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THE LINEARIZATION OF THE PRODUCT OF CONTINUOUS q-JACOBI POLYNOMIALS

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1. Introduction. The problem of linearizing the product of two Jacobi polynomials, $P_m^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(x)$, and to establish the conditions for the non-negativity of the coefficients has been of considerable interest for many years. Explicit non-negative representations were sought and found by many authors [**7**, **8**, **13**, **14**], but only in the special case $\alpha = \beta$, although Hylleraas [**14**] succeeded in finding a formula in another case $\alpha = \beta + 1$. Gasper [**9**, **10**] found the necessary and sufficient conditions for the non-negativity of the linearization coefficients by exploiting a recurrence relation obtained by Hylleraas for the above-mentioned product. Koornwinder [**16**] approached the same problem from a different point of view and managed to find a non-negative integral expression to these coefficients when $-\frac{1}{2} \leqq \beta \leqq \alpha$. However, an exact formula in a hypergeometric series form for general α, β has been very elusive so far, in spite of the fact that all computation of special cases seemed to indicate that such a formula should exist. Gasper always believed in its existence and so did the author. Our efforts finally paid off in the discovery of the elusive formula [**17**] that turns out to be a non-negative multiple of a very well-poised two-balanced ${}_9F_8(1)$ and the non-negativity of the expression is self-evident in the case $0 \leqq \alpha + \beta$, $\beta \leqq \alpha$.

This rather fortunate turn of events raises the next immediate question: is there a q-analogue? Rogers' linearization formula [**19**] for the continuous q-ultraspherical polynomials has been known for a long time, for which a computational proof has been found by Gasper [**11**] very recently. It seems reasonable to expect that an extension of the author's ${}_9F_8$ formula might exist.

The first difficulty in working with the q-analogues of the Jacobi polynomials in the context of the linearization problem is to decide which is the right analogue. There are the ${}_2\phi_1$ polynomials of Hahn [**12**], ${}_3\phi_2$ polynomials of Andrews and Askey [**2**] and also the recently found ${}_4\phi_3$ polynomials of Askey and Wilson [**4**]. The order of difficulty in dealing with these polynomials may appear to be progressively greater from a ${}_2\phi_1$ to a ${}_4\phi_3$, but in fact, the continuous q-Jacobi polynomials defined by Askey-Wilson seem to have the most beautiful properties and to be the most appropriate for our purposes.

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A basic hypergeometric series in r upper and s lower parameters is defined by

$$ {}_r\phi_s\left[\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix}; q, z\right] = \sum_{k=0}^{\infty} \frac{(a_1;q)_k(a_2;q)_k \ldots (a_r;q)_k}{(q;q)_k(b_1;q)_k(b_2;q)_k \ldots (b_s;q)_k} z^k $$

where the products $(a;q)_k$ along with some of their properties are defined in the Appendix. The general ${}_4\phi_3$ polynomials as defined by Askey and Wilson are:

$$ (1.2) \qquad p_n(x) \equiv p_n(x;a;b,c,d) = {}_4\phi_3\left[\begin{matrix} q^{-n}, q^{n-1}abcd, ae^{i\theta}, ae^{-i\theta} \\ ab, \quad ac, \quad ad \end{matrix}; q, q\right], $$

where $x = \cos\theta$ and $n = 0, 1, \ldots$. With x and a fixed, $p_n(x)$ is obviously symmetrical with respect to a permutation of b, c and d. To see the other symmetries of $p_n(x)$ that are not so obvious we need to make use of the q-analogue of Whipple's transformation formula for a balanced ${}_4F_3(1)$. This analogue is implicitly contained in Watson's analogue of Whipple's transform [**6**, p. 69] and has been worked out explicitly by Askey and Ismail [**3**]. For the sake of completeness, however, we have worked out this basic formula in the Appendix. Thus, by applying (A.15) one can easily see that

$$ \begin{aligned} p_n(x;a;b,c,d) &= \frac{(bc;q)_n(bd;q)_n}{(ac;q)_n(ad;q)_n}\left(\frac{a}{b}\right)^n p_n(x;b;a,c,d) \\ (1.3) \qquad &= \frac{(cd;q)_n(cb;q)_n}{(ab;q)_n(ad;q)_n}\left(\frac{a}{c}\right)^n p_n(x;c;b,a,d) \\ &= \frac{(dc;q)_n(db;q)_n}{(ab;q)_n(ac;q)_n}\left(\frac{a}{d}\right)^n p_n(x;d;b,c,a). \end{aligned} $$

These relations show what happens when we interchange the first parameter with any of the other three. Let us now consider a special case. Let us set $a = -d$, $b = -c$. Then the last line above gives

$$ (1.4) \qquad p_n(x;a;b,-a,-b) = (-1)^n p_n(x;-a;b,-b,a). $$

However, the polynomial on the right is the same as one obtains from (1.2) by replacing θ by $\pi - \theta$ (i.e., $x \to -x$) with $c = -b$ and $d = -a$. Thus

$$ (1.5) \qquad p_n(-x;a;b,-b,-a) = (-1)^n p_n(x;a;b,-a,-b). $$

The analogies with the Jacobi polynomials are not quite obvious in the general formula (1.2). There are, in fact, a number of different ways of specializing the parameters a, b, c, d to show the connection. For example, one could set

$$a = q^{\alpha/2+1/4},\ b = q^{\alpha/2+3/4},\ c = -q^{\beta/2+1/4},\ -q^{\beta/2+3/4}.$$

In the limit $q \to 1-$ this leads to the same Jacobi polynomials as does the one with

(1.6) $\quad a = \sqrt{q} = -d,\ b = q^{\alpha+1/2},\ c = -q^{\beta+1/2}.$

As we shall see later the second choice is more suitable for the linearization problem. With this specialization, (1.3) gives

$$(1.7)\quad p_n(x;\sqrt{q};q^{\alpha+1/2}, -q^{\beta+1/2}, -\sqrt{q}) = \frac{(q^{\beta+1};q)_n(-q^{\alpha+1};q)_n}{(q^{\alpha+1};q)_n(-q^{\beta+1};q)_n} \times (-1)^n p_n(x; -\sqrt{q}; q^{\alpha+1/2}; -q^{\beta+1/2}, \sqrt{q}).$$

However,

$$(1.8)\quad p_n(x; -\sqrt{q}; q^{\alpha+1/2}, -q^{\beta+1/2}, \sqrt{q})$$
$$= {}_4\phi_3\left[\begin{matrix} q^{-n}, q^{n+\alpha+\beta+1}, -\sqrt{q}\,e^{i\theta}, -\sqrt{q}\,e^{-i\theta} \\ -q^{\alpha+1}, \quad q^{\beta+1}, \quad -q \end{matrix}; q, q\right]$$
$$= {}_4\phi_3\left[\begin{matrix} q^{-n}, q^{n+\alpha+\beta+1}, \sqrt{q}\,e^{i(\pi-\theta)}, \sqrt{q}\,e^{-i(\pi-\theta)} \\ -q^{\alpha+1}, \quad q^{\beta+1}, \quad -q \end{matrix}; q, q\right]$$
$$= p_n(-x;\sqrt{q}; -q^{\alpha+1/2}, q^{\beta+1/2}, -\sqrt{q}).$$

The continuous q-Jacobi polynomials may now be defined as

$$(1.9)\quad P_n^{(\alpha,\beta)}(x;q) = \frac{(q^{\alpha+1};q)_n(-q^{\beta+1};q)_n}{(q;q)_n(-q;q)_n} p_n(x;\sqrt{q};q^{\alpha+1/2}, -q^{\beta+1/2}, -\sqrt{q}).$$

(1.7) and (1.8) then provide us the relationship

$$(1.10)\quad P_n^{(\alpha,\beta)}(-x;q) = (-1)^n P_n^{(\beta,\alpha)}(x;q).$$

As $q \to 1$ this gives the exact analogue of the well-known symmetry property of the Jacobi polynomials. Apart from orthogonality this is perhaps the most important property of the Jacobi polynomials, as far as the linearization problem is concerned.

Askey and Wilson showed that the polynomials $p_n(x;a;b,c,d)$ are orthogonal with respect to the weight function

$$(1.11)\quad w(x) \equiv w(x;a,b,c,d)$$
$$= (1-x^2)^{-1/2}\prod_{k=0}^{\infty}\frac{(1-2xq^k+q^{2k})(1-2xq^{k+1/2}+q^{2k+1})}{(1-2axq^k+a^2q^{2k})(1-2bxq^k+b^2q^{2k})}\cdot \times \frac{(1+2xq^k+q^{2k})(1+2xq^{k+1/2}+q^{2k+1})}{(1-2cxq^k+c^2q^{2k})(1-2dxq^k+d^2q^{2k})}$$

By applying Sears' basic theorem on general q-integrals [**20**, **21**], one can prove that

$$(1.12)\quad h_0 = \int_{-1}^{1} dx\, w(x; a, b, c, d)$$
$$= \kappa \frac{(abcd; q)_\infty}{(ab; q)_\infty (ac; q)_\infty (ad; q)_\infty (bc; q)_\infty (bd; q)_\infty (cd; q)_\infty},$$

where

$$(1.13)\quad \kappa = 2\pi[(\sqrt{q}; q)_\infty(-\sqrt{q}; q)_\infty(-q; q)_\infty]^2/(q; q)_\infty.$$

For the integral to exist it is obvious from the product in the denominators of (1.12) and (1.13) that the absolute values of q, a, b, c and d must be less than 1.

One can then show that

$$(1.14)\quad \int_{-1}^{1} dx\, w(x)\, p_m(x; a; b, c, d) p_n(x; a; b, c, d) = h_n \delta_{mn},$$

with

$$(1.15)\quad h_n = h_0 \frac{a^{2n}(q; q)_n (cd; q)_n (bd; q)_n (bc; q)_n}{(abcdq^{-1}; q)_n (ab; q)_n (ac; q)_n (ad; q)_n} \cdot \frac{1 - abcdq^{-1}}{1 - abcdq^{2n-1}}.$$

In the special case when a, b, c, d are given by (1.6) the total weight h_0 may be written in a somewhat suggestive notation by using F. H. Jackson's [**5**, **15**] definition of the q-gamma function

$$(1.16)\quad \Gamma_q(x) = \frac{(q; q)_\infty}{(q^x; q)_\infty} (1 - q)^{1-x}, \quad 0 < q < 1.$$

After a bit of manipulation one gets

$$(1.17)\quad h_0 = \frac{2\pi \Gamma_q(\alpha + 1)\Gamma_q(\beta + 1)}{\Gamma_q(\frac{1}{2})\Gamma_q(\frac{1}{2})\Gamma_q(\alpha + \beta + 2)}$$
$$\times \left[\frac{(-\sqrt{q}; q)_\infty(-\sqrt{q}; q)_\infty(-q; q)_\infty}{(-q^{\alpha+1}; q)_\infty(-q^{\beta+1}; q)_\infty(-q^{\alpha+\beta+1}; q)_\infty}\right].$$

If $\alpha + \frac{1}{2}$ and $\beta + \frac{1}{2}$ were non-negative integers then the expression within the brackets could be written as $(-\sqrt{q}; q)_{\alpha+1/2}(-\sqrt{q}; q)_{\beta+1/2}$-$(-q; q)_{\alpha+\beta}$. For other values of α, β with $\operatorname{Re}\alpha > -1$, $\operatorname{Re}\beta > -1$ we may still use the same notation provided we understand $(q^a; q)_b$ to mean

$$(1.18)\quad (q^a; q)_b = {}_1\phi_0(q^{-b}; q; q^{a+b}).$$

Since $\lim_{q\to 1-} \Gamma_q(x) = \Gamma(x)$ and $\lim_{q\to 1-} (-q^c; q)_d = 2^d$, c real, we get

$$(1.19)\quad \lim_{q\to 1-} h_0 = 2^{2(\alpha+\beta+1)}\Gamma(\alpha + 1)\Gamma(\beta + 1)/\Gamma(\alpha + \beta + 2).$$

The reason we get a $2^{2(\alpha+\beta+1)}$ rather than the usual $2^{\alpha+\beta+1}$ is that the limit of $w(x; \sqrt{q}, q^{\alpha+1/2}, -q^{\beta+1/2}, -\sqrt{q})$ as $q \to 1-$ is not just $(1-x)^\alpha(1+x)^\beta$ but has an additional factor $2^{\alpha+\beta+1}$.

