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ON ORTHOGONAL POLYNOMIALS WITH RESPECT TO A POSITIVE DEFINITE MATRIX OF MEASURES

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ABSTRACT. In this paper, we prove that any sequence of polynomials $(p_n)_n$ for which $\mathrm{dgr}(p_n) = n$ which satisfies a $(2N+1)$-term recurrence relation is orthogonal with respect to a positive definite $N \times N$ matrix of measures. We use that result to prove asymptotic properties of the kernel polynomials associated to a positive measure or a positive definite matrix of measures. Finally, some examples are given.

1. **Introduction.** In this paper we introduce an extension to the theory of orthogonal polynomials with respect to a positive measure, by considering orthogonal polynomials with respect to a positive definite matrix of measures. A $N \times N$ matrix of measures $\mu = (\mu_{i,j})_{i,j=1}^N$ ($\mu_{i,j}$ being complex Borel measures) will be called *positive definite* if for any Borel set A the numerical matrix $\big(\mu_{i,j}(A)\big)_{i,j=1}^N$ is positive semidefinite (in particular these numerical matrices are hermitian).

We associate with every positive definite matrix of measures an inner product (possibly degenerate) defined on the linear space of complex polynomials $\mathbb{P}$ as follows: given a natural number m, $0 \leq m \leq N-1$, we define the operators $R_{N,m}: \mathbb{P} \longrightarrow \mathbb{P}$ by

$$R_{N,m}(p) = \sum_n \frac{p^{(nN+m)}(0)}{(nN+m)!} t^n \tag{1.1}$$

i.e. for every m, the operator $R_{N,m}$ takes from the polynomial p just those powers t^k, for which $k \equiv m \pmod N$ and then changes t^{nN+m} to t^n.

Using these operators, we define the inner product B_μ associated with μ by

$$B_\mu(p,q) = \sum_{m,m'=1}^N \int R_{N,m-1}(p)\overline{R_{N,m'-1}(q)}\, d\mu_{m,m'} \quad \text{for } p, q \in \mathbb{P}. \tag{1.2}$$

We say that the polynomials $(p_n)_n$ are orthogonal (resp. orthonormal) with respect to the matrix of measures μ if they are orthogonal (resp. orthonormal) with respect to the inner product B_μ.

An important result in the classical theory of orthogonal polynomials is the so-called Favard Theorem [F] (although the result seems to be known already to Stieltjes, Chebyshev, and others) which establishes a close relationship between orthogonality with respect to a positive measure and three term recurrence relations:

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THEOREM (FAVARD). *Let $(p_n)_n$ be a sequence of polynomials for which* $\operatorname{dgr}(p_n) = n$ *and which satisfies the following three term recurrence relation:*

$$tp_n(t) = a_{n+1}p_{n+1}(t) + b_n p_n(t) + a_n p_{n-1}(t),$$

where $(a_n)_n$ and $(b_n)_n$ are real sequences and $p_{-1}(t) = 0$. Then, there exists a positive measure with respect to which the polynomials $(p_n)_n$ are orthonormal.

As the main result in this paper, we extend Favard's Theorem establishing a close relationship between orthogonality with respect to a positive definite $N \times N$ matrix of measures and $(2N+1)$-term recurrence relations: we say that a sequence of polynomials $(p_n)_n$ for which $\operatorname{dgr}(p_n) = n$ satisfies a (symmetric) $(2N+1)$-term recurrence relation if the following formula

$$t^N p_n(t) = c_{n,0}p_n(t) + \sum_{l=1}^{N}\Big(c_{n,l}p_{n-l}(t) + c_{n+l,l}p_{n+l}(t)\Big) \tag{1.3}$$

holds, where $(c_{n,l})_n$ are real sequences for $l = 0, \ldots, N$ and $p_l(t) = 0$ if $l < 0$.

