{a}}(\mathcal{Z})$. We have a CAR representation

$$\mathcal{Y} \ni y \mapsto \phi(y) = a^*(z) + a(z) \in B_{\mathrm{h}}\big(\Gamma_{\mathrm{a}}(\mathcal{Z})\big), \quad y = (z, \overline{z}). \tag{16.82}$$

As in Def. 13.4, we call (16.82) the *Fock CAR representation.*

Note that j-creation, resp. j-annihilation operators defined for the CAR representation (16.82) coincide with the usual creation, resp. annihilation operators $a^*(z)$, resp. $a(z)$. Likewise, a vector $\Psi \in \Gamma_{\mathrm{a}}(\mathcal{Z})$ is a j-vacuum for (16.82) iff it is proportional to Ω.

We can also consider another Kähler anti-involution j_1, not necessarily equal to j. The following theorem describes the vacua in $\Gamma_{\mathrm{a}}(\mathcal{Z})$ corresponding to j_1 . It is essentially a restatement of parts of Thm. 16.36.

Theorem 16.57 (1) *Let $c \in B_{\mathrm{a}}^2(\overline{\mathcal{Z}}, \mathcal{Z})$. Let j_1 be the Kähler anti-involution determined by c, as in Subsect. 16.1.10. Then Ω_c is the unique vector satisfying the following conditions:*

(i) $\|\Omega_c\| = 1$,

(ii) $(\Omega_c|\Omega) > 0$,

(iii) Ω_c *is a vacuum for* j_1.

(2) *The statement (1)(iii) is equivalent to*

$$\big(a(z) - a^*(c\overline{z})\big)\Omega_c = 0, \quad z \in \mathcal{Z}. \tag{16.83}$$

16.4.3 Unitary equivalence of Fock CAR representations

Suppose that we are given a real Hilbert space $(\mathcal{Y}, \nu)$ endowed with two Kähler structures, defined e.g. by two Kähler anti-involutions. Each Kähler structure determines the corresponding Fock CAR representation. In this subsection we will prove a necessary and sufficient condition for the equivalence of these two representations.

Theorem 16.58 (The Shale–Stinespring theorem about Fock representations) *Let $\mathcal{Z}$, $\mathcal{Z}_1$ be the holomorphic subspaces of $\mathbb{C}\mathcal{Y}$ corresponding to the Kähler*

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anti-involutions j *and* j_1. *Let*

$$\mathcal{Y} \ni y \mapsto \phi(y) \in B_{\mathrm{h}}\big(\Gamma_{\mathrm{a}}(\mathcal{Z})\big), \tag{16.84}$$

$$\mathcal{Y} \ni y \mapsto \phi_1(y) \in B_{\mathrm{h}}\big(\Gamma_{\mathrm{a}}(\mathcal{Z}_1)\big) \tag{16.85}$$

be the corresponding Fock representations of CAR. Then the following statements are equivalent:

(1) *There exists a unitary operator* $W : \Gamma_{\mathrm{a}}(\mathcal{Z}) \to \Gamma_{\mathrm{a}}(\mathcal{Z}_1)$ *such that*

$$W\phi(y) = \phi_1(y)W, \quad y \in \mathcal{Y}. \tag{16.86}$$

(2) $\mathrm{j} - \mathrm{j}_1$ *is Hilbert–Schmidt (or any of the equivalent conditions of Prop. 16.31 hold).*

Proof Let a_1^*, a_1, Ω_1 denote the creation and annihilation operators and the vacuum for the representation (16.85).

(2)⇒(1). Assume that $\mathrm{j} - \mathrm{j}_1 \in B^2(\mathcal{Y})$. We know, by Prop. 16.31 (4), that there exists $r \in O_{\mathrm{j}}(\mathcal{Y})$ such that $\mathrm{j}_1 = r\mathrm{j}r^{\#}$. Thus, by Thm. 16.43, there exists $U_r \in U\big(\Gamma_{\mathrm{a}}(\mathcal{Z})\big)$ such that $U_r\phi(y)U_r^* = \phi(ry)$.