The principal aim of this investigation is a non-negative representation of the coefficients g_k in the expansion

$$(1.20)\quad p_m(x; a; b, c, d)p_n(x; a; b, c, d) = \sum_k (g_k/h_k)p_k(x; a; b, c, d)$$

in the case (1.6) generally, and, for the special case $\alpha = \beta$, in particular.

The orthogonality relation (1.14) gives us the coefficients g_k as an integral:

$$(1.21)\quad g_k \equiv g_k(m, n; a, b, c, d) = \int_{-1}^{1} dx w(x)p_m(x)p_n(x)p_k(x).$$

To facilitate the integration it is convenient to use the first two identities in (1.3). Thus

$$(1.22)\quad g_k = \frac{(bc; q)_m(bd; q)_m(cd; q)_n(cb; q)_n}{(ac; q)_m(ad; q)_m(ab; q)_n(ad; q)_n}\cdot\frac{a^{m+n}}{b^m c^n} f_k,$$

where

$$(1.23)\quad f_k \equiv f_k(m, n; a, b, c, d) = \int_{-1}^{1} dx w(x)p_m(x; b; a, c, d)p_n(x; c; b, a, d)p_k(x; a; b, c, d).$$

Before proceeding any further it is important to point out that, as a consequence of the symmetry properties (1.4) and (1.5) as well as the easily verifiable fact that $w(-x; a, b, -b, -a) = w(x; a, b, -b, -a)$, we have

$$(1.24)\quad g_k(m, n; a, b, -b, -a) = (-1)^{m+n+k}g_k(m, n; a, b, -b, -a)$$

which implies that $g_k(m, n; a, b, -b, -a)$ vanishes if $m + n + k$ is odd. This corresponds to the familiar ultraspherical case and will provide guidance later.

Using (1.2) and (1.11) one can easily see that

$$(1.25)\quad f_k = \sum_\lambda \sum_\mu \sum_\nu \frac{(q^{-k}; q)_\lambda(q^{k-1}abcd; q)_\lambda(q^{-m}; q)_\mu(q^{m-1}abcd; q)_\mu \times (q^{-n}; q)_\nu(q^{n-1}abcd; q)_\nu}{(q; q)_\lambda(q; q)_\mu(q; q)_\nu(ab; q)_\lambda(ac; q)_\lambda(ad; q)_\lambda(ba; q)_\mu \times (bc; q)_\mu(bd; q)_\mu} \times \frac{q^{\lambda+\mu+\nu}}{(ca; q)_\nu(cb; q)_\nu(cd; q)_\nu} I(\lambda, \mu, \nu),$$

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where

$$(1.26)\quad I(\lambda,\mu,\nu) = \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} \times \prod_{i=0}^{\infty} \frac{(1-2xq^i+q^{2i})(1-2xq^{i+1/2}+q^{2i+1}) \times (1+2xq^i+q^{2i})(1+2xq^{i+1/2}+q^{2i+1})}{(1-2axq^{\lambda+i}+a^2q^{2\lambda+2i})(1-2bxq^{\mu+i}+b^2q^{2\mu+2i}) \times (1-2cxq^{\nu+i}+q^{2\nu+2i}c^2)(1-2dxq^i+d^2q^{2i})}$$

$$= \int_{-1}^{1} dxw(x;aq^\lambda,bq^\mu,cq^\nu,d)$$

$$= \kappa \frac{(abcdq^{\lambda+\mu+\nu};q)_\infty}{(abq^{\lambda+\mu};q)_\infty(acq^{\lambda+\nu};q)_\infty(adq^\lambda;q)_\infty(bcq^{\mu+\nu};q)_\infty(bdq^\mu;q)_\infty \times (cdq^\nu;q)_\infty}$$

$$= h_0 \cdot \frac{(ab;q)_{\lambda+\mu}(ac;q)_{\lambda+\nu}(bc;q)_{\mu+\nu}(ad;q)_\lambda(bd;q)_\mu(cd;q)_\nu}{(abcd;q)_{\lambda+\mu+\nu}},$$

by (1.12) and (A.1). Hence

$$(1.27)\quad f_k = h_0 \sum_{\lambda=0}^{k}\sum_{\mu=0}^{m} \frac{(q^{-k};q)_\lambda(q^{k-1}abcd;q)_\lambda(q^{-m};q)_\mu \times (q^{m-1}abcd;q)_\mu(ab;q)_{\lambda+\mu}q^{\lambda+\mu}}{(q;q)_\lambda(q;q)_\mu(ab;q)_\lambda(ab;q)_\mu(abcd;q)_{\lambda+\mu}} \cdot {}_4\phi_3\left[\begin{matrix} q^{-n}, q^{n-1}abcd, acq^\lambda, & bcq^\mu \\ ac, & bc, abcdq^{\lambda+\mu} \end{matrix}; q, q\right].$$

Since this ${}_4\phi_3$ is balanced we may use (A.15) to obtain

$${}_4\phi_3[\quad] = \frac{(q^{-\lambda-\mu};q)_n(ad;q)_n}{(abcdq^{\lambda+\mu};q)_n(bc;q)_n}(bcq^{\lambda+\mu})^n \times {}_4\phi_3\left[\begin{matrix} q^{-n}, q^{n-1}abcd, q^{-\lambda}, a/bq^{-\mu} \\ ac, \ ad, \ q^{-\lambda-\mu} \end{matrix}; q, q\right].$$

The factor $(q^{-\lambda-\mu};q)_n$ on the right implies that f_k vanishes unless $\lambda + \mu \geqq n$. It follows by symmetry that f_k vanishes unless k, m, n satisfy the triangle inequalities. Let us then set

$$m = n - s, k = s + j$$

so that

$$0 \leqq s \leqq n, 0 \leqq j \leqq 2n - 2s.$$

Since $0 \leqq \mu \leqq n - s$ and $\lambda + \mu \geqq n$, λ can have values only between

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s and $s+j$. Hence

$$(1.28)\quad g_{s+j} \equiv g_{s+j}(n-s, n; a, b, c, d)$$
$$= h_0 \frac{(bc;q)_{n-s}(bd;q)_{n-s}(cd;q)_n b^s a^{2n-s}}{(ac;q)_{n-s}(ad;q)_{n-s}(ab;q)_n(abcd;q)_n} \sum_{\lambda=s}^{s+j} \sum_{\mu=n-\lambda}^{n-s} [\quad]$$
$$= h_0 \frac{(bc;q)_{n-s}(bd;q)_{n-s}(cd;q)_n b^s a^{2n-s}}{(ac;q)_{n-s}(ad;q)_{n-s}(ab;q)_n(abcd;q)_n}$$
$$\times \sum_{\lambda=0}^{j} \sum_{\mu=0}^{\lambda} \frac{(q^{-s-j};q)_{\lambda+s}(q^{s+j-1}abcd;q)_{\lambda+s}}{(q;q)_{\lambda+s}(q;q)_{n-s-\lambda+\mu}}$$
$$\cdot \frac{(q^{s-n};q)_{n-s-\lambda+\mu}(q^{n-s-1}abcd;q)_{n-s-\lambda+\mu}(ab;q)_{n+\mu} \times (q^{-n-\mu};q)_n q^{(n+1)(\mu+n)}}{(ab;q)_{s+\lambda}(ab;q)_{n-s-\lambda+\mu}(abcdq^n;q)_{n+\mu}}$$
$$\cdot {}_4\phi_3\left[\begin{matrix} q^{-n}, q^{n-1}abcd, q^{-\lambda-s}, (a/b)q^{\lambda+s-n-\mu} \\ ac, \quad ad, \quad q^{-n-\mu} \end{matrix}; q, q\right].$$

By using the identities (A.2) to (A.6) and simplifying we obtain

$$(1.29)\quad g_{s+j} = h_0 \frac{(q;q)_n(bc;q)_{n-s}(bd;q)_{n-s}(cd;q)_n \times (q^{s-n};q)_{n-s}(q^{n-s-1}abcd;q)_{n-s}}{(q;q)_{n-s}(ab;q)_{n-s}(ac;q)_{n-s}(ad;q)_{n-s} \times (q;q)_s(ab;q)_s(abcd;q)_{2n}}$$
$$\cdot (q^{-s-j};q)_s(q^{s+j-1}abcd;q)_s(-1)^n q^{n(n+1)/2} b^s a^{2n-s} e_j,$$

where

$$(1.30)\quad e_j = \sum_{\lambda=0}^{j} \frac{(q^{-j};q)_\lambda(q^{j+2s-1}abcd;q)_\lambda(q^{s-n};q)_\lambda(q^{s-n+1}/ab;q)_\lambda}{(q;q)_\lambda(q^{s+1};q)_\lambda(abq^s;q)_\lambda(q^{2s-2n+2}/abcd;q)_\lambda}\left(\frac{q^2}{cd}\right)^\lambda$$
$$\cdot \sum_{\mu=0}^{\lambda} \frac{(q^{-\lambda};q)_\mu(q^{2n-2s-\lambda-1}abcd;q)_\mu(q^{n+1};q)_\mu(abq^n;q)_\mu}{(q;q)_\mu(q^{n-s-\lambda+1};q)_\mu(abq^{n-s-\lambda};q)_\mu(abcdq^{2n};q)_\mu} q^\mu$$
$$\cdot {}_4\phi_3\left[\begin{matrix} q^{-n}, q^{n-1}abcd, q^{-\lambda-s}, (a/b)q^{\lambda+s-n-\mu} \\ ac, \quad ad, \quad q^{-n-\mu} \end{matrix}; q, q\right].$$

It is obvious that we must be able to carry out at least one summation before any progress can be made towards our final goal. To achieve that end we first transform the balanced ${}_4\phi_3$ by (A.15)

$$(1.31)\quad {}_4\phi_3[\quad] = \frac{(bcq^{n-\lambda-s};q)_{\lambda+s}(bdq^{n-\lambda-s};q)_{\lambda+s}}{(ac;q)_{\lambda+s}(ad;q)_{\lambda+s}}\left(\frac{a}{b}q^{\lambda+s-n}\right)^{\lambda+s}$$
$$\cdot {}_4\phi_3\left[\begin{matrix} q^{-\mu}, q^{-\lambda-s}, (b/a)q^{-\lambda-s}, q^{n-1}abcd \\ q^{-n-\mu}, \quad bcq^{n-\lambda-s}, bdq^{n-\lambda-s} \end{matrix}; q, q\right].$$

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Next, we transform the ${}_4\phi_3$ on the right to a very well-poised ${}_8\phi_7$ by (A.14): Thus

$$(1.32)\quad {}_4\phi_3[\quad] = {}_4\phi_3\left[\begin{matrix}(b/a)q^{-\lambda-s}, q^{-\lambda-s}, q^{n-1}abcd, \quad q^{-\mu} \\ q^{-n-\mu}, bcq^{n-\lambda-s}, bdq^{n-\lambda-s}; q, q\end{matrix}\right]$$

$$= \frac{(q^{n-s-\lambda+1}; q)_\mu (abcdq^{2n}; q)_\mu}{(q^{n+1}; q)_\mu (abcdq^{2n-\lambda-s}; q)_\mu}\, {}_8\phi_7\left[\begin{matrix} abcdq^{2n-\lambda-s-1}, q\sqrt{abcdq^{2n-\lambda-s-1}}, \\ \sqrt{abcdq^{2n-\lambda-s-1}}, \end{matrix}\right.$$

$$-q\sqrt{abcdq^{2n-\lambda-s-1}}, acq^n, adq^n, q^{-\lambda-s}, q^{n-1}abcd, q^{-\mu}$$

$$-\sqrt{abcdq^{2n-\lambda-s-1}}, bdq^{n-\lambda-s}, bcq^{n-\lambda-s}, abcdq^{2n}, q^{n+1-\lambda-s},$$

$$\left. abcdq^{2n-\lambda-s+\mu}; q, \frac{b}{a} q^{n+1-\lambda-s+\mu}\right].$$

