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DISSECTIONS OF QUOTIENTS OF THETA-FUNCTIONS

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We prove a general theorem on dissections of quotients of theta-functions. As corollaries, we establish six q-series identities that were conjectured by M.D. Hirschhorn.

1. INTRODUCTION

An N-dissection of a q-series $F(q)$ with integral powers is a representation of the form

$$F(q) = \sum_{k=0}^{N-1} q^k F_k(q^k),$$

where $F_k(q)$ is a series in q with integral powers.

Ramanujan was most probably the first person to give dissections of q-series identities. In his lost notebook [**6**], he recorded dissections of the generating function of cranks and the generating function of ranks. Since Ramanujan's time, and inspired by his partition congruences, a great deal of work has been done by many people on identities and partition theorems obtained through dissection techniques.

For $|q| < 1$ and any integer n, set

$$(a;q)_n = \prod_{k=0}^{\infty} \frac{1 - aq^k}{1 - aq^{n+k}}$$

and

$$(a;q)_\infty = \lim_{n\to\infty} (a;q)_n.$$

Next, we define Ramanujan's theta-function by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \qquad |ab| < 1, \tag{1.1}$$

which satisfies the Jacobi triple product identity ([**3**, p. 35, Entry 19])

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{1.2}$$

We also define

$$f(-q) := (q;q)_\infty.$$

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Recently, working jointly with Sellers on overpartitions, Hirschhorn conjectured a total of six identities, (3.1) – (3.6) below, which are dissections of quotients of theta-functions. Hirschhorn then communicated these conjectures in [**4**] to the author. Thus our primary aim of this paper is to prove a theorem which yields these six conjectures as special cases.

In Section 2, we prove a general theorem (Theorem 2.1), from which all of (3.1) – (3.6) follow as corollaries. The six conjectures, (3.1) – (3.6) are then stated and proved in Section 3. In Section 4 of this paper, we give an alternative proof of (3.1) and (3.2), which involves the reciprocal of the quintuple product identity (4.1).

2. A general theorem

We prove the following theorem, which is an N-dissection of a quotient of theta-functions, where N is any positive integer.

Theorem 2.1. *Let N be any positive integer. Then for $|q^N| < |x| < 1$, we have*

(2.1)

$$\frac{(-x;q^N)_\infty(-(q^N/x);q^N)_\infty}{(x;q^N)_\infty((q^N/x);q^N)_\infty} = 2\frac{(-q^N;q^N)^2_\infty}{(q^N;q^N)^2_\infty}\sum_{k=0}^{N-1} x^k \frac{(-x^N q^{kN};q^{N^2})_\infty(-q^{N(N-k)}x^{-N};q^{N^2})_\infty(q^{N^2};q^{N^2})^2_\infty}{(x^N;q^{N^2})_\infty(q^{N^2}x^{-N};q^{N^2})_\infty(-q^{Nk};q^{N^2})_\infty(-q^{N(N-k)};q^{N^2})_\infty}.$$

Proof: Recall Ramanujan's famous ${}_1\psi_1$ summation formula

$$\sum_{n=-\infty}^{\infty}\frac{(a;q)_n}{(b;q)_n}z^n = \frac{(az;q)_\infty((q/az);q)_\infty((b/a);q)_\infty(q;q)_\infty}{(z;q)_\infty((b/az);q_\infty)((q/a);q)_\infty(b;q)_\infty},$$

which is valid for $|b/a| < |z| < 1$. (See [**1, 2**], [**3**, p. 32, Entry 17] and [**5**] for proofs.) Letting $(a,b,z) = (y, yq, x)$, we obtain the useful corollary

$$\sum_{n=-\infty}^{\infty}\frac{x^n}{1-yq^n} = \frac{(xy;q)_\infty((q/xy);q)_\infty(q;q)^2_\infty}{(x;q)_\infty((q/x);q)_\infty(y;q)_\infty((q/y);q)_\infty}. \tag{2.2}$$

Next, we specialise (2.2) by letting $y = -1$ and replacing q by q^N, where N is any positive integer, to deduce that

$$\sum_{n=-\infty}^{\infty}\frac{x^n}{1+q^{Nn}} = \frac{(-x;q^N)_\infty(-(q^N/x);q^N)_\infty(q^N;q^N)^2_\infty}{2(x;q^N)_\infty((q^N/x);q^N)_\infty(-q^N;q^N)^2_\infty}. \tag{2.3}$$