Note that $r_{\mathbb{C}} \in U(\mathbb{C}\mathcal{Y})$ and $r_{\mathbb{C}}\mathcal{Z} = \mathcal{Z}_1$. Set $u := r_{\mathbb{C}}\big|_{\mathcal{Z}}$. Then $u \in U(\mathcal{Z}, \mathcal{Z}_1)$. Note that $\Gamma(u) \in U\big(\Gamma_{\mathrm{a}}(\mathcal{Z}), \Gamma_{\mathrm{a}}(\mathcal{Z}_1)\big)$, and

$$\Gamma(u)a^*(z)\Gamma(u)^* = a_1^*(uz), \quad \Gamma(u)a(z)\Gamma(u)^* = a_1(uz), \quad z \in \mathcal{Z}.$$

Hence, $\Gamma(u)\phi(y)\Gamma(u)^* = \phi_1(ry)$. Therefore, $W := \Gamma(u)U_r^*$ satisfies (16.86).

(1)⇒(2). Suppose that the representations (16.84) and (16.85) are equivalent with the help of the operator $W \in U\big(\Gamma_{\mathrm{a}}(\mathcal{Z}_1), \Gamma_{\mathrm{a}}(\mathcal{Z})\big)$. Let $\mathcal{Y}_{\mathrm{sg}} := \mathrm{Ker}(\mathrm{j} + \mathrm{j}_1)$ and $\mathcal{Y}_{\mathrm{reg}} := \mathcal{Y}_{\mathrm{sg}}^{\perp}$. Let $\mathcal{Z}_{\mathrm{sg}} := \frac{\mathbb{1} - \mathrm{i}\mathrm{j}_{\mathbb{C}}}{2}\mathcal{Y}_{\mathrm{sg}}$, $\mathcal{Z}_{\mathrm{reg}} := \frac{\mathbb{1} - \mathrm{i}\mathrm{j}_{\mathbb{C}}}{2}\mathcal{Y}_{\mathrm{reg}}$.

Clearly,

$$a_1(w)\Omega_1 = 0, \quad w \in \mathcal{Z}_1,$$

and

$$Wa_1(w)W^* = W\phi_1(\overline{w})W^* = \phi(\overline{w}), \quad w \in \mathcal{Z}_1.$$

Hence,

$$\phi(\overline{w})W\Omega_1 = 0, \quad w \in \mathcal{Z}_1.$$

Hence, in particular,

$$a^*(z)W\Omega_1 = 0, \quad z \in \mathcal{Z}_{\mathrm{sg}}.$$

Lemma 16.46 implies that $\mathcal{Z}_{\mathrm{sg}}$ is finite-dimensional. Let $(w_1, \dots, w_n)$ be an o.n. basis of $\mathcal{Z}_{\mathrm{sg}}$. Set $\Psi := a^*(w_1) \cdots a^*(w_n)W\Omega_1$. Let $c \in Cl_{\mathrm{a}}(\overline{\mathcal{Z}}, \mathcal{Z})$ be defined as in (16.30). Then

$$\big(a(z) - a^*(c\overline{z})\big)\Psi = 0, \quad z \in \mathcal{Z}_{\mathrm{reg}},$$
$$a(z)\Psi = 0, \quad z \in \mathcal{Z}_{\mathrm{sg}}.$$

By Thm. 16.36 (2), this implies that $c \in B^2(\overline{\mathcal{Z}}, \mathcal{Z})$. Hence, $\mathrm{j} - \mathrm{j}_1 \in B^2(\mathcal{Y})$. □

16.4.4 Fock sector of a CAR representation

Theorem 16.59 *Set*

$$\mathcal{H}^\pi := \mathrm{Span}^{\mathrm{cl}}\Big\{ \prod_{i=1}^{n} a^{\pi *}(z_i)\Psi \ : \ \Psi \in \mathcal{K}^\pi, \ z_1, \dots, z_n \in \mathcal{Z}, \ n \in \mathbb{N} \Big\}. \qquad (16.87)$$

(1) $\mathcal{H}^\pi$ *is invariant under* $\phi^\pi(y)$, $y \in \mathcal{Y}$.