We now write this as a sum over an index i, plug it in (1.31) and then in (1.30). The sum over μ in (1.30) now reads

$$\sum \frac{(abcdq^{2n-\lambda-s-1}; q)_i (q\sqrt{\ }; q)_i (-q\sqrt{\ }; q)_i (acq^n; q)_i (adq^n; q)_i \times (q^{-\lambda-s}; q)_i}{(q; q)_i (\sqrt{\ }; q)_i (-\sqrt{\ }; q)_i (bdq^{n-\lambda-s}; q)_i (bcq^{n-\lambda-s}; q)_i (abcdq^{2n}; q)_i}$$

$$\cdot \frac{(q^{n-1}abcd; q)_i}{(q^{n+1-s-\lambda}; q)_i} \left(\frac{b}{a} q^{n+1-\lambda-s}\right)^i$$

$$\times \sum_{\mu=0}^{\lambda} \frac{(q^{-\lambda}; q)_\mu (q^{2n-2s-\lambda-1}abcd; q)_\mu (abq^n; q)_\mu}{(q; q)_\mu (abq^{n-s-\lambda}; q)_\mu (abcdq^{2n-\lambda-s}; q)_{\mu+i}} \cdot (q^{-\mu}; q)_i (q^{i+1})^\mu.$$

However,

$$(1.33)\quad \sum_{\mu=0}^{\lambda} [\quad] = (-1)^i q^{i/2(i+1)} \frac{(q^{-\lambda}; q)_i (q^{2n-2s-\lambda-1}abcd; q)_i (abq^n; q)_i}{(abq^{n-s-\lambda}; q)_i (abcdq^{2n-\lambda-s}; q)_{2i}}$$

$$\cdot {}_3\phi_2\left[\begin{matrix} q^{i-\lambda}, abcdq^{2n-2s-\lambda-1+i}, abq^{n+i} \\ abcdq^{2n-\lambda-s+2i}, abq^{n-\lambda-s+i}; q, q\end{matrix}\right]$$

$$= (-1)^i q^{1/2 i(i+1)} \frac{(q^{-\lambda}; q)_i (q^{2n-2s-\lambda-1}abcd; q)_i (abq^n; q)_i}{(abq^{n-s-\lambda}; q)_i (abcdq^{2n-\lambda-s}; q)_{\lambda+i}}$$

$$\cdot \frac{(q^{s+1+i}; q)_{\lambda-i} (cdq^{n-\lambda-s+i}; q)_{\lambda-i}}{(q^{s-n+1}/ab; q)_{\lambda-i}}$$

$$= \frac{(q^{s+1}; q)_\lambda (cdq^{n-\lambda-s}; q)_\lambda}{(abcdq^{2n-\lambda-s}; q)_\lambda (q^{s-n+1}/ab; q)_\lambda}$$

$$\cdot \frac{(q^{-\lambda}; q)_i (q^{2n-2s-\lambda-1}abcd; q)_i (abq^n; q)_i}{(q^{s+1}; q)_i (cdq^{n-\lambda-s}; q)_i (abcdq^{2n-s}; q)_i} \left(\frac{q^{\lambda+s-n+1}}{ab}\right)^i.$$

It has been possible to carry out the summation because the ${}_3\phi_2$ above is balanced and hence summable by the q-analogue of Pfaff-Saalschutz theorem (A.12). The last line in (1.33) has been obtained by

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repeated use of the identities (A.2-A.6). The sum over μ in (1.30) finally reduces to

(1.34) $$\frac{(q^{s+1};q)_\lambda(cdq^{n-s-\lambda};q)_\lambda}{(q^{s-n+1}/ab;q)_\lambda(abcdq^{2n-s-\lambda};q)_\lambda}\,{}_{10}\phi_9\left[\begin{matrix}abcdq^{2n-\lambda-s-1}, q\sqrt{\ }, -q\sqrt{\ }, abq^n, acq^n, adq^n, abcdq^{n-1}, q^{-\lambda-s}, abcdq^{2n-2s-\lambda-1}, q^{-\lambda} \\ \sqrt{\ }, -\sqrt{\ }, cdq^{n-\lambda-s}, bdq^{n-\lambda-s}, bcq^{n-\lambda-s}, q^{n+1-\lambda-s}, abcdq^{2n}, q^{s+1}, abcdq^{2n-s}\end{matrix};q,q^2/a^2\right].$$

This ${}_{10}\phi_9$ is obviously very well-poised but not balanced unless $a=\pm\sqrt{q}$. And unless it is balanced there is no known transformation formula for a general very well-poised ${}_{10}\phi_9$. So it seems necessary for computational purposes that we choose

(1.35) $$a=\sqrt{q}.$$

Since our main objective is to solve the linearization problem for the q-Jacobi polynomials defined by (1.6) let us also set

(1.36) $$d=-\sqrt{q}.$$

(If we had chosen $a=-\sqrt{q}$ then d would have to be equal to $\sqrt{q}$.)

Then, by simplifying the coefficients, we get

(1.37) $$e_j=\frac{(bc;q)_n(-b\sqrt{q};q)_n(q^{s-n+1/2}/b)^s}{(c\sqrt{q};q)_s(-q;q)_s(bc;q)_{n-s}(-b\sqrt{q};q)_{n-s}}$$
$$\cdot\sum_{\lambda=0}^{j}\frac{(q^{-j};q)_\lambda(-bcq^{j+2s};q)_\lambda(q^{s-n};q)_\lambda}{(q;q)_\lambda(bq^{s+1/2};q)_\lambda(cq^{s+1/2};q)_\lambda}$$
$$\cdot\frac{(q^{s-n+1}/bc;q)_\lambda(-q^{s-n+1/2}/b;q)_\lambda(-q^{s-n+1/2}/c;q)_\lambda}{(-q^{s+1};q)_\lambda(-q^{2s-2n+1}/bc;q)_\lambda(-q^{s-2n}/bc;q)_\lambda}q^\lambda$$
$$\times\,{}_{10}\phi_9\left[\begin{matrix}-bcq^{2n-s-\lambda}, q\sqrt{\ }, -q\sqrt{\ }, -bcq^{2n-2s-\lambda}, -bcq^n, -q^{n+1}, q^{-\lambda-s}, bq^{n+1/2}, cq^{n+1/2}, q^{-\lambda} \\ \sqrt{\ }, -\sqrt{\ }, q^{s+1}, q^{n-s+1-\lambda}, bcq^{n-s-\lambda}, -bcq^{2n+1}, -cq^{n-s+\lambda+1/2}, -bq^{n-s-\lambda+1/2}, -bcq^{2n-s+1}\end{matrix};q,q\right].$$

We now apply the transformation formula (A.16) in such a way that the parameters $q^{-\lambda}$, $q^{-\lambda-s}$, $bq^{n+1/2}$, $cq^{n+1/2}$ remain unchanged on the top, the idea being that $q^{-\lambda}$ and $q^{-\lambda-s}$ leave the terminating character of the series intact while the terms $bq^{n+1/2}$ and $cq^{n+1/2}$ become a positive and negative pair in the special case $b=-c$. Recalling that our main interest is in the linearization problem with the parameters specialized by (1.6) we now substitute $b=q^{\alpha+1/2}$, $c=-q^{\beta+1/2}$ in the formulas above and obtain a reformulation of the problem in the following form:

$$(1.38)\quad p_{n-s}(x;\sqrt{q};q^{\alpha+1/2},-q^{\beta+1/2},-\sqrt{q})p_n(x;\sqrt{q};q^{\alpha+1/2},-q^{\beta+1/2},-\sqrt{q}) = \sum_{j=0}^{2n-2s} b_j p_{s+j}(x;\sqrt{q};q^{\alpha+1/2},-q^{\beta+1/2},-\sqrt{q})$$

where

$$(1.39)\quad b_j = \mu_j F_j$$

with

$$(1.40)\quad \mu_j = q^{n-s-j(s+1)} \frac{(q;q)_n(q^{\beta+1};q)_n(q^{\alpha+\beta+1};q)_{2s}(q^{\alpha+\beta+1+n-s};q)_{n-s} \times (-q^{\alpha+1};q)_n(-q^{\alpha+\beta+1};q)_n}{(q;q)_s(q^{\beta+1};q)_s(q^{\alpha+1};q)_{n-s}(q^{\alpha+\beta+2};q)_{2n}(-q;q)_{n-s} \times (-q^{\alpha+1};q)_s(-q^{\beta+1};q)_{n-s}(-q^{\alpha+\beta+1};q)_s}$$
$$\cdot \frac{(q^{\alpha+s+1};q)_j(q^{\alpha+\beta+1+2s};q)_j(-q^{s+1};q)_j(-q^{\beta+s+1};q)_j}{(q;q)_j(q^{\beta+s+1};q)_j(-q^{\alpha+s+1};q)_j(-q^{\alpha+\beta+s+1};q)_j} \cdot \frac{1-q^{\alpha+\beta+1+2s+2j}}{1-q^{\alpha+\beta+1}},$$

and

$$(1.41)\quad F_j = \sum_{\lambda=0}^{j} \frac{(q^{-j};q)_\lambda(q^{j+2s+\alpha+\beta+1};q)_\lambda(q^{\alpha+n+1};q)_\lambda(-q^{\beta+n+1};q)_\lambda \times (q^{s-n-\alpha-\beta};q)(-q^{s-n};q)_\lambda}{(q;q)_\lambda(q^{\alpha+\beta+2+2n};q)_\lambda(q^{\alpha+s+1};q)_\lambda(-q^{\beta+s+1};q)_\lambda \times (q^{2s-2n-\alpha-\beta};q)_\lambda(-1;q)_\lambda} \cdot q^\lambda K_\lambda,$$

$$(1.42)\quad K_\lambda = {}_{10}\Phi_9\left[\begin{matrix} -q^{-\lambda}, q\sqrt{-q^{-\lambda}}, -q\sqrt{-q^{-\lambda}}, & q^{-\lambda-s}, -q^{-\lambda-s}, & q^{\beta+n+1}, \\ \sqrt{-q^{-\lambda}}, -\sqrt{-q^{-\lambda}}, & -q^{s+1}, \quad q^{s+1}, & -q^{-\beta-n-\lambda}, \end{matrix}\right.$$
$$\left.\begin{matrix} -q^{\alpha+n+1}, & -q^{s-n-\alpha-\beta}, & q^{s-n}, & q^{-\lambda} \\ q^{-\alpha-n-\lambda}, & q^{n-s+1+\alpha+\beta-\lambda}, & -q^{n-s+1-\lambda}, & -q \end{matrix}; q, q\right].$$

The ${}_{10}\Phi_9$ series above, which we have denoted by K_λ for notational brevity, has the following properties: (i) it is very well-poised and 2-balanced; (ii) as $q \to 1-$ it reduces to a balanced ${}_4F_3$ series; (iii) when $\alpha = \pm\beta$ it acquires an additional matching character in that six of the ten numerator parameters occur in positive and negative pairs. It is this third property that enables us to recognize it as the basic analogue of a nearly-poised ${}_4F_3$ which has a known transformation formula in terms of another balanced ${}_4F_3$. The formula is, in fact, a special case of eq. (1) of § 4.7 in [**6**] of which a q-analogue has recently been found [**18**]. Using this q-analogue one could make further transformations of the sums in (1.41) and eventually obtain linearization formulae in the ultraspherical case $\alpha = \beta$ as well as the case $\alpha = -\beta$. However, in the general case $\alpha \neq \pm\beta$ it is not possible to transform K_λ to a form that has this matching property and so the transformation formula of [**18**] cannot be applied. One would hope that the steps followed in [**17**] to obtain the coefficients in the case of ordinary Jacobi polynomials might have their q-analogues and thus one would find a basic version of the results obtained therein. It turns out, however, that the procedures followed in [**17**] do not lead to any meaningful result in the q-Jacobi case and so an alternative approach needs to be found. Fortunately, we have been able to find just such an approach that leads us to the final result:

$$(1.43)\quad b_j = A_{n,s}\frac{(q^{\alpha+\beta+1+2s};q)_j(-q^{s+1};q)_j(-q^{\beta+s+1};q)_j(q^{2s-2n};q)_j \times (q^{2\alpha+2\beta+2n+2};q)_j(q^{\alpha-\beta};q)_j}{(q;q)_j(-q^{\alpha+s+1};q)_j(-q^{\alpha+\beta+s+1};q)_j(q^{\alpha+\beta+2+2n};q)_j \times (q^{2\beta+2s+2};q)_j(q^{2s-2n-\alpha-\beta};q)_j}$$

$$\cdot\frac{1-q^{\alpha+\beta+1+2s+2j}}{1-q^{\alpha+\beta+1}}\,q^{n-s-\alpha j}\,{}_{10}\Phi_9\left[\begin{matrix}p^{\beta+s+1/2},\ p^{1+1/2(\beta+s+1/2)},\ -p^{1+1/2(\beta+s+1/2)},\ p^{\beta+1/2},\ p^{\beta+n+1},\ p^{s-n-\alpha},\\ p^{1/2(\beta+s+1/2)},\ -p^{1/2(\beta+s+1/2)},\ p^{s+1},\ p^{s-n+1/2},\ p^{\alpha+\beta+n+3/2},\end{matrix}\right.$$

$$\left.\begin{matrix}p^{((\alpha+\beta+1)/2)+s+(j/2)},\ p^{((\alpha+\beta+2)/2)+s+(j/2)},\ p^{(1-j)/2},\ p^{-(j/2)}\\ p^{((\beta-\alpha)/2)+1-(j/2)},\ p^{((\beta-\alpha)/2)+1/2-(j/2)},\ p^{\beta+s+1+(j/2)},\ p^{\beta+s+3/2+(j/2)}\end{matrix};p,p\right]$$

where $p = q^2$ and

$$(1.44)\quad A_{n,s} = (q;q)_n(q^{\beta+1};q)_n(q^{\alpha+\beta+1};q)_{2s}(q^{\alpha+\beta+1};q)_{2n-2s}(-q^{\alpha+1};q)_n \times (-q^{\alpha+\beta+1};q)_n/\{(q;q)_s(q^{\alpha+1};q)_{n-s}(q^{\alpha+\beta+1};q)_{n-s}(q^{\beta+1};q)_s \times (q^{\alpha+\beta+2};q)_{2n}(-q;q)_{n-s}(-q^{\beta+1};q)_{n-s}(-q^{\alpha+1};q)_s \times (-q^{\alpha+\beta+1};q)_s\}.$$