We multiply (2.3) by $2\big((-q^N;q^N)^2_\infty\big)/\big((q^N;q^N)^2_\infty\big)$ to obtain

$$\begin{aligned}\frac{(-x;q^N)_\infty(-(q^N/x);q^N)_\infty}{(x;q^N)_\infty((q^N/x);q^N)_\infty} &= 2\frac{(-q^N;q^N)^2_\infty}{(q^N;q^N)^2_\infty}\sum_{n=-\infty}^{\infty}\frac{x^n}{1+q^{Nn}}\\ &= 2\frac{(-q^N;q^N)^2_\infty}{(q^N;q^N)^2_\infty}\sum_{k=0}^{N-1}x^k\sum_{n=-\infty}^{\infty}\frac{x^{Nn}}{1+q^{N^2n+Nk}}.\end{aligned} \tag{2.4}$$

Finally, we apply (2.2) in each of the infinite sums in (2.4), and this completes the proof of the theorem. ∎

3. Conjectures of Hirschhorn and their proofs.

Corollary 3.1. *We have*

$$\sum_{n=-\infty}^{\infty} q^{(3n^2+n)/2} \Big/ \sum_{n=-\infty}^{\infty} (-1)^n q^{(3n^2+n)/2} \tag{3.1}$$
$$= \frac{1}{(q^3;q^{18})_\infty^6 (q^6;q^{18})_\infty^3 (q^{12};q^{18})_\infty^3 (q^{15};q^{18})_\infty^6} + \frac{2q}{(q^3;q^{18})_\infty^5 (q^6;q^{18})_\infty^3 (q^9;q^{18})_\infty^2 (q^{12};q^{18})_\infty^3 (q^{15};q^{18})_\infty^5} + \frac{4q^2}{(q^3;q^{18})_\infty^4 (q^6;q^{18})_\infty^3 (q^9;q^{18})_\infty^4 (q^{12};q^{18})_\infty^3 (q^{15};q^{18})_\infty^4}$$

and

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{(3n^2+n)/2} \Big/ \sum_{n=-\infty}^{\infty} q^{(3n^2+n)/2} \tag{3.2}$$
$$= \frac{1}{(q^3;q^{18})_\infty^2 (q^6;q^{18})_\infty (q^{12};q^{18})_\infty (q^{15};q^{18})_\infty^2} - \frac{2q}{(q^3;q^{18})_\infty (q^6;q^{18})_\infty (q^9;q^{18})_\infty^2 (q^{12};q^{18})_\infty (q^{15};q^{18})_\infty}.$$

Corollary 3.2. *We have*

(3.3)
$$\sum_{n=-\infty}^{\infty} q^{(5n^2+n)/2} \Big/ \sum_{n=-\infty}^{\infty} (-1)^n q^{(5n^2+n)/2}$$
$$= 1/(q^5;q^{50})_\infty^4 (q^{10};q^{50})_\infty^4 (q^{15};q^{50})_\infty^6 (q^{20};q^{50})_\infty (q^{30};q^{50})_\infty (q^{35};q^{50})_\infty^6 (q^{40};q^{50})_\infty^4 (q^{45};q^{50})_\infty^4$$
$$+ 4q^6/(q^5;q^{50})_\infty^4 (q^{10};q^{50})_\infty^2 (q^{15};q^{50})_\infty^4 (q^{20};q^{50})_\infty^3 (q^{25};q^{50})_\infty^4 (q^{30};q^{50})_\infty^3 (q^{35};q^{50})_\infty^4 (q^{40};q^{50})_\infty^2 (q^{45};q^{10})_\infty^4$$
$$+ 2q^2/(q^5;q^{50})_\infty^3 (q^{10};q^{50})_\infty^5 (q^{15};q^{50})_\infty^6 (q^{25};q^{50})_\infty^2 (q^{35};q^{50})_\infty^6 (q^{40};q^{50})_\infty^5 (q^{45};q^{50})_\infty^3$$
$$+ 2q^3/(q^5;q^{50})_\infty^4 (q^{10};q^{50})_\infty^3 (q^{15};q^{50})_\infty^5 (q^{20};q^{50})_\infty^2 (q^{25};q^{50})_\infty^2 (q^{30};q^{50})_\infty^2 (q^{35};q.5^{50})_\infty^5 (q^{40};q^{50})_\infty^3 (q^{45};q^{10})_\infty^4$$
$$+ 2q^4/(q^5;q^{50})_\infty^5 (q^{10};q^{50})_\infty (q^{15};q^{50})_\infty^4 (q^{20};q^{50})_\infty^4 (q^{25};q^{50})_\infty^2 (q^{30};q^{50})_\infty^4 (q^{35};q^{50})_\infty^4 (q^{40};q^{50})_\infty (q^{45};q^{50})_\infty^5,$$