(2) *There exists a unique unitary operator*

$$U^\pi : \mathcal{K}^\pi \otimes \Gamma_{\mathrm{a}}(\mathcal{Z}) \to \mathcal{H}^\pi$$

satisfying

$$U^\pi \Psi \otimes a^*(z_1) \cdots a^*(z_n)\Omega = a^{\pi *}(z_1) \cdots a^{\pi *}(z_n)\Psi, \ \ \Psi \in \mathcal{K}^\pi, \quad z_1, \dots, z_n \in \mathcal{Z}.$$

(3)

$$U^\pi \mathbb{1} \otimes \phi(y) = \phi^\pi(y) U^\pi, \quad y \in \mathcal{Y}.$$

(4) *If there exists an isometry* $U : \Gamma_{\mathrm{a}}(\mathcal{Z}) \to \mathcal{H}$ *such that* $U\phi(y) = \phi^\pi(y)U$, $y \in \mathcal{Y}$, *then* $\mathrm{Ran}\, U \subset \mathcal{H}^\pi$.

(5) $\mathcal{H}^\pi$ *depends on* j *only through its equivalence class w.r.t. the relation*

$$\mathrm{j}_1 \sim \mathrm{j}_2 \ \Leftrightarrow \ \mathrm{j}_1 - \mathrm{j}_2 \in B^2(\mathcal{Y}). \qquad (16.88)$$

Definition 16.60 *Introduce the equivalence relation (16.88) in the set of Kähler anti-involutions on* $\mathcal{Y}$. *Let* $[\mathrm{j}]$ *denote the equivalence class w.r.t. this relation. Then the space* $\mathcal{H}^\pi$ *defined in (16.87) is called the* $[\mathrm{j}]$-Fock sector of the representation ϕ^π.

Proof of Thm. 16.59. Clearly, $\mathcal{H}^\pi$ is invariant under $\phi^\pi(y)$, $y \in \mathcal{Y}$. We define $U^\pi : \mathcal{K}^\pi \otimes \Gamma_{\mathrm{a}}^{\mathrm{al}}(\mathcal{Z}) \to \mathcal{H}$ such that the identity in (2) holds. Clearly, U^π is isometric and extends to a unitary map from $\mathcal{K}^\pi \otimes \Gamma_{\mathrm{a}}(\mathcal{Z})$ to $\mathcal{H}^\pi$ satisfying (3). If U is as in (4), then $U\mathbb{C}\Omega \subset \mathcal{K}^\pi$, which shows that $\mathrm{Ran}\, U \subset \mathcal{H}^\pi$. The proof of (5) is identical to the bosonic case. □

As in Subsect. 11.4.4, we have the following proposition:

Proposition 16.61 *Let* j *be a Kähler anti-involution on* $\mathcal{Y}$. *If the CAR representation* ϕ^π *is irreducible and* $\mathcal{K}^\pi \neq \{0\}$, *then* ϕ^π *is unitarily equivalent to the* $[\mathrm{j}]$-*Fock CAR representation.*

16.4.5 Number operator of a CAR representation

As in Subsect. 11.4.5, we discuss the notion of the number operator associated with a CAR representation and a Kähler anti-involution.

Definition 16.62 *We define the* number operator N^π *associated with the CAR representation* ϕ^π *and the Kähler anti-involution* j *by*

$$N^\pi := U^\pi(\mathbb{1} \otimes N)U^{\pi *}, \quad \mathrm{Dom}\, N := U^\pi \mathcal{K}^\pi \otimes \mathrm{Dom}\, N.$$

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As in Subsect. 11.4.5, it is convenient to give an alternative definition of N^π using the number quadratic form.