It is easy to see that in the limit $q \to 1-$ (1.43) reduces to (1.9) of [**17**]. The non-negativity of b_j is quite clear in this formula when $-\frac{1}{2} \leqq \beta \leqq \alpha$. However, in any larger region the terms of the series above are not necessarily all non-negative and so further transformations need to be made to show that the b_j's are indeed non-negative if $-1 < \beta \leqq \alpha$ and $\alpha + \beta + 1 \geqq 0$. Since this requires a bit of manipulation we shall carry out the calculations and state the results in Section 4. The derivation of (1.43) from (1.40) and (1.41) will be carried out in Sections 2 and 3.

2. Transformation of K_λ. There are two key steps in the proof of (1.43). The first one is to rewrite the ${}_{10}\Phi_9$ series in (1.42) as a ${}_{12}\Phi_{11}$ by introducing a matching parameter so that six of the top parameters occur in positive and negative pairs. This can be done in four different ways, but the one we have chosen to work with is shown in the following

$$(2.1)\qquad K_\lambda = {}_{12}\Phi_{11}\left[\begin{matrix} -q^{-\lambda}, q\sqrt{-q^{-\lambda}}, -q\sqrt{-q^{-\lambda}}, q^{-\lambda-s}, -q^{-\lambda-s}, q^{\beta+n+1}, \\ \sqrt{-q^{-\lambda}}, -\sqrt{-q^{-\lambda}}, -q^{s+1}, q^{s+1}, -q^{-\beta-n-\lambda}, \end{matrix}\right.$$
$$\begin{matrix} -q^{\beta+n+1}, q^{-\lambda}, -q^{\alpha+n+1}, q^{-\beta-n-\lambda}, -q^{s-n-\alpha-\beta}, \\ q^{-\beta-n-\lambda}, -q, q^{-\alpha-n-\lambda}, -q^{\beta+n+1}, q^{\alpha+\beta+1+n-s-\lambda}, \end{matrix}$$
$$\left.\begin{matrix} q^{s-n} \\ -q^{n-s+1-\lambda}; q, q \end{matrix}\right].$$

It is obvious that it is the parameter $-q^{\beta+n+1}$ we have chosen to match $q^{\beta+n+1}$ and that the series is really a ${}_{10}\Phi_9$. Using the identity (A.7) we can easily see that

$$(2.2)\qquad \frac{(-q^{-\lambda};q)_k(q^{-\lambda};q)_k(q\sqrt{-q^{-\lambda}};q)_k(-q\sqrt{-q^{-\lambda}};q)_k \times (q^{-\lambda-s};q)_n(-q^{-\lambda-s};q)_n(q^{\beta+n+1};q)_k(-q^{\beta+n+1};q)_k}{(q;q)_k(-q;q)_k(\sqrt{-q^{-\lambda}};q)_k(-\sqrt{-q^{-\lambda}};q)_k(q^{s+1};q)_k \times (-q^{s+1};q)_k(-q^{-\beta-n-\lambda};q)_k(q^{-\beta-n-\lambda};q)_k}$$
$$= \frac{(p^{-\lambda};p)_k(1+p^{k-\lambda/2})(p^{-\lambda-s};p)_k(p^{\beta+n+1};p)_k}{(p;p)_k(1+p^{-\lambda/2})(p^{s+1};p)_k(p^{-\beta-n-\lambda};p)_k},$$

where the symbol p is used throughout the paper to stand for q^2.

Now we apply (A.12) to get the easily verified identity

$$(2.3)\qquad \frac{(p^{-\lambda-s};p)_k(p^{\beta+n+1};p)_k}{(p^{s+1};p)_k(p^{-\beta-n-\lambda};p)_k}$$
$$= p^{(\beta+n-s)k}\sum_{r=0}^{k}\frac{(p^{-k};p)_r(p^{k-\lambda};p)_r(p^{s-n-\beta};p)_r}{(p;p)_r(p^{s+1};p)_r(p^{-\beta-n-\lambda};p)_r}p^r.$$

The factor $p^{(\beta+n-s)k}$ appears in front because of the identity (A.6) which enables us to write $(p^{s+1+\lambda-k};p)_k/(p^{\beta+n+1+\lambda-k};p)_k$ as

$$p^{(s-n-\beta)k}(p^{-\lambda-s};p)_k/(p^{-\beta-n-\lambda};p)_k.$$

Using (2.3) in (2.2) and then in (2.1) we get

$$K_\lambda = \sum_r \frac{(p^{s-n-\beta};p)_r p^r}{(p;p)_r(p^{s+1};p)_r(p^{-\beta-n-\lambda};p)_r}$$
$$\times \sum_k \frac{(-q^{\alpha+n+1};q)_k(q^{-\beta-n-\lambda};q)_k}{(q^{-\alpha-n-\lambda};q)_k(-q^{\beta+n+1};q)_k}$$
$$\cdot\frac{(-q^{s-n-\alpha-\beta};q)_k(q^{s-n};q)_k}{(q^{\alpha+\beta+1+n-s-\lambda};q)_k(-q^{n-s+1-\lambda};q)_k}$$
$$\times q^k\frac{(p^{-k};p)_r(p^{-\lambda};p)_{r+k}(1+p^{k-\lambda/2})}{(p;p)_k(1+p^{-\lambda/2})}p^{(\beta+n-s)k}.$$

Note the sum vanishes unless $k \geqq r$. So we make a transformation $k \to k + r$, carry out some simplifications and use the identity

$$(p^{-k-r}; p)_r = (-1)^r q^{-2k-r(r+1)} (p; p)_{k+r}/(p; p)_k$$

to get

$$(2.4)\quad K_\lambda = \sum_r \frac{(p^{-\lambda}; p)_{2r}(p^{s-n-\beta}; p)_r(-q^{\alpha+n+1}; q)_r(q^{-\beta-n-\lambda}; q)_r \times (-q^{s-n-\alpha-\beta}; q)_r(q^{s-n}; q)_r}{(p; p)_r(p^{s+1}; p)_r(p^{-\beta-n-\lambda}; p)_r(q^{-\alpha-n-\lambda}; q)_r \times (-q^{\beta+n+1}; q)_r(q^{\alpha+\beta+1+n-s-\lambda}; q)_r(-q^{n-s+1+\lambda}; q)_r}$$

$$\cdot \frac{1 + p^{r-\lambda/2}}{1 + p^{-\lambda/2}} (-1)^r q^{2(\beta+1+n-s)r-r^2} {}_8\Phi_7\left[\begin{matrix} -q^{2r-\lambda}, q\sqrt{-q^{2r-\lambda}}, -q\sqrt{-q^{2r-\lambda}}, q^{2r-\lambda}, -q^{\alpha+n+1+r}, q^{r-\beta-n-\lambda}, -q^{s-n-\alpha-\beta+r}, q^{s-n+r} \\ \sqrt{-q^{2r-\lambda}}, -\sqrt{-q^{2r-\lambda}}, -q, q^{r-\alpha-n-\lambda}, -q^{\beta+n+1+r}, q^{\alpha+\beta+1+n-s-\lambda+r}, -q^{n-s+1-\lambda+r} \end{matrix}; q, q^{2\beta+2n-2s+1-2r}\right].$$

The ${}_8\Phi_7$ series in (2.4) has the same structure as the one in (A.14) and so if we use the correspondence

$$a \to -q^{2r-\lambda}, b \to -q^{\alpha+n+1+r}, c \to q^{r-\beta-n-\lambda}, d \to -q^{s-n-\alpha-\beta-r},$$

$$e \to q^{s-n+r} \quad \text{and} \quad f \to q^{2r-\lambda},$$

then the ${}_8\Phi_7$ in (2.4) transforms (after using (A.6)) to

$$(2.5)\quad \frac{(q^{2s-2n-\alpha-\beta+2r}; q)_{\lambda-2r}(-1; q)_{\lambda-2r}}{(q^{s-n-\alpha-\beta+r}; q)_{\lambda-2r}(-q^{s-n+r}; q)_{\lambda-2r}} {}_4\Phi_3\left[\begin{matrix} q^{2r-\lambda}, q^{\beta-\alpha}, -q^{s-n-\alpha-\beta+r}, q^{s-n+r} \\ q^{r-\alpha-n-\lambda}, -q^{\beta+n+1+r}, q^{2s-2n-\alpha-\beta+2r} \end{matrix}; q, q\right].$$

However, using (A.3) the coefficient of the ${}_4\Phi_3$ above simplifies to

$$\frac{(q^{2s-2n-\alpha-\beta}; q)_\lambda(-1; q)_\lambda}{(-q^{s-n}; q)_\lambda(q^{s-n-\alpha-\beta}; q)_\lambda}$$

$$\cdot \frac{(q^{s-n-\alpha-\beta}; q)_r(-q^{s-n}; q)_r(q^{\alpha+\beta+1+n-s-\lambda}; q)_r(-q^{n-s+1-\lambda}; q)_r}{(q^{2s-2n-\alpha-\beta}; q)_{2r}(-q^{1-\lambda}; q)_{2r}}$$

$$\times (-1)^r q^{r^2-(\alpha+\beta+2n-2s)r}.$$

Using this in (2.5) and going back to (2.4) we finally obtain

$$(2.6)\qquad K_\lambda = \frac{(q^{2s-2n-\alpha-\beta};q)_\lambda(-1;q)_\lambda}{(-q^{s-n};q)_\lambda(q^{s-n-\alpha-\beta};q)_\lambda}$$
$$\times \sum_{r=0}^{[\lambda/2]} \frac{(p^{-\lambda/2};p)_r(p^{(1-\lambda)/2};p)_r(p^{s-n-\beta};p)_r(p^{s-n-\alpha-\beta};p)_r}{(p;p)_r(p^{s+1};p)_r(q^{2s-2n-\alpha-\beta};q)_{2r}}$$
$$\times \frac{(p^{s-n};p)_r(-q^{\alpha+n+1};q)_r}{(-q^{-\beta-n-\lambda};q)_r(q^{-\alpha-n-\lambda};q)_r(-q^{\beta+n+1};q)_r} q^{r(\beta-\alpha+2)}$$
$$\times {}_4\Phi_3\left[\begin{matrix} q^{2r-\lambda}, & q^{\beta-\alpha}, & q^{s-n+r}, & -q^{s-n-\alpha-\beta+r} \\ & q^{2s-2n-\alpha-\beta+2r}, & q^{r-\alpha-n-\lambda}, & -q^{\beta+n+1+r}\end{matrix}; q, q\right].$$

It should be pointed out that the use of (2.5) leads to (2.6) only after some simplifications and particularly the application of yet another identity (A.8).