THEOREM 1. *Let $(p_n)_n$ be a sequence of polynomials for which* $\operatorname{dgr}(p_n) = n$ *and which satisfies a $(2N+1)$-term recurrence relation. Then there exists a positive definite $N \times N$ matrix of measures with respect to which the polynomials $(p_n)_n$ are orthonormal.*

Here, we should compare this theorem with the results in [D]. There, we proved that the operator of multiplication by t^N is symmetric for an inner product B if and only if B is the inner product defined by a $N \times N$ matrix of measures. Let us note that the operator of multiplication by t^N is symmetric for B if and only if the sequence of orthonormal polynomials for B satisfies a $(2N+1)$-term recurrence relation. However, in [D], we did not give any positivity conditions on the matrix of measures μ, as Theorem 1 gives. These positivity conditions on the matrix of measures are essential to extend the inner product from the space of polynomials to a Hilbert space of functions. Thus, for a positive definite $N \times N$ matrix of measures $\mu = (\mu_{i,j})_{i,j=1}^N$, we consider the Hilbert space (which is denoted also by $L^2(\mu)$) of vector valued functions $f\colon \mathbb{R} \longrightarrow \mathbb{C}^N$ for which there exists a sequence of simple functions $(\phi_n)_n$, $\phi_n = (\phi_{n,1}, \ldots, \phi_{n,N})$ which tends $\mu_{i,j}$ a.e. to f for $i,j = 1, \ldots, N$ and which is a Cauchy sequence with respect to μ, *i.e.*,

$$\sum_{i,j=1}^{N} \int \Big(\phi_{n,i}(t) - \phi_{m,i}(t)\Big)\Big(\overline{\phi_{n,j}(t) - \phi_{m,j}(t)}\Big)\, d\mu_{i,j}(t) \longrightarrow 0, \quad n,m \longrightarrow \infty.$$

The space of scalar polynomials $\mathbb{P}$ is included in this Hilbert space of vector valued functions by using the operators $R_{N,m}$:

$$\begin{aligned} R_N\colon \mathbb{P} &\longrightarrow L^2(\mu) \\ p &\longrightarrow \Big(R_{N,m}(p)\Big)_{m=0}^{N-1}. \end{aligned}$$

Hence, we have that the operator R_N is an isometry between the linear space of scalar polynolmials $\mathbb{P}$ (with the inner product defined by the positive definite matrix of measures μ) and the subspace of vector valued polynomials $\mathbb{P}^N$ of $L^2(\mu)$.

To complete this paper, we apply Theorem 1 to prove new asymptotic properties of the kernel polynomials associated with a positive measure. Indeed, let $(p_n)_n$ be the sequence of orthonormal polynomials with respect to a positive measure ρ. The kernel polynomials are defined by $k_n(z,w) = \sum_{k=0}^{n} p_k(z)\overline{p_k(w)}$. The following asymptotic property for these kernel polynomials is very well-known: for every $x \in \mathbb{R}$

$$\frac{1}{\sum_{n=0}^{\infty} |p_n(x)|^2} = \sup\Big\{\nu(\{x\}) : \nu \text{ is a positive measure with respect to which the polynomials } (p_n)_n \text{ are orthonormal}\Big\}.$$

But similar formulas for $x \in \mathbb{C} \setminus \mathbb{R}$ or for the derivatives of the polynomials do not seem to be known. Here we shall show such formulas: for a positive measure ρ, let us put

$$A_\rho = \{\nu : \ \nu \text{ is a positive measure which defines the same inner product on } \mathbb{P} \text{ as } \rho\ \}.$$

Let us put B_ρ for this inner product, *i.e.*, $B_\rho(p,q) = \int p(t)\overline{q(t)}\,d\rho$. For $\nu \in A_\rho$ and $N \in \mathbb{N}$, we define the measures $\nu_{i,j}$ as follows: if N is odd then $\nu_{i,j} = \nu$, and if N is even, then $\nu_{i,j}$ is the measure with support in $[0,+\infty)$ defined by $\nu_{i,j}(A) = \nu(A)+(-1)^{i+j}\nu(-A)$. Now, we set $\mu_{\nu,i,j}$ the measure with density $t^{\frac{i+j-2}{N}}$ with respect to the image measure $\nu_{i,j}^{\psi} = \nu_{i,j}\psi^{-1}$ where $\psi(t) = t^{\frac{1}{N}}$. Then it is not hard to see that the matrix of measures $\mu_\nu = (\mu_{\nu,i,j})_{i,j=1}^{N}$ is positive definite and that its associated inner product B_{μ_ν} (see (1.2)) is just the same as B_ρ. This means that orthogonal polynomials with respect to a positive measure are a particular case of orthogonal polynomials with respect to a positive definite matrix of measures. It should be noticed that it could be possible that the inner product B_ρ was obtained from another positive definite $N \times N$ matrix of measures σ different from those μ_ν, $\nu \in A_\rho$ defined above. Actually, such a matrix of measures plays a fundamental role in understanding the behaviour of $\frac{1}{\sum_n |p_n^{(N)}(x)|^2}$ for $x \in \mathbb{R}$, or $\frac{1}{\sum_n |p_n(xw)|^2}$ for w satisfying $w^N = -1$ and $x \in \mathbb{R}$. To illustrate this, we mention the following results:

$$\frac{1}{\sum_n |p_n'(0)|^2} = \sup\Big\{\mu_{22}(\{0\}) - \frac{|\mu_{12}|^2(\{0\})}{\mu_{11}(\{0\})} : \mu = (\mu_{ij})_{i,j=1}^{2} \text{ is a positive definite } 2\times 2 \text{ matrix of measures with respect to which the polynomials } (p_n)_n \text{ are orthonormal}\Big\}$$

$$\frac{1}{\sum_n |p_n(xi)|^2} = \sup\Big\{\min\Big\{\mu_{11}(\{-x^2\}), \frac{\mu_{11}(\{-x^2\})\mu_{22}(\{-x^2\}) - |\mu_{12}(\{-x^2\})|^2}{x^2\mu_{11}(\{-x^2\}) + \mu_{22}(\{-x^2\}) - 2x\Im\mu_{12}(\{-x^2\})}\Big\} : \mu = (\mu_{ij})_{i,j=1}^{2} \text{ is a positive definite } 2\times 2 \text{ matrix of measures with respect to which the polynomials } (p_n)_n \text{ are orthonormal}\Big\}.$$

Thus, this theory of orthogonal polynomials with respect to a positive definite matrix of measures is going to be interesting not only for its own sake, but also because of their applications to the classical theory. Here, we should say that in a subsequent paper, a very close relationship between polynomials satisfying a $(2N+1)$-term recurrence relation and $N \times N$ matrix orthogonal polynomials will be established.

In Section 2, we have put together some basic definitions about recurrence relations and inner products. Section 3 contains an extension of Theorem 1, its proof and some remarks about density of polynomials in the Hilbert space $L^2(\mu)$ for a positive definite matrix of measures μ. Section 4 contains some results about positive semidefinite matrices. Section 5 contains the results about kernel polynomials mentioned above and the extensions of these results for orthonormal polynomials with respect to a positive definite matrix of measures. Finally, Section 6 contains some examples. Thus, we show a new family of Brenke type polynomials which are orthogonal with respect to a positive definite 2x2 matrix of measures. They show that the class of Brenke type polynomials which are orthogonal with respect to a positive definite 2×2 matrix of measures is wider than that of Brenke type polynomials orthogonal with respect to a positive measure (see [Ch, p. 167]).

2. **Preliminaries.** As usual, $\mathbb{P}$ will denote the linear space of complex polynomials. It is well-known that there is a bijection between inner products B defined on $\mathbb{P}$ and positive definite Hermitian matrices $(a_{i,j})_{i,j}$ which are characterized by $\Delta_n = \det\big((a_{i,j})_{i,j=0}^n\big) > 0$ for all $n \in \mathbb{N}$. In fact, this bijection is given by $a_{i,j} = B(t^i, t^j)$.

We can obtain an expression for the sequence of orthonormal polynomials $(p_n)_n$ with respect to an inner product B:

$$p_n(t) = \frac{1}{\sqrt{\Delta_{n-1}\Delta_n}} \begin{vmatrix} a_{0,0} & \cdots & a_{0,n} \\ \vdots & \ddots & \vdots \\ a_{n-1,0} & \cdots & a_{n-1,n} \\ 1 & \cdots & t^n \end{vmatrix}.$$

The inner product B can be defined from its sequence of orthonormal polynomials $(p_n)_n$ as follows. If $p(t) = \sum_n \alpha_n p_n(t)$ and $q(t) = \sum_n \beta_n p_n(t)$, then $B(p,q) = \sum_n \alpha_n \bar{\beta}_n$. Conversely, every sequence of polynomials $(p_n)_n$ for which $\operatorname{dgr} p_n = n$ defines a unique inner product B by putting $B(p,q) = \sum_n \alpha_n \bar{\beta}_n$, if $p(t) = \sum_n \alpha_n p_n(t)$ and $q(t) = \sum_n \beta_n p_n(t)$, so that $(p_n)_n$ are orthonormal with respect to B.