(3.4)
$$\sum_{n=-\infty}^{\infty} q^{(5n^2+3n)/2} \Big/ \sum_{n=-\infty}^{\infty} (-1)^n q^{(5n^2+3n)/2}$$

$$
\begin{aligned}
&= 1/(q^5;q^{50})_\infty^6(q^{10};q^{50})_\infty(q^{15};q^{50})_\infty^4(q^{20};q^{50})_\infty^4(q^{30};q^{50})_\infty^4(q^{35};q^{50})_\infty^4(q^{40};q^{50})_\infty(q^{45};q^{50})_\infty^6 \\
&\quad + 2q/(q^5;q^{50})_\infty^4(q^{10};q^{50})_\infty^4(q^{15};q^{50})_\infty^5(q^{20};q^{50})_\infty(q^{25};q^{50})_\infty^2(q^{30};q^{50})_\infty(q^{35};q^{50})_\infty^5 \\
&\qquad\qquad (q^{40};q^{50})_\infty^4(q^{45};q^{50})_\infty^4 \\
&\quad + 2q^2/(q^5;q^{50})_\infty^5(q^{10};q^{50})_\infty^2(q^{15};q^{50})_\infty^4(q^{20};q^{50})_\infty^3(q^{25};q^{50})_\infty^2(q^{30};q^{50})_\infty^3(q^{35};q^{50})_\infty^4 \\
&\qquad\qquad (q^{40};q^{50})_\infty^2(q^{45};q^{50})_\infty^5 \\
&\quad + 2q^3/(q^5;q^{50})_\infty^6(q^{15};q^{50})_\infty^3(q^{20};q^{50})_\infty^5(q^{25};q^{50})_\infty^2(q^{30};q^{50})_\infty^5(q^{35};q^{50})_\infty^3(q^{45};q^{50})_\infty^6 \\
&\quad + 4q^4/(q^5;q^{50})_\infty^4(q^{10};q^{50})_\infty^3(q^{15};q^{50})_\infty^4(q^{20};q^{50})_\infty^2(q^{25};q^{50})_\infty^4(q^{30};q^{50})_\infty^2(q^{35};q^{50})_\infty^4 \\
&\qquad\qquad (q^{40};q^{50})_\infty^3(q^{45};q^{50})_\infty^4,
\end{aligned}
$$

(3.5)

$$
\begin{aligned}
&\sum_{n=-\infty}^{\infty}(-1)^n q^{(5n^2+n)/2} \Big/ \sum_{n=-\infty}^{\infty} q^{(5n^2+n)/2} \\
&= 1/(q^5;q^{50})_\infty^4(q^{15};q^{50})_\infty^2(q^{20};q^{50})_\infty^3(q^{30};q^{50})_\infty^3(q^{35};q^{50})_\infty^2(q^{45};q^{50})_\infty^4 \\
&\quad - 2q^2/(q^5;q^{50})_\infty^3(q^{10};q^{50})_\infty(q^{15};q^{50})_\infty^2(q^{20};q^{50})^2(q^{25};q^{50})_\infty^2(q^{30};q^{50})^2(q^{35};q^{50})_\infty^2 \\
&\qquad\qquad (q^{40};q^{50})_\infty(q^{45};q^{50})_\infty^3 \\
&\quad - 2q^3/(q^5;q^{50})_\infty^2(q^{10};q^{50})_\infty^2(q^{15};q^{50})_\infty^3(q^{20};q^{50})_\infty(q^{25};q^{50})_\infty^2(q^{30};q^{50})_\infty(q^{35};q^{50})_\infty^3 \\
&\qquad\qquad (q^{40};q^{50})_\infty^2(q^{45};q^{10})_\infty^2 \\
&\quad + 2q^4/(q^5;q^{50})_\infty^3(q^{15};q^{50})_\infty^2(q^{20};q^{50})_\infty^3(q^{25};q^{50})_\infty^2(q^{30};q^{50})_\infty^3(q^{35};q^{50})_\infty^2(q^{45};q^{50})_\infty^3,
\end{aligned}
$$