This is the form of K_λ that seems most appropriate for use in (1.41). Note that the right hand side of (2.6) contains two groups of products, one with base q and the other with base $p = q^2$. The question may arise at this point what we may have possibly gained by transforming a single series in (1.42) to a double series in (2.6). One only has to look closely at the form of the ${}_4\Phi_3$ in (2.6) to realize that it becomes 1 in the special case $\alpha = \beta$ and K_λ becomes a multiple of a balanced ${}_4\Phi_3$ with base p and thus (2.6) constitutes a quadratic transformation of (1.42) in the ultraspherical case. It is not difficult to show that with this special form of K_λ for $\alpha = \beta$ the double series in (1.41) can be summed exactly and the linearization coefficients b_j do reduce to ratios of products when j is even and to zero when j is odd. So, by transforming K_λ of (1.42) to the double series in (2.6) we are really looking for an expansion of b_j in a series in $q^{\beta-\alpha}$.

It is obvious that a direct substitution of (2.6) in (1.41) is not going to render any of the series in the triple sum summable unless further transformations are made. So far we have concentrated on an "inner transformation" of F_j, that is, a transformation of the inner sum to a more suitable form. For the ultraspherical case this is sufficient to produce a summable form. However, in the general situation one needs to carry out an "outer transformation" of the λ-series in (1.41), hoping to get some cancellations, before the triple series can be seen to drop into summable forms. This rather strenuous work is carried out in the following section.

3. Computation of F_j. Since

$$(p^{-\lambda/2};p)_r(p^{(1-\lambda)/2};p)_r = (q^{-\lambda};q)_{2r}$$

by (A.8), and

$$(p^{s-n};p)_r = (q^{s-n};q)_r(-q^{s-n};q)_r$$

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we get, on using (2.6) in (1.41),

$$(3.1)\qquad F_j = \sum_r \sum_l \frac{(p^{s-n-\beta};p)_r(p^{s-n-\alpha-\beta};p)_r(-q^{s-n};q)_r \times (-q^{\alpha+n+1};q)_r(q^{\beta-\alpha};q)_l(q^{s-n};q)_{r+l}}{(p;p)_r(p^{s+1};p)_r(q^{2s-2n-\alpha-\beta};q)_{2r+l}} \times (-q^{\beta+n+1};q)_{r+l} \cdot (-q^{s-n-\alpha-\beta+r};q)_l\, q^{r(\beta-\alpha+2)+l}\gamma_{r,l,j},$$

where

$$(3.2)\qquad \gamma_{r,l,j} = \sum_{\lambda=0}^{j} \frac{(q^{-j};q)_\lambda(q^{j+2s+\alpha+\beta+1};q)_\lambda(q^{\alpha+n+1};q)_\lambda(-q^{\beta+n+1};q)_\lambda}{(q;q)_\lambda(q^{\alpha+\beta+2+2n};q)_\lambda(q^{\alpha+s+1};q)_\lambda(-q^{\beta+s+1};q)_\lambda} \cdot \frac{(q^{-\lambda};q)_{2r+l}\, q^\lambda}{(q^{-\alpha-n-\lambda};q)_{r+l}(-q^{-\beta-n-\lambda};q)_r}.$$

Since the terms of this series vanish unless $\lambda \geqq 2r + l$ we make a variable change $\lambda \to 2r + l + \lambda$ and simplify, by using (A.5) and (A.6).

$$(3.3)\qquad \gamma_{r,l,j} = \frac{(q^{-j};q)_{2r+l}(q^{j+2s+\alpha+\beta+1};q)_{2r+l}(q^{\alpha+n+1-r-l};q)_{2r+l} \times (-q^{\beta+n+1-r};q)_{2r+l}}{(q^{-\alpha-n};q)_{r+l}(-q^{-\beta-n};q)_r(q^{\alpha+\beta+2+2n};q)_{2r+l} \times (q^{\alpha+s+1};q)_{2r+l}(-q^{\beta+s+1};q)_{2r+l}} \cdot (-1)^l q^{1/2(2r+l)(2r+l+1)}\, {}_4\Phi_3\left[\begin{matrix} q^{2r+l-j},\ q^{j+2s+\alpha+\beta+1+2r+l},\ q^{\alpha+n+1+r},\ -q^{\beta+n+1+r+l} \\ q^{\alpha+s+1+2r+l},\ -q^{\beta+s+1+2r+l},\ q^{\alpha+\beta+2+2n+2r+l} \end{matrix};q,q\right].$$

Note that the ${}_4\Phi_3$ series above is balanced and hence we may apply (A.15) to make a further transformation. The second key step that we alluded to in Section 2 consists of using (A.15) once in (3.3) by choosing the parameters judiciously, simplifying and rewriting $\gamma_{r,l,j}$ back into the form (3.2). In the first stage of calculations we transform the ${}_4\Phi_3$ above to

$$\frac{(-q^{\alpha+s+1+2r+l};q)_{j-2r-l}(q^{2s-2n+2r+l};q)_{j-2r-l}}{(-q^{\beta+s+1+2r+l};q)_{j-2r-l}(q^{\alpha+\beta+2+2n+2r+l};q)_{j-2r-l}}(-q^{\beta+1+2n-s})^{j-2r-l} \times {}_4\Phi_3\left[\begin{matrix} q^{2r+l-j},\ q^{j+2s+\alpha+\beta+1+2r+l},\ q^{s-n+r+l},\ -q^{s-n+r+\alpha-\beta} \\ q^{\alpha+s+1+2r+l},\ -q^{\alpha+s+1+2r+l},\ q^{2s-2n+2r+l} \end{matrix};q,q\right].$$

The crucial feature of this transformation is that two of the denominator parameters occur as a matching pair. This is the "outer transformation" that we alluded to at the end of Section 2. When we transfer this to (3.3) and simplify, we get

$$\text{(3.4)}\qquad \gamma_{r,l,j} = \frac{(q^{2s-2n};q)_j(-q^{\alpha+s+1};q)_j}{(q^{\alpha+\beta+2+2n};q)_j(-q^{\beta+s+1};q)_j} \times (-q^{\beta+1+2n-s})^j q^{1/2(2r+l)(2r+l-1-2\beta+2s-4n)}$$
$$\cdot \frac{(q^{-j};q)_{2r+l}(q^{j+2s+\alpha+\beta+1};q)_{2r+l}(q^{\alpha+n+1-r-l};q)_{2r+l} \times (-q^{\beta+n+1-r};q)_{2r+l}}{(q^{-\alpha-n};q)_{r+l}(-q^{-\beta-n};q)_r(q^{\alpha+s+1};q)_{2r+l}(-q^{\alpha+s+1};q)_{2r+l} \times (q^{2s-2n};q)_{2r+l}}$$
$$\cdot {}_4\Phi_3\left[\begin{matrix} q^{2r+l-j},\ q^{j+2s+\alpha+\beta+1+2r+l}, & q^{s-n+r+l}, & -q^{s-n+r+\alpha-\beta} \\ q^{\alpha+s+1+2r+l}, & -q^{\alpha+s+1+2r+l}, & q^{2s-2n+2r+l}\end{matrix};q,q\right].$$

However, by (A.6),

$$\begin{aligned}&(q^{\alpha+n+1-r-l};q)_{2r+l}/(q^{-\alpha-n};q)_{r+l}\\ &= (q^{\alpha+n+1-r-l};q)_{r+l}(q^{\alpha+n+1};q)_r/(q^{-\alpha-n};q)_{r+l}\\ &= (-q^{\alpha+n+1})^{r+l}q^{-1/2(r+l)(r+l+1)}(q^{\alpha+n+1};q)_r\end{aligned}$$

and

$$\begin{aligned}&(-q^{\beta+n+1-r};q)_{2r+l}/(q^{-\beta-n};q)_r\\ &= (-q^{\beta+n+1-r};q)_r(-q^{\beta+n+1};q)_{r+l}/(q^{-\beta-n};q)_r\\ &= q^{(\beta+n+1)r-1/2r(r+1)}(-q^{\beta+n+1};q)_{r+l}.\end{aligned}$$

Hence

$$\text{(3.5)}\qquad \gamma_{r,l,j} = \delta_j(-1)^{r+l}q^{\epsilon(r,l)}(q^{\alpha+n+1};q)_r(-q^{\beta+n+1};q)_{r+l} \cdot \frac{(q^{-j};q)_{2r+l}(q^{j+2s+\alpha+\beta+1};q)_{2r+l}}{(q^{\alpha+s+1};q)_{2r+l}(-q^{\alpha+s+1};q)_{2r+l}(q^{2s-2n};q)_{2r+l}}$$
$$\cdot {}_4\Phi_3\left[\begin{matrix} q^{2r+l-j},\ q^{j+2s+\alpha+\beta+1+2r+l}, & -q^{s-n+r+\alpha-\beta}, & q^{s-n+r+l} \\ q^{\alpha+s+1+2r+l}, & -q^{\alpha+s+1+2r+l}, & q^{2s-2n+2r+l}\end{matrix};q,q\right]$$

where

$$\text{(3.6)}\qquad \delta_j = \frac{(q^{2s-2n};q)_j(-q^{\alpha+s+1};q)_j}{(q^{\alpha+\beta+2+2n};q)_j(-q^{\beta+s+1};q)_j}(-q^{\beta+1+2n-s})^j$$

and

$$\text{(3.7)}\qquad \epsilon(r,l) = \tfrac{1}{2}(2r+l)(2r+l-1+2s-4n-2\beta) + \tfrac{1}{2}(r+l)(2\alpha+2n+1-r-l) + \tfrac{1}{2}r(2\beta+2n+1-r).$$

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It can be easily verified that

$$(3.8)\qquad \frac{(-q^{s-n+\alpha-\beta-r-l};q)_{2r+l}(q^{s-n-r};q)_{2r+l}}{(-q^{\beta-\alpha+1+n-s};q)_{r+l}(q^{n-s+1};q)_r(-q^{s-n+\alpha-\beta};q)_r(q^{s-n};q)_{r+l}} = (-1)^r q^{-1/2(r+l)(r+l+1+2n-2s+2\beta-2\alpha)-1/2r(r+2n-2s+1)}.$$

Using this identity in (3.5) and simplifying the powers of q we have

$$(3.9)\qquad \gamma_{r,l,j} = \delta_j \frac{(q^{\alpha+n+1};q)_r(-q^{\beta+n+1};q)_{r+l}}{(-q^{s-n+\alpha-\beta};q)_r(q^{s-n};q)_{r+l}}\xi_{r,l,j},$$

where

$$(3.10)\qquad \xi_{r,l,j} = \frac{(q^{-j};q)_{2r+l}(q^{j+n+\alpha+\beta+1};q)_{2r+l} \times (-q^{s-n+\alpha-\beta-r-l};q)_{2r+l}(q^{s-n-r};q)_{2r+l}}{(-q^{\beta-\alpha+1+n-s};q)_{r+l}(q^{n-s+1};q)_r \times (q^{\alpha+s+1};q)_{2r+l}(-q^{\alpha+s+1};q)_{2r+l}(q^{2s-2n};q)_{2r+l}} \cdot (-1)^l q^{1/2(2r+l)(2r+l+1)}\, {}_4\Phi_3\left[\begin{matrix} q^{2r+l-j}, & q^{j+2s+\alpha+\beta+1+2r+l}, & -q^{s-n+\alpha-\beta+r}, & q^{s-n+r+l} \\ & q^{\alpha+s+1+2r+l}, & -q^{\alpha+s+1+2r+l}, & q^{2s-2n+2r+l} \end{matrix}; q, q\right].$$

Note that the right hand side has essentially the same form as the right hand side of (3.3) and so one can immediately transform $\xi_{r,l,j}$ to the form of (3.2). Thus

$$(3.11)\qquad \xi_{r,l,j} = \sum_{\lambda=0}^{j} \frac{(q^{-j};q)_\lambda(q^{j+2s+\alpha+\beta+1};q)_\lambda(-q^{s-n+\alpha-\beta};q)_\lambda(q^{s-n};q)_\lambda}{(q;q)_\lambda(q^{\alpha+s+1};q)_\lambda(-q^{\alpha+s+1};q)_\lambda(q^{2s-2n};q)_\lambda} \cdot \frac{(q^{-\lambda};q)_{2r+l}q^\lambda}{(-q^{n-s+1+\beta-\alpha-\lambda};q)_{r+l}(q^{n-s+1-\lambda};q)_r}.$$