Let h be a real polynomial of degree N. We say that the sequence of polynomials $(p_n)_n$ with $\operatorname{dgr} p_n = n$, satisfies a (symmetric) $(2N+1)$ term recurrence relation defined by h if the following formula holds:

$$h(t)p_n(t) = c_{n,0}p_n(t) + \sum_{l=1}^{N}\Big(c_{n,l}p_{n-l}(t) + c_{n+l,l}p_{n+l}(t)\Big), \tag{2.1}$$

where $(c_{n,l})_n$ are real sequences for $l = 0, \ldots, N$ (of course, if $l < 0$ then $c_{n,l} = p_l = 0$). Because of the condition on the degree of the polynomial p_n, we have $c_{n,N} \neq 0$ for all $n \in \mathbb{N}$. Let us notice that for $N = 1$ and $h(t) = t$, we get the classical three term recurrence relation which characterizes the orthonormal polynomials with respect to a positive measure.

It is not hard to see that the inner products whose sequence of orthonormal polynomials satisfies a $(2N+1)$-term recurrence relation defined by a polynomial h are the same as those for which the operator of multiplication by h is symmetric, *i.e.*, $B(hp,q) = B(p,hq)$ for all $p, q \in \mathbb{P}$. We shall use this fact often.

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For a real number x, we consider the basis of $\mathbb{P}$ given by

$$\{(t-x)^k h^n(t) : k = 0, 1, \dots, N-1, n \geq 0\}.$$

For a non-negative integer m, for which $0 \leq m \leq N-1$, we define the operators $R_{h,x,m}$ by

$$R_{h,x,m} \colon \mathbb{P} \longrightarrow \mathbb{P}$$

$$R_{h,x,m}(p) = \sum_{n\geq 0} a_{m,n}(t-x)^n \quad \text{if } p(t) = \sum_{\substack{k=0,1,\dots,N-1\\ n\in\mathbb{N}}} a_{k,n}(t-x)^k h(t)^n. \tag{2.2}$$

Let us notice that for $h(t) = (t-x)^N$

$$R_{N,x,m}(p) = \sum_n \frac{p^{(nN+m)}(x)}{(nN+m)!} t^n.$$

For $x = 0$ and $h(t) = t^N$ these operators coincide with the operators $R_{N,m}$ defined in the Introduction of this paper.

From [D] we get, after a straightforward reformulation, the following orthogonality conditions for a sequence of polynomials satisfying a $(2N+1)$-term recurrence relation defined by the polynomial h:

THEOREM A. *Let $(p_n)_n$ be a sequence of polynomials for which* $\operatorname{dgr}(p_n) = n$ *and which satisfies a $(2N+1)$-term recurrence relation like (2.1). For every real number x, there exists a $N \times N$ matrix of measures such that the bilinear form defined by*

$$B_\mu(p,q) = \sum_{m,m'=1}^{N} \int R_{h,x,m-1}(p)\overline{R_{h,x,m'-1}(q)}\, d\mu_{m.m'} \quad \textit{for } p,q \in \mathbb{P} \tag{2.3}$$

is an inner product on $\mathbb{P}$ and $(p_n)_n$ is the sequence of orthonormal polynomials with respect to B.

According to this theorem, we define

DEFINITION 2.1. An inner product B is said to be (h,x)-*defined* by the matrix of measures $\mu = (\mu_{i,j})_{i,j=1}^N$ if it is defined from this matrix of measures by the expression (2.3). For $h(t) = (t-x)^N$, we will say x-defined by μ, and for $h(t) = t^N$ and $x = 0$ we simply say that B is defined by the matrix of measures μ. Analogously, the orthogonal (orthonormal) polynomials with respect to this inner product B are said to be (h,x)-*orthogonal* ((h,x)-*orthonormal*), x-orthogonal (x-orthonormal) if $h(t) = (t-x)^N$ or simply orthogonal (orthonormal) if $h(t) = t^N$ and $x = 0$, with respect to the matrix of measures μ.