and

(3.6)

$$
\begin{aligned}
&\sum_{n=-\infty}^{\infty}(-1)^n q^{(5n^2+3n)/2} \Big/ \sum_{n=-\infty}^{\infty} q^{(5n^2+3n)/2} \\
&= 1/(q^5;q^{50})_\infty^2(q^{10};q^{50})_\infty^3(q^{15};q^{50})_\infty^4(q^{35};q^{50})_\infty^4(q^{40};q^{50})_\infty^3(q^{45};q^{50})_\infty^2 \\
&\quad - 2q/(q^5;q^{50})_\infty^2(q^{10};q^{50})_\infty^3(q^{15};q^{50})_\infty^3(q^{25};q^{50})_\infty^2(q^{35};q^{50})_\infty^3(q^{40};q^{50})_\infty^3(q^{45};q^{50})_\infty^2 \\
&\quad + 2q^2/(q^5;q^{50})_\infty^3(q^{10};q^{50})_\infty(q^{15};q^{50})_\infty^2(q^{20};q^{50})_\infty^2(q^{25};q^{50})_\infty^2(q^{30};q^{50})_\infty^2(q^{35};q^{50})_\infty^2 \\
&\qquad\qquad (q^{40};q^{50})_\infty(q^{45};q^{10})_\infty^3 \\
&\quad - 2q^3/(q^5;q^{50})_\infty^2(q^{10};q^{50})_\infty^2(q^{15};q^{50})_\infty^3(q^{20};q^{30})_\infty(q^{25};q^{50})_\infty^2(q^{30};q^{50})_\infty(q^{35};q^{50})_\infty^3 \\
&\qquad\qquad (q^{40};q^{50})_\infty^2(q^{45};q^{50})_\infty^2.
\end{aligned}
$$

We now begin our proof of Corollaries 3.1 and 3.2.

PROOF OF (3.1): Letting $(x, N) = (q, 3)$ in Theorem 2.1, we obtain

$$
\begin{aligned}
&\frac{(-q;q^3)_\infty(-q^2;q^3)_\infty}{(q;q^3)_\infty(q^2;q^3)_\infty} \\
&\quad = 2\frac{(-q^3;q^3)_\infty^2}{(q^3;q^3)_\infty^2}\sum_{k=0}^{2} q^k \frac{(-q^{3k+3};q^9)_\infty(-q^{9-3k-3};q^9)_\infty(q^9;q^9)_\infty^2}{(q^3;q^9)_\infty(q^6;q^9)_\infty(-q^{3k};q^9)_\infty(-q^{9-3k};q^9)_\infty}
\end{aligned}
$$

$$\begin{aligned}
&= \frac{(-q^3;q^3)^2_\infty}{(q^3;q^3)^2_\infty} \frac{(-q^3;q^9)_\infty(-q^6;q^9)_\infty(q^9;q^9)^2_\infty}{(q^3;q^9)_\infty(q^6;q^9)_\infty(-q^9;q^9)^2_\infty} \\
&\quad + 2q\frac{(-q^3;q^3)^2_\infty}{(q^3;q^3)^2_\infty} \frac{(-q^6;q^9)_\infty(-q^3;q^9)_\infty(q^9;q^9)^2_\infty}{(q^3;q^9)_\infty(q^6;q^9)_\infty(-q^3;q^9)_\infty(-q^6;q^9)_\infty} \\
&\quad + 4q^2\frac{(-q^3;q^3)^2_\infty}{(q^3;q^3)^2_\infty} \frac{(-q^9;q^9)^2_\infty(q^9;q^9)^2_\infty}{(q^3;q^9)_\infty(q^6;q^9)_\infty(-q^6;q^9)_\infty(-q^3;q^9)_\infty} \\
&= \frac{(-q^3;q^3)^3_\infty}{(q^3;q^9)^3_\infty(q^6;q^9)^3_\infty(-q^9;q^9)^3_\infty} + \frac{2q(-q^3;q^3)^2_\infty}{(q^3;q^9)^3_\infty(q^6;q^9)^3_\infty} + \frac{4q^2(-q^3;q^3)_\infty(-q^9;q^9)^3_\infty}{(q^3;q^9)^3_\infty(q^6;q^9)^3_\infty} \\
&= \frac{1}{(q^3;q^9)^3_\infty(q^6;q^9)^3_\infty(q^3;q^{18})^3_\infty(q^{15};q^{18})^3} + \frac{2q}{(q^3;q^9)^3_\infty(q^6;q^9)^3_\infty(q^3;q^6)^2_\infty} \\
&\quad + \frac{4q^2}{(q^3;q^9)^3_\infty(q^6;q^9)^3_\infty(q^3;q^6)_\infty(q^9;q^{18})^3_\infty}.
\end{aligned}$$