The upshot of the whole calculation is that we are now getting some cancellations: the factors $(-q^{\beta+n+1};q)_{r+l}$ and $(q^{s-n};q)_{r+l}$ in (3.9) cancelling with their counterparts in (3.1). Thus, substituting (3.11) in (3.9) and finally in (3.1) we obtain

$$(3.12)\qquad F_j = \delta_j \sum_{\lambda=0}^{j} \frac{(q^{-j};q)_\lambda(q^{j+2s+\alpha+\beta+1};q)_\lambda(-q^{s-n+\alpha-\beta};q)_\lambda(q^{s-n};q)_\lambda}{(q;q)_\lambda(q^{\alpha+s+1};q)_\lambda(-q^{\alpha+s+1};q)_\lambda(q^{2s-2n};q)_\lambda} q^\lambda \times \sum_{r=0}^{[\lambda/2]} \frac{(p^{-\lambda/2};p)_r(p^{(1-\lambda)/2};p)_r(p^{s-n-\alpha-\beta};p)_r(p^{s-n-\beta};p)_r \times (p^{\alpha+n+1};p)_r(-q^{s-n};q)_r}{(p;p)_r(p^{s+1};p)_r(q^{2s-2n-\alpha-\beta};q)_{2r}(-q^{s-n+\alpha-\beta};q)_r \times (q^{n-s+1-\lambda};q)_r(-q^{n-s+1+\beta-\alpha-\lambda};q)_r} \times p^{r+1/2r(\beta-\alpha)}\, {}_3\Phi_2\left[\begin{matrix} q^{2r-\lambda}, & q^{\beta-\alpha}, & -q^{s-n-\alpha-\beta+r} \\ & -q^{n-s+1+\beta-\alpha-\lambda+r}, & q^{2s-2n-\alpha-\beta+2r} \end{matrix}; q, q\right].$$

The pleasant outcome is that the ${}_3\Phi_2$ series is balanced and hence summable by (A.12). Thus

$$ {}_3\Phi_2[\quad] = \frac{(-q^{s-n+r}; q)_{\lambda-2r}(q^{2s-2n-2\beta+2r}; q)_{\lambda-2r}}{(q^{2s-2n-\alpha-\beta+2r}; q)_{\lambda-2r}(-q^{s-n+\alpha-\beta+r}; q)_{\lambda-2r}}. $$

Now,

$$ \frac{(p^{s-n-\beta}; p)_r(-q^{s-n}; q)_r}{(q^{2s-2n-\alpha-\beta}; q)_{2r}(-q^{s-n+\alpha-\beta}; q)_r} \cdot \frac{(-q^{s-n+r}; q)_{\lambda-2r}(q^{2s-2n-2\beta+2r}; q)_{\lambda-2r}}{(q^{2s-2n-\alpha-\beta+2r}; q)_{\lambda-2r}(-q^{s-n+\alpha-\beta+r}; q)_{\lambda-2r}} $$

$$ = \frac{(q^{2s-2n-2\beta}; q)_\lambda}{(q^{2s-2n-\alpha-\beta}; q)_\lambda(p^{1/2+s-n-\beta}; p)_r} \cdot \frac{(-q^{s-n}; q)_{\lambda-r}}{(-q^{s-n+\alpha-\beta}; q)_{\lambda-r}} $$

$$ = \frac{(q^{2s-2n-2\beta}; q)_\lambda(-q^{s-n}; q)_\lambda}{(q^{2s-2n-\alpha-\beta}; q)_\lambda(-q^{s-n+\alpha-\beta}; q)_\lambda} \cdot \frac{(-q^{n-s+1+\beta-\alpha-\lambda}; q)_r}{(p^{1/2+s-n-\beta}; p)_r(-q^{n-s+1-\lambda}; q)_r} q^{(\alpha-\beta)r}. $$

Using this in (3.12) we get

$$ (3.13) \quad F_j = \delta_j \sum_{\lambda=0}^{j} \frac{(q^{-j}; q)_\lambda(q^{j+2s+\alpha+\beta+1}; q)_\lambda(p^{s-n}; p)_\lambda(q^{2s-2n-2\beta}; q)_\lambda}{(q; q)_\lambda(q^{2s-2n}; q)_\lambda(p^{\alpha+s+1}; p)_\lambda(q^{2s-2n-\alpha-\beta}; q)_\lambda} q^\lambda \cdot {}_4\Phi_3\left[\begin{matrix} p^{-\lambda/2}, p^{(1-\lambda)/2}, & p^{s-n-\alpha-\beta}, & p^{\alpha+n+1} \\ p^{s+1}, & p^{1/2+s-n-\beta}, & p^{n-s+1-\lambda}; p, p \end{matrix}\right]. $$

Observe that the ${}_4\Phi_3$ series in p is also balanced. Applying (A.14) we now obtain

$$ (3.14) \quad {}_4\Phi_3[\quad] = \frac{(p^{n-s+1/2-[\lambda/2]}; p)_{[\lambda/2]}(p^{-\alpha-s-\lambda}; p)_{[\lambda,2]}}{(p^{n-s+1-\lambda}; p)_{[\lambda/2]}(p^{-\alpha-s-1/2-[\lambda/2]}; p)_{[\lambda/2]}} $$

$$ \times {}_8\Phi_7\left[\begin{matrix} p^{\alpha+s+1/2}, p^{1+1/2(\alpha+s+1/2)}, & -p^{1+1/2(\alpha+s+1/2)}, & p^{\alpha+1/2}, \\ p^{1/2(\alpha+s+1/2)}, & -p^{1/2(\alpha+s+1/2)}, & p^{s+1}, \end{matrix}\right. $$

$$ \left.\begin{matrix} p^{\alpha+\beta+n+1}, & p^{\alpha+n+1}, & p^{(1-\lambda)/2}, & p^{-\lambda/2} \\ p^{1/2+s-n-\beta}, & p^{s-n+1/2}, & p^{\alpha+s+1+\lambda/2}, & p^{\alpha+s+3/2+\lambda/2}; p, p^{2s-2n-\alpha-\beta+\lambda} \end{matrix}\right], $$

where $[x]$ is the usual greatest integer function. Using the identities (A.6) and (A.8) we can show that the coefficient of ${}_8\Phi_7$ above simplifies to

$$ (p^{\alpha+s+1}; p)_\lambda(q^{2s-2n}; q)_\lambda/(p^{s-n}; p)_\lambda(q^{2\alpha+2s+2}; q)_\lambda. $$

Hence

$$(3.15)\quad F_j = \delta_j \sum_{\lambda=0}^{j} \frac{(q^{-j};q)_\lambda (q^{j+2s+\alpha+\beta+1};q)_\lambda (q^{2s-2n-2\beta};q)_\lambda}{(q;q)_\lambda (q^{2s-2n-\alpha-\beta};q)_\lambda (q^{2\alpha+2s+2};q)_\lambda} q^\lambda$$

$$\times \sum_{k=0}^{[\lambda/2]} \frac{(p^{\alpha+s+1/2};p)_k (p^{1/2(\alpha+s+1/2)+1};p)_k (-p^{1/2(\alpha+s+1/2)+1};p)_k \times (p^{\alpha+1/2};p)_k}{(p;p)_k (p^{1/2(\alpha+s+1/2)};p)_k (-p^{1/2(\alpha+s+1/2)};p)_k (p^{s+1};p)_k}$$

$$\cdot \frac{(p^{\alpha+\beta+n+1};p)_k (p^{\alpha+n+1};p)_k (q^{-\lambda};q)_{2k}}{(p^{1/2+s-n-\beta};p)_k (p^{s-n+1/2};p)_k (q^{2\alpha+2s+2+\lambda};q)_{2k}} p^{(\lambda+2s-2n-\alpha-\beta)k}$$

$$= \delta_j \sum_k \frac{(p^{\alpha+s+1/2};p)_k (p^{1/2(\alpha+s+1/2)+1};p)_k (-p^{1/2(\alpha+s+1/2)+1};p)_k \times (p^{\alpha+1/2};p)_k}{(p;p)_k (p^{1/2(\alpha+s+1/2)};p)_k (-p^{1/2(\alpha+s+1/2)};p)_k (p^{s+1};p)_k}$$

$$\cdot \frac{(p^{\alpha+\beta+n+1};p)_k (p^{\alpha+n+1};p)_k}{(p^{1/2+s-n-\beta};p)_k (p^{s-n+1/2};p)_k} p^{(2s-2n-\alpha-\beta)k} \eta_k,$$

where

$$(3.16)\quad \eta_k = \sum_{\lambda=0}^{j} \frac{(q^{-j};q)_\lambda (q^{j+2s+\alpha+\beta+1};q)_\lambda (q^{2s-2n-2\beta};q)_\lambda}{(q;q)_\lambda (q^{2s-2n-\alpha-\beta};q)_\lambda (q^{2\alpha+2s+2};q)_{\lambda+2k}} (q^{-\lambda};q)_{2k} \cdot q^{\lambda(2k+1)}.$$

The sum in (3.16) vanishes unless $j \geqq 2k$. So we make the transformation $\lambda \to \lambda + 2k$ and get

$$(3.17)\quad \eta_k = \frac{(q^{-j};q)_{2k} (q^{j+2s+\alpha+\beta+1};q)_{2k} (q^{2s-2n-2\alpha};q)_{2k}}{(q^{2s-2n-\alpha-\beta};q)_{2k} (q^{2\alpha+2s+2};q)_{4k}} q^{k(2k+1)}$$

$$\cdot {}_3\Phi_2\left[\begin{matrix} q^{2k-j},\ q^{j+2s+\alpha+\beta+1+2k},\ q^{2s-2n-2\beta+2k} \\ q^{2\alpha+2s+2+4k},\ q^{2s-2n-\alpha-\beta+2k} \end{matrix}; q, q\right].$$

Happily, the ${}_3\Phi_2$ here is also balanced and so, by applying (A.12) once again, we get its sum as

$$\frac{(q^{2\alpha+2\beta+2n+2+2k};q)_{j-2k} (q^{\alpha-\beta+1-j+2k};q)_{j-2k}}{(q^{2\alpha+2s+2+4k};q)_{j-2k} (q^{\alpha+\beta+1+2n-2s-j};q)_{j-2k}}.$$

We now substitute this in (3.17), apply (A.3) and simplify to get

$$(3.18)\quad \eta_k = \frac{(q^{2\alpha+2\beta+2n+2};q)_j (q^{\alpha-\beta+1-j};q)_j}{(q^{2\alpha+2s+2};q)_j (q^{\alpha+\beta+1+2n-2s-j};q)_j}$$

$$\cdot \frac{(q^{-j};q)_{2k} (q^{j+2s+\alpha+\beta+1};q)_{2k} (q^{2s-2n-2\beta};q)_{2k}\, q^{2(\alpha+\beta+1+2n-2s)k}}{(q^{2\alpha+2\beta+2n+2};q)_{2k} (q^{\alpha-\beta+1-j};q)_{2k} (q^{2\alpha+2s+2+j};q)_{2k}}.$$

Finally, we break up each of the products of the type $(a;q)_{2k}$ into two factors by (A.8) and then substitute in (3.15) which yields

$$(3.19)\quad F_j = \frac{(q^{2\alpha+2\beta+2n+2};q)_j(q^{\alpha-\beta+1-j};q)_j(q^{2s-2n};q)_j(-q^{\alpha+s+1};q)_j \times (-q^{\beta+1+2n-s})^j}{(q^{\alpha+\beta+2+2n};q)_j(q^{2\alpha+2s+2};q)_j(q^{\alpha+\beta+1+2n-2s-j};q)_j(-q^{\beta+s+1};q)_j}$$

$$\cdot {}_{10}\Phi_9\left[\begin{matrix} p^{\alpha+s+1/2}, p^{1+1/2(\alpha+s+1/2)}, -p^{1+1/2(\alpha+s+1/2)}, p^{\alpha+1/2}, p^{\alpha+n+1}, \\ p^{1/2(\alpha+s+1/2)}, -p^{1/2(\alpha+s+1/2)}, p^{s+1}, p^{s-n+1/2}, \end{matrix}\right.$$

$$\begin{matrix} p^{s-n-\beta}, p^{((\alpha+\beta+1)/2)+s+j/2}, p^{((\alpha+\beta+2)/2)+s+j/2}, p^{(1-j)/2}, \\ p^{\alpha+\beta+n+3/2}, p^{((\alpha-\beta)/2)+1-j/2}, p^{((\alpha-\beta)2)+1/2-j/2}, p^{\alpha+s+1+j/2}, \end{matrix}$$

$$\left.\begin{matrix} p^{-j/2} \\ p^{\alpha+s+3/2+j/2}; p, p \end{matrix}\right].$$