With every $(2N+1)$-term recurrence relation like (2.1), we associate a $(2N+1)$-banded infinite symmetric matrix by putting the sequences $(c_{n,l})_n$ which appear in the recurrence relation on the diagonals of the matrix, *i.e.*, we define the matrix $J = (j_{n,m})_{n,m\in\mathbb{N}}$ by

$$j_{n,m} = \begin{cases} c_{\max\{n,m\},|n-m|} & \text{if } |n-m| \leq N \\ 0 & \text{if } |n-m| > N. \end{cases}$$

This matrix will be called the *N-Jacobi matrix* for the recurrence relation (the classical case corresponds to $N = 1$). It should be noticed that this N-Jacobi matrix J does not depend on the polynomial h which appears in the recurrence relation. We say that J is the (h, N)-Jacobi matrix associated with a sequence of polynomials $(p_n)_n$ which satisfies the $(2N+1)$-term recurrence relation defined by the polynomial h if J is the N-Jacobi matrix for that recurrence relation.

It should be noticed that given a $(2N + 1)$-banded infinite symmetric matrix, J for which $j_{n,n+N} \neq 0$, $n \in \mathbb{N}$, a real polynomial h of degree N and a set of N polynomials $(p_n)_{n=0}^{N-1}$ with $\operatorname{dgr} p_n \leq n$, one can define a sequence of polynomials $(p_n)_{n\in\mathbb{N}}$ which satisfies a $(2N + 1)$-term recurrence relation defined by h and for which the N-Jacobi matrix is J. Only when $\operatorname{dgr} p_n = n$ for $n = 0, 1, \ldots, N - 1$ (and since $\operatorname{dgr} h = N$, that implies $\operatorname{dgr} p_n = n$ for all $n \in \mathbb{N}$), this sequence of polynomials is the sequence of orthonormal polynomials $(h, 0)$-defined by a matrix of measures μ.

3. **Orthogonal polynomials with respect to a positive definite matrix of measures.** As we wrote in the introduction, since the matrix of measures which appears in Theorem A (see Section 2 of this paper) have no positivity conditions, one does not expect to be able to extend the inner product from the space of polynomials $\mathbb{P}$ to a space of functions which should be complete for the topology generated by this inner product. So, first of all, we improve Theorem A by giving a positivity condition on the matrix of measures which will show to be sufficient in order to extend the inner product to a natural Hilbert space of functions.

THEOREM 3.1. *Let h be a real polynomial of degree N, x a real number and $(p_n)_n$ a sequence of polynomials for which* $\operatorname{dgr}(p_n) = n$ *and which satisfies a $(2N+1)$-term recurrence relation defined by the polynomial h (see (2.1)). Then there exists a positive definite $N \times N$ matrix of measures μ such that the polynomials $(p_n)_n$ are (h, x)-orthonormal with respect to μ.*

PROOF. Let us consider the Hilbert space of complex square summable sequences ℓ^2, and the N-Jacobi matrix J for the recurrence relation of the sequence of polynomials $(p_n)_n$. We also put J for the operator defined by this matrix on ℓ^2 with domain the subspace of finite sequences, *i.e.*,

$$J\colon \ell^2 \longrightarrow \ell^2$$

$$J\big((a_n)_n\big) = (a_0, a_1, \ldots)J.$$

Because the matrix J is symmetric, J is a symmetric operator with dense domain. As usual, $(e_n)_n$ will denote the canonical basis of ℓ^2.

In the space of polynomials $\mathbb{P}$, we consider the inner product B defined by the sequence of polynomials $(p_n)_n$, *i.e.*, $B(p, q) = \sum_n \alpha_n \bar{\beta}_n$ if $p(t) = \sum_n \alpha_n p_n(t)$ and $q(t) = \sum_n \beta_n p_n(t)$. Then the operator $T\colon \mathbb{P} \longrightarrow \ell^2$ defined by $T(p) = \sum_n \alpha_n e_n$ if $p(t) = \sum_n \alpha_n p_n(t)$ is an isometry from the linear space of polynomials $\mathbb{P}$ endowed with the inner product B to the domain of J (*i.e.*, the linear subspace of finite sequences), and from the $(2N+1)$-term recurrence relation defined by the polynomial h, it follows that

$$T^{-1} \circ J \circ T = h \tag{3.1}$$

on $\mathbb{P}$.