Definition 16.63 *We define the* number quadratic form n^π *by*

$$n^\pi(\Phi) := \sup_{\mathcal{V}} n^\pi_{\mathcal{V}}(\Phi), \qquad \Phi \in \mathcal{H},$$

where $\mathcal{V}$ runs over finite-dimensional subspaces of $\mathcal{Z}$,

$$n^\pi_{\mathcal{V}}(\Phi) := \sum_{i=1}^{\dim \mathcal{V}} \|a^\pi(v_i)\Phi\|^2,$$

$(v_1, \ldots, v_{\dim \mathcal{V}})$ *being an o.n. basis of $\mathcal{V}$.*

Theorem 16.64 *Let n^π be the number quadratic form associated with W^π,* j. *Then* $\operatorname{Dom} n^\pi = \operatorname{Dom}(N^\pi)^{\frac{1}{2}}$ *and*

$$n^\pi(\Phi) = (\Phi|N^\pi\Phi), \qquad \Phi \in \operatorname{Dom} N^\pi.$$

In particular, $\mathcal{H}^\pi = (\operatorname{Dom} n^\pi)^{\mathrm{cl}}$.

The proof of Thm. 16.64 is completely analogous to Thm. 11.52. Lemmas 11.54 and 11.55 extend to the fermionic case if we replace Lemma 11.53 by the simpler Lemma 16.65 below.

We denote by $\tilde{N}^\pi$ the self-adjoint operator (with a possibly non-dense domain) associated with the quadratic form n^π.

Lemma 16.65 *The operators $a^\pi(z)$ preserve* $(\operatorname{Dom} \tilde{N}^\pi)^{\mathrm{cl}}$, *and if F is a bounded Borel function, one has*

$$a^\pi(z)F(\tilde{N}^\pi - \mathbb{1}) = F(\tilde{N}^\pi)a^\pi(z), \quad z \in \mathcal{Z}.$$

Proof Let us suppress the superscript π to simplify notation. Considering first the quadratic forms $n_{\mathcal{V}}$ for $\mathcal{V}$ finite-dimensional, we easily obtain

$$n(a(z)\Phi) + n(a^*(z)\Phi) = \|z\|^2 n(\Phi) - 2\|a(z)\Phi\|^2 + \|z\|^2\|\Phi\|^2, \quad \Phi \in \operatorname{Dom} n,$$

which implies that $a(z)$, $a^*(z)$ preserve $\operatorname{Dom}(\tilde{N}^{\frac{1}{2}})$. Similarly, we obtain

$$\big(\Phi \mid \tilde{N}a(z)\Psi\big) = \big(\Phi \mid a(z)\tilde{N}\Psi\big) - \big(\Phi \mid a(z)\Psi\big), \quad \Phi, \Psi \in \operatorname{Dom} \tilde{N}^{\frac{1}{2}}.$$

This implies that $a(z) : \operatorname{Dom} \tilde{N} \to \operatorname{Dom} \tilde{N}$ and

$$a(z)(\tilde{N} - \mathbb{1}) = \tilde{N}a(z). \tag{16.89}$$

From (16.89), we get that $a(z)(\tilde{N} - \lambda\mathbb{1})^{-1} = (\tilde{N} + \mathbb{1} - \lambda\mathbb{1})^{-1}a(z)$, which completes the proof of the lemma. □

16.5 Notes

The Shale–Stinespring theorem comes from Shale–Stinespring (1964).

Infinite-dimensional analogs of the Pin representation seem to have been first noted by Lundberg (1976).

Among early works describing implementations of orthogonal transformations on Fock spaces let us mention the books by Berezin (1966) and by Friedrichs (1953). They give concrete formulas for the implementation of Bogoliubov transformations in fermionic Fock spaces. Related problems were discussed, often independently, by other researchers, such as Ruijsenaars (1976, 1978).

A comprehensive monograph about the CAR is the book by Plymen–Robinson (1994).

The book by Neretin (1996) and a review article by Varilly–Gracia-Bondia (1994) describe the infinite-dimensional Pin group.

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