For general values of α, β the ${}_{10}\Phi_9$ in (3.19) cannot be summed and so the summation process ends here. However, the terms of this series do not have any definite sign pattern and so we look for a transformation that will do the job. F. H. Jackson's transformation, (A.16), between two well-poised balanced ${}_{10}\Phi_9$'s provides the answer for it:

$$(3.20)\quad {}_{10}\Phi_9\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, d, \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, \end{matrix}\right.$$

$$\left.\begin{matrix} B, C, D, q^{-k} \\ aq/B, aq/C, aq/D, aq^{k+1}; q, q \end{matrix}\right]$$

$$= \frac{(aq;q)_k(aq/BC;q)_k(aq/BD;q)_k(aq/CD;q)_k}{(aq/B;q)_k(aq/C;q)_k(aq/D;q)_k(aq/BCD;q)_k}$$

$$\times\ {}_{10}\Phi_9\left[\begin{matrix} a', q\sqrt{a'}, -q\sqrt{a'}, \frac{ba'}{a}, \frac{ca'}{a}, \frac{da'}{a}, \\ \sqrt{a'}, -\sqrt{a'}, \frac{aq}{b}, \frac{aq}{c}. \frac{aq}{d}, \end{matrix}\right.$$

$$\left.\begin{matrix} B, C, D, q^{-k} \\ a'q/B, a'q/C, a'q/D, a'q^{k+1}; q, q \end{matrix}\right]$$

where

$$(3.21)\quad a^3q^2 = bcdBCDq^{-k}$$

and

$$(3.22)\quad a' = a^2q/bcd.$$

The reason for writing the formula in this fashion is that it provides a clue as to how to use it in any given problem. Note that on the right hand side the last four top parameters, namely, B, C, D and q^{-k} remain unchanged while the three bottom parameters on the left remain as they are on the right hand side. The new parameter a' is constructed by combining a with the top three parameters b, c, d according to the prescription (3.22). Equation (3.21) simply expresses the balancedness of the series. We may now apply (3.20) directly to (3.19) by taking a as $p^{\alpha+s+1/2}$, b as $p^{\alpha+1/2}$, c as $p^{\alpha+n+1}$ and d as $p^{s-n-\beta}$. Thus we get, after simplifying the products in the coefficient,

$$(3.23)\quad {}_{10}\Phi_9\left[\begin{matrix} p^{\alpha+s+1/2}, & p^{1+1/2(\alpha+s+1/2)}, & -p^{1+1/2(\alpha+s+1/2)}, & p^{\alpha+1/2}, & p^{\alpha+n+1}, \\ & p^{1/2(\alpha+s+1/2)}, & -p^{1/2(\alpha+s+1/2)}, & p^{s+1}, & p^{s-n+1/2}, \end{matrix}\right.$$

$$\begin{matrix} p^{s-n-\beta}, & p^{((\alpha+\beta+1)/2)+s+j/2}, & p^{((\alpha+\beta+2)/2)+s+j/2}, & p^{(1-j)/2}, \\ p^{\alpha+\beta+n+3/2}, & p^{((\alpha-\beta)/2)+1-j/2}, & p^{((\alpha-\beta)/2)+1/2-j/2}, & p^{\alpha+s+1+j/2}, \end{matrix}$$

$$\left.\begin{matrix} p^{-j/2} \\ p^{\alpha+s+3/2+j/2}; & p, p \end{matrix}\right]$$

$$= \frac{(q^{2\alpha+2s+2}; q)_j (q^{\alpha-\beta}; q)_j (p^{\beta+s+1}; p)_j}{(q^{2\beta+2s+2}; q)_j (q^{\alpha-\beta+1-j}; q)_j (p^{\alpha+s+1}; p)_j} q^{-1/2j(j-1)}$$

$$\times\ {}_{10}\Phi_9\left[\begin{matrix} p^{\beta+s+1/2}, & p^{1+1/2(\beta+s+1/2)}, & -p^{1+1/2(\beta+s+1/2)}, & p^{\beta+1/2}, \\ & p^{1/2(\beta+s+1/2)}, & -p^{1/2(\beta+s+1/2)}, & p^{s+1}, \end{matrix}\right.$$

$$\begin{matrix} p^{\beta+n+1}, & p^{s-n-\alpha}, & p^{((\alpha+\beta+1)/2)+s+j/2}, & p^{((\alpha+\beta+2)/2)+s+j/2}, \\ p^{s-n+1/2}, & p^{\alpha+\beta+n+3/2}, & p^{((\beta-\alpha)/2)+1-j/2}, & p^{((\beta-\alpha)/2)+1/2-j/2}, \end{matrix}$$

$$\left.\begin{matrix} p^{(1-j)/2}, & p^{-j/2} \\ p^{\beta+s+1+(j/2)}, & p^{\beta+s+3/2+(j/2)}; & p, p \end{matrix}\right].$$

This, combined with (3.19), (1.40) and (1.39) immediately yields (1.43).

4. Remarks on special cases and the non-negativity of b_j. It is quite clear that the left hand side of (3.23) equals 1 when $\alpha = -\frac{1}{2}$. The ${}_{10}\Phi_9$ on the right hand side of (3.23) also becomes 1 when $\beta = -\frac{1}{2}$. Thus the coefficients in the linearization formula (1.38) break up into ratios of products when $\alpha = -\frac{1}{2}$ or $\beta = -\frac{1}{2}$. Also, when $\alpha = \beta$ the ${}_{10}\Phi_9$ on the right of (3.23) becomes a very well-poised terminating balanced ${}_8\Phi_7$, provided j is even, which can be summed by Jackson's theorem [**21**, p. 247].

The non-negativity of the coefficients is quite clear in (1.43) provided $-\frac{1}{2} \leqq \beta \leqq \alpha$. However, if $-1 < \beta < -\frac{1}{2}$ the parameter $p^{\beta+1/2}$ produces a negative factor $1 - p^{\beta+1/2}$ and hence the terms are not all non-negative. To bring about a non-negative expression in any extended region we need a transformation that is a q-analogue of Bailey's formula [**6**, eq(1), § 7.6, p. 63]. This analogue is easily obtained from (3.20) by transforming the ${}_{10}\Phi_9$ on the right hand side once more. We first construct a parameter, say a'' by taking one of the parameters ba'/a, ca'/a, da'/a and any two of the parameters B, C, D. If, in particular, we choose ba'/a and C, D we get

$$a'' = aa'^2q/CDba' = aa'q/bCD = a^3q^2/b^2cdCD = B/b^{q^{-k}}.$$

Thus

$$(4.1)\qquad {}_{10}\Phi_9\left[\begin{matrix} a', q\sqrt{a'}, -q\sqrt{a'}, ba'/a, & C, & D, ca'/a, da'/a, & B, \\ \sqrt{a'}, \ -\sqrt{a'}, \ aq/b, a'q/C, a'q/D, & aq/c, & aq/d, a'q/B, \end{matrix} \begin{matrix} q^{-k} \\ a'q^{k+1} \end{matrix}; q, q\right]$$

$$= \frac{(a'q;q)_k(a^2q/cda';q)_k(aq/cB;q)_k(aq/dB;q)_k}{(aq/c;q)_k(aq/d;q)_k(a'q/B;q)_k(a'q/Bcda';q)_k}$$

$$\times {}_{10}\Phi_9\left[\begin{matrix} (B/b)q^{-k}, q\sqrt{(B/b)q^{-k}}, -q\sqrt{(B/b)q^{-k}}, Bq^{-k}/a, \\ \sqrt{(B/b)q^{-k}}, \ -\sqrt{(B/b)q^{-k}}, \ aq/b, \end{matrix} \begin{matrix} (BC/ba')q^{-k}, (BD/ba')q^{-k}, & ca'/a, & da'/a, \\ a'q/C, & a'q/D, Baq^{1-k}/bca', & Baq^{1-k}/bda', \end{matrix} \begin{matrix} B, & q^{-k} \\ q^{1-k}/b, & Bq/b \end{matrix}; q, q\right].$$

This, combined with (3.20) through (3.22) as well as (A.6) produces the following formula:

$$(4.2)\qquad {}_{10}\Phi_9\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, & b, & c, & d, \\ \sqrt{a}, \ -\sqrt{a}, aq/b, aq/c, aq/d, \end{matrix} \begin{matrix} B, & C, & D, & q^{-k} \\ aq/B, aq/C, aq/D, aq^{k+1} \end{matrix}; q, q\right]$$

$$= \frac{(aq;q)_k(aq/BC;q)_k(aq/BD;q)_k(aq/cB;q)_k(aq/dB;q)_k \times (b;q)_k}{(aq/c;q)_k(aq/d;q)_k(aq/B;q)_k(aq/C;q)_k(aq/D;q)_k \times (b/B;q)_k} B^k$$

$$\times {}_{10}\Phi_9\left[\begin{matrix} (B/b)q^{-k}, q\sqrt{(B/b)q^{-k}}, -q\sqrt{(B/b)q^{-k}}, & B, \\ \sqrt{(B/b)q^{-k}}, \ -\sqrt{(B/b)q^{-k}}, q^{1-k}/b, \end{matrix} \begin{matrix} aq/bc, & aq/bd, & aq/bC, & aq/bD, Bq^{-k}/a, & q^{-k} \\ Bcq^{-k}/a, Bdq^{-k}/a, BCq^{-k}/a, BDq^{-k}/a, & aq/b, Bq/b \end{matrix}; q, q\right],$$

while the parameters are, of course, still connected by (3.21).

In order to apply this to (1.43) it is convenient to take the ${}_{10}\Phi_9$ on the left hand side of (3.23) rather than that on the right hand side. First consider the case when j is even. We choose $p^{\alpha+1/2}$ as B and $p^{(1-j)/2}$ as b. Then (4.2) gives the left hand side of (3.23) as

$$(4.3)\quad \frac{(p^{\alpha+s+3/2};p)_{j/2}(p^{s-n-\alpha};p)_{j/2}(p^{\beta+n+1};p)_{j/2}(p^{1/2-((\alpha+\beta)/2)-j/2};p)_{j/2} \times (p^{-((\alpha+\beta)/2)-j/2};p)_{j/2}(p^{(1-j)/2};p)_{j/2}\,p^{(\alpha+1/2)j/2}}{(p^{s+1};p)_{j/2}(p^{s-n+1/2};p)_{j/2}(p^{\alpha+\beta+n+3/2};p)_{j/2}(p^{((\alpha-\beta)/2)+1-j/2};p)_{j/2} \times (p^{((\alpha-\beta)/2)+1/2-j/2};p)_{j/2}(p^{-\alpha-j/2};p)_{j/2}}$$

$$\times\, {}_{10}\Phi_9\left[\begin{matrix} p^{\alpha}, p^{1+\alpha/2}, -p^{1+\alpha/2}, p^{\alpha+1/2}, p^{(\alpha-\beta)/2}, p^{((\alpha-\beta+1)/2)+1/2}, \\ p^{\alpha/2}, -p^{\alpha/2}, p^{1/2}, p^{((\alpha+\beta)/2)+1}, p^{(\alpha+\beta+1)/2}, \end{matrix}\right.$$

$$\left.\begin{matrix} p^{s-n+j/2}, p^{\alpha+\beta+n+1+j/2}, p^{-s-j/2}, p^{-j/2} \\ p^{\alpha+n+1-s-j/2}, p^{-\beta-n-j/2}, p^{\alpha+s+1+j/2}, p^{\alpha+1+j/2}; p, p \end{matrix}\right].$$

On the other hand, if j is odd then we take $p^{1/2-j/2}$ as p^{-k}, $q^{\alpha+1/2}$ as B and $p^{-j/2}$ as b. Then we get the left hand side of (3.23) in the form

$$(4.4)\quad \frac{(p^{\alpha+s+3/2};p)_{(j-1)/2}(p^{s-n-\alpha};p)_{(j-1)/2}(p^{\beta+n+1};p)_{(j-1)/2} \times (p^{1/2-((\alpha+\beta)/2)-(j/2)};p)_{(j-1)/2}(p^{-((\alpha+\beta)/2)-(j/2)};p)_{(j-1)/2} \times (p^{-j/2};p)_{(j-1)/2}p^{(\alpha+1/2)((j-1)/2)}}{(p^{s+1};p)_{(j-1)/2}(p^{s-n+1/2};p)_{(j-1)/2}(p^{\alpha+\beta+n+3/2};p)_{(j-1)/2} \times (p^{((\alpha-\beta)/2)+1-(j/2)};p)_{(j-1)/2}(p^{((\alpha-\beta)/2\,+1/2-(j/2)};p)_{(j-1)/2} \times (p^{-\alpha-1/2-(j/2)};p)_{(j-1)/2}}$$

$$\cdot\, {}_{10}\Phi_9\left[\begin{matrix} p^{\alpha+1}, p^{3/2+\alpha/2}, -p^{3/2+\alpha/2}, p^{\alpha+1/2}, p^{((\alpha-\beta)/2)+1}, p^{((\alpha-\beta)/2)+1/2}, \\ p^{1/2+\alpha/2}, -p^{1/2+\alpha/2}, p^{3/2}, p^{((\alpha+\beta)/2)+1}, p^{((\alpha+\beta)/2)+3/2}, \end{matrix}\right.$$

$$\left.\begin{matrix} p^{s-n+1/2+j/2}, p^{\alpha+\beta+n+3/2+j/2}, p^{1/2-s-j/2}, p^{(1-j)/2} \\ p^{\alpha+3/2+n-s-j/2}, p^{1/2-\beta-n-j/2}, p^{\alpha+3/2+s+j/2}, p^{\alpha+3/2+j/2}; p, p \end{matrix}\right].$$

In both cases one can easily see that the terms of the ${}_{10}\Phi_9$ series are non-negative if $0 \leqq \alpha + \beta + 1$ and $-1 < \beta \leqq \alpha$. One can obtain different expressions for b_j depending on whether j is even or odd by using (4.3) or (4.4) in (3.23) and going back to (1.43). It is simple to verify that the coefficients also remain non-negative in the extended region.