□

PROOF OF (3.2): Let $(x,N) = (-q,3)$ in Theorem 2.1 and proceed as in the proof of (3.1). □

PROOF OF (3.3): Let $(x,N) = (q,5)$ in Theorem 2.1 and proceed as in the proof of (3.1). □

PROOF OF (3.4): Let $(x,N) = (q^2,5)$ in Theorem 2.1 and proceed as in the proof of (3.1). □

PROOF OF (3.5): Let $(x,N) = (-q,5)$ in Theorem 2.1 and proceed as in the proof of (3.1). □

PROOF OF (3.6): Let $(x,N) = (-q^2,5)$ in Theorem 2.1 and proceed as in the proof of (3.1). □

4. RECIPROCAL OF THE QUINTUPLE PRODUCT IDENTITY.

In this section, we provide a completely different proof of Corollary 3.1. First, we express the reciprocal of the quintuple product identity [**3**, p. 80, equation (38.2)] in the form (4.1), as a three dissection.

THEOREM 4.1. (Quintuple Product Identity.) *With* $f(a,b)$ *defined by* (1.1),

$$f(P^3Q, Q^5/P^3) - P^2 f(Q/P^3, P^3Q^5) = f(-Q^2)\frac{f(-P^2,-Q^2/P^2)}{f(PQ,Q/P)}. \tag{4.1}$$

Recall the elementary identity

$$\frac{1}{A-B} = \frac{A^2+AB+B^2}{A^3-B^3}. \tag{4.2}$$

We set $A = f(P^3Q, Q^5/P^3)$ and $B = P^2 f(Q/P^3, P^3Q^5)$ in (4.2). Replacing P by $\omega^k P$ in (4.1), $k = 0,1,2$, where ω is a primitive cube root of unity, and multiplying all three results together, we find that

$$A^3 - B^3 = \prod_{k=0}^{2}\{f(\omega^{3k}P^3Q, Q^5/\omega^{3k}P^3) - \omega^{2k}P^2 f(Q/\omega^{3k}P^3, \omega^{3k}P^3Q^5)\} \tag{4.3}$$

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$$=f^3(-Q^2)\frac{f(-P^6,-Q^6/P^6)}{f(P^3Q^3,Q^3/P^3)}.$$

Thus, by (4.1), (4.2) and (4.3), we have obtained the following theorem.

THEOREM 4.2. *With $f(a,b)$ defined by (1.1),*

(4.4)

$$f^2(-Q^2)\frac{f(PQ,Q/P)}{f(-P^2,-Q^2/P^2)}=\frac{f(P^3Q^3,Q^3/P^3)f^2(P^3Q,Q^5/P^3)}{f(-P^6,-Q^6/P^6)} + P^2\frac{f(P^3Q^3,Q^3/P^3)f(P^3Q,Q^5/P^3)f(Q/P^3,P^3Q^5)}{f(-P^6,-Q^6/P^6)} + P^4\frac{f(P^3Q^3,Q^3/P^3)f^2(Q/P^3,P^3Q^5)}{f(-P^6,-Q^6/P^6)}.$$

We now give a different proof of (3.2) and (3.1) using Theorems 4.1 and 4.2, respectively.

SECOND PROOF OF (3.2): Apply (4.1) with $P=-q^{1/2}$ and $Q=q^{3/2}$ and divide by $f(-q^3)$. ∎

SECOND PROOF OF (3.1): Apply (4.4) with $P=-q^{1/2}$ and $Q=q^{3/2}$ and divide by $f^2(-q^3)$. ∎

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