For the ultraspherical case $\alpha = \beta$ it is obvious from (3.19) and (4.4) that b_j vanishes when j is odd while (4.3) combined with (3.19) and

(1.40) gives the following value of b_j when j is even:

$$(4.5)\qquad b_j = \frac{(q;q)_n(q^{\beta+1};q)_n(q^{2\beta+1};q)_{2s}(q^{2\beta+1+n-s};q)_{n-s} \times (-q^{\beta+1};q)_n(-q^{2\beta+1};q)_n}{(q;q)_s(q^{\beta+1};q)_s(-q^{\beta+1};q)_s(q^{\beta+1};q)_{n-s}(-q^{\beta+1};q)_{n-s} \times (q^{2\beta+2};q)_{2n}(-q;q)_{n-s}(-q^{2\beta+1};q)_s}$$

$$\cdot \frac{(q^{2\beta+2s+1};q)_j(-q^{s+1};q)_j}{(q;q)_j(-q^{2\beta+s+1};q)_j} \cdot \frac{1-q^{2\beta+2s+2j+1}}{1-q^{2\beta+1}}$$

$$\times \frac{(q^{4\beta+2n+2};q)_j(q^{2s-2n};q)_j(-q^{\beta+1+2n-s})^j}{(q^{2\beta+2n+2};q)_j(q^{2\beta+2s+2};q)_j(q^{2\beta+1+2n-2s-j};q)_j}$$

$$\times \frac{(p^{\beta+s+3/2};p)_{j/2}(p^{s-n-\beta};p)_{j/2}(p^{\beta+n+1};p)_{j/2}}{(p^{1/2-\beta-j/2};p)_{j,2}(p^{1/2-j};p)_{j,2}\, p^{(\beta+1/2)(j/2)}} \frac{}{(p^{s+1};p)_{j/2}(p^{s-n+1/2};p)_{j/2}(p^{2\beta+n+3/2};p)_{j/2}}\, q^{n-s-j(s+1)}.$$

This result is essentially the same as that of Rogers [**19**] who used a method of induction to prove his formula by taking the following representation of the q-ultraspherical polynomials

$$(4.6)\qquad c_n(\cos\theta;\beta|q) = \frac{(\beta;q)_n}{(q;q)_n}\, e^{in\theta}\, {}_2\Phi_1\left[\begin{matrix} q^{-n}, & \beta \\ q^{1-n}, & \beta^{-1} \end{matrix}; q, q\beta^{-1}e^{2i\theta}\right].$$

For further comments on Rogers' formula see [**3**].

We would like to point out another special case when the b_j's become ratios of products. Note that the ${}_{10}\Phi_9$'s in (4.3) and (4.4) become ${}_8\Phi_7$'s when $\alpha = \beta + 1$, and, as we said before, such ${}_8\Phi_7$'s are summable. This provides a q-analogue of Hylleraas' result [**14**] for this particular case.

Appendix. Throughout the paper we have made use of a large number of basic identities, summation and transformation theorems that we would like to state in this appendix. First, we list a set of relations between products of the type $(a;q)_n$. These relations are given in [**21**, p. 241-242].

$$(A.1)\qquad (a;q)_n = (1-a)(1-aq)(1-aq^2)\ldots(1-aq^{n-1}),\ (a;q)_0 = 1;$$

$$(a;q)_\infty = \prod_{r=0}^{\infty}(1-aq^r) = (a;q)_n(aq^n;q)_\infty.$$

$$(A.2)\qquad (a;q)_{m+n} = (a;q)_m(aq^m;q)_n.$$

$$(A.3)\qquad (a;q)_{m-n} = (-a)^{-n}\frac{(a;q)_m}{(q^{1-m};q)_n}\, q^{1/2n(n+1)-mn}.$$

$$(A.4)\qquad (aq^{-n};q)_{N-n} = (a;q)_N(q/a;q)_n(-a)^{-n}q^{1/2n(3n+1)-2Nn}/(q^{1-N}/a;q)_{2n}.$$

$$(A.5)\qquad (aq^{-n};q)_N = \frac{(a;q)_N(q/a;q)_n}{(q^{1-N}/a;q)_n}\, q^{-Nn}.$$

$$(A.6)\qquad (aq^{-n};q)_n = (-a)^n q^{-1/2n(n+1)}(q/a;q)_n.$$

In these relations it is assumed that m, n, N are non-negative integers with values such that expressions on the right hand side of each relation remains meaningful. The q-analogue of the duplication formula for gamma functions consists of two relations

(A.7) $$(a; q^2)_n = (\sqrt{a}; q)_n(-\sqrt{a}; q)_n.$$

(A.8) $$(a; q)_{2n} = (a; q^2)_n(aq; q^2)_n.$$

Next we state a very important summation theorem in basic hypergeometric series, due to F. H. Jackson [**21**, p. 96]

(A.9) $$ {}_8\phi_7\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, & b, & c, & d, & e, & q^{-n} \\ \sqrt{a}, -\sqrt{a}, & aq/b, & aq/c, & aq/d, & aq/e, & aq^{n+1} \end{matrix}; q, q\right] = \frac{(aq; q)_n(aq/bc; q)_n(aq/bd; q)_n(aq/cd; q)_n}{(aq/b; q)_n(aq/c; q)_n(aq/d; q)_n(aq/bcd; q)_n},$$

provided

(A.10) $$a^2q = bcdeq^{-n}.$$

This last property implies that the product of the parameters in the denominator in the ${}_8\Phi_7$ above is q times that of the parameters in the numerator, that is, the series is balanced. This is the q-analogue of Dougall's summation theorem [**6**, p. 26] for a very well-poised and 2-balanced ${}_7F_6(1)$ series.

As consequences of this basic theorem we get the analogue of a ${}_5F_4$ (1) series:

(A.11) $$ {}_6\phi_5\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, & b, & c, & q^{-n} \\ \sqrt{a}, -\sqrt{a}, & aq/b, & aq/c, & aq^{n+1} \end{matrix}; 1, aq^{n+1}/bc\right] = \frac{(aq; q)_n(aq/bc; q)_n}{(aq/b; q)_n(aq/c; q)_n},$$

and the sum of a balanced terminating ${}_3\phi_2$ series;

(A.12) $$ {}_3\phi_2\left[\begin{matrix} b, c, & q^{-n} \\ d, & bcq^{1-n}/d \end{matrix}; q, q\right] = \frac{(d/b; q)_n(d/c; q)_n}{(d; q)_n(d/bc; q)_n}.$$

As mentioned in [**21**, p. 96] the q-analogue of Dixon's theorem for a well-poised ${}_3F_2(1)$ series is not ${}_3\phi_2$ but a ${}_4\phi_3$ given by

(A.13) $$ {}_4\phi_3\left[\begin{matrix} a, -q\sqrt{a}, & b, & c \\ -\sqrt{a}, & aq/b, & aq/c \end{matrix}; q, q\sqrt{a}/bc\right] = \frac{(aq; q)_\infty(q\sqrt{a}/b; q_\infty(q\sqrt{a}/c; q)_\infty(aq/bc; q)_\infty}{(aq/b; q)_\infty(aq/c; q)_\infty(q\sqrt{a}; q)_\infty(q\sqrt{a}/bc; q)_\infty}.$$

Let us now turn to the transformation theorems. In this paper we have made repeated use of the following formula, which is due to Watson [**21**, p. 100] and is the q-analogue of Whipple's well-known formula connecting a balanced ${}_4F_3(1)$ with a very well-poised ${}_7F_6(1)$:

$$\text{(A.14)} \quad {}_4\phi_3\left[\begin{matrix} aq/bc, & d, & e, & f \\ & def/a, & aq/b, & aq/c \end{matrix}; q, q\right]$$
$$= \frac{(aq/d;q)_\infty(aq/e;q)_\infty(aq/f;q)_\infty(aq/def;q)_\infty}{(aq;q)_\infty(aq/ef;q)_\infty(aq/fd;q)_\infty(aq/de;q)_\infty}$$
$$\times {}_8\phi_7\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, & b, & c, & d, & e, & f \\ \sqrt{a}, & -\sqrt{a}, & aq/b, & aq/c, & aq/d, & aq/e, & aq/f \end{matrix}; q, \frac{a^2q^2}{bcdef}\right]$$

provided one of the parameters d, e, f is of the form q^{-n}, n a non-negative integer. If, in particular, $f = q^{-n}$, then, by (A.1), the coefficient on the right hand side of (A.14) simplifies to

$$\frac{(aq/d;q)_n(aq/e;q)_n}{(aq;q)_n(aq/de;q)_n}.$$

By considering an interchange of the parameters b, c, d, e which leaves the ${}_8\phi_7$ unchanged but changes the ${}_4\phi_3$ it is easy to deduce the analogue of Whipple's transformation formula between two terminating and balanced ${}_4\phi_3$'s:

$$\text{(A.15)} \quad {}_4\phi_3\left[\begin{matrix} q^{-n}, q^{n-1}abcd, & ae, & af \\ & ab, ace, adf \end{matrix}; q, q\right] = \frac{(bd/e;q)_n(bc/f;q)_n}{(ace;q)_n(adf;q)_n}\left(\frac{aef}{b}\right)^n$$
$$\times {}_4\phi_3\left[\begin{matrix} q^{-n}, q^{n-1}abcd, & b/e, & b/f \\ & ab, bd/e, bc/f \end{matrix}; q, q\right].$$

The last transformation formula is due to F. H. Jackson [**21**, p. 102] connecting a terminating very well-poised balanced ${}_{10}\phi_9$ series with another:

$$\text{(A.16)} \quad {}_{10}\phi_9\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, & b, & c, & d, & e, & f, & \frac{a^3q^{n+2}}{bcdef}, q^{-n} \\ \sqrt{a}, & -\sqrt{a}, & aq/b, & aq/c, & aq/d, & aq/e, & aq/f, & \frac{bcdefq^{-n-1}}{a^2}, aq^{n+1} \end{matrix}; q, q\right]$$
$$= \frac{(aq;q)_n(aq/ef;q)_n\left(\frac{bcdfq^{-n-1}}{a^2};q\right)_n\left(\frac{bcdeq^{-n-1}}{a^2};q\right)_n}{(aq/e;q)_n(aq/f;q)_n\left(\frac{bcdefq^{-n-1}}{a^2};q\right)_n\left(\frac{bcdq^{-n-1}}{a^2};q\right)_n}$$
$${}_{10}\phi_9\left[\begin{matrix} k, q\sqrt{k}, -q\sqrt{k}, & kb/a, & kc/a, & kd/a, & e, & f, & a^3q^{n+2}/bcdef, & q^{-n} \\ \sqrt{k}, & -\sqrt{k}, & aq/b, & aq/c, & aq/d, & kq/e, & kq/f, & kbcdefq^{-n-1}/a^3, & kq^{n+1} \end{matrix}; q, q\right]$$

where $k = a^2q/bcd$.

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