We define the operator $\mathcal{K} = J + x\,\mathrm{Id}$ (Id is the identity operator) in ℓ^2 with domain the subspace of finite sequences. Clearly, $\mathcal{K}$ is symmetric with dense domain, and since the $(2N+1)$-term recurrence relation has real coefficients, the deficiency indices of $\mathcal{K}$ are equal ([DS], p. 1231), so $\mathcal{K}$ has a selfadjoint extension, which we denote by $\mathcal{K}$ as well. Let us put E for the resolution of the identity of the operator $\mathcal{K}$. Finally, we put $x_m = T\big((t-x)^{m-1}\big)$ for $m = 1, \dots, N$, and define the matrix of measures $\mu = (\mu_{i,j})_{i,j=1}^{N}$, by $\mu_{i,j} = E_{x_i,x_j}$ for $i,j = 1, \dots, N$.

Firstly, we are going to prove that μ is a positive definite matrix of measures. Let A be a Borel set. For $\alpha_1, \dots, \alpha_N$ complex numbers, we put $u = \sum_{n=1}^{N} \alpha_n x_n$. Then we have

$$\sum_{i,j=1}^{N} \alpha_i \bar{\alpha}_j \mu_{i,j}(A) = \sum_{i,j=1}^{N} \alpha_i \bar{\alpha}_j E_{x_i,x_j}(A) = E_{u,u}(A) \geq 0.$$

The last inequality follows because E is a resolution of the identity, and so the measures $E_{u,u}$ are positive for all $u \in \ell^2$. Hence, the matrix of measures μ is positive definite.

Finally, we prove that B is the inner product (h,x)-defined by μ. Let us put $p(t) = (t-x)^k h(t)^n$ and $q(t) = (t-x)^{k'} h(t)^{n'}$ for $0 \leq k, k' \leq N-1$ and $n, n' \in \mathbb{N}$. Then, from the definition (2.2) of the operators $R_{h,x,m}$, (3.1) and the definition of the matrix of measures μ, we have

$$\begin{aligned}
\sum_{m,m'=1}^{N} & \int R_{h,x,m-1}(p)\overline{R_{h,x,m'-1}(q)}\, d\mu_{m,m'} \\
&= \int R_{h,x,k}\big((t-x)^k h(t)^n\big)\overline{R_{h,x,k'}\big((t-x)^{k'} h(t)^{n'}\big)}\, d\mu_{k+1,k'+1} \\
&= \int (t-x)^{n+n'}\, d\mu_{k+1,k'+1} \\
&= \int (t-x)^{n+n'}\, dE_{x_{k+1},x_{k'+1}} \\
&= \langle J^{n+n'} x_{k+1}, x_{k'+1} \rangle = \big\langle J^{n+n'} T\big((t-x)^k\big), T\big((t-x)^{k'}\big)\big\rangle \\
&= \big\langle T\big((t-x)^k\big) h(t)^{n+n'}, T\big((t-x)^{k'}\big)\big\rangle \\
&= B\big((t-x)^k h(t)^{n+n'}, (t-x)^{k'}\big) = B\big((t-x)^k h(t)^n, (t-x)^{k'} h(t)^{n'}\big) \\
&= B\big(p(t), q(t)\big).
\end{aligned}$$

Since $(t-x)^k h(t)^n$, $k = 0, 1, \dots, N-1$, $n \in \mathbb{N}$ is a basis of $\mathbb{P}$, the theorem is proved. ∎

The positive definiteness condition on the matrix of measures allows to us to extend the inner product for simple functions:

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LEMMA 3.2. *Let* $\mu = (\mu_{i,j})_{i,j=1}^N$ *be a positive definite matrix of measures. Then*

$$\sum_{i,j=1}^{N} \int f_i(t)\overline{f_j(t)}\, d\mu_{i,j}(t) \geq 0 \tag{3.2}$$

for all simple functions $f_1, \ldots, f_N$.

PROOF. We can write $f_i(t) = \sum_{j=n_i}^{n_{i+1}-1} a_j \chi_{A_j}(t)$, for $i = 1, 2, \ldots, N$ and $0 = n_1 < n_2 < \cdots < n_{N+1}$. Let us denote $\mathcal{A}$ the set of all subsets of $\{0, 1, 2, \ldots, n_{N+1}\}$. For every $B \in \mathcal{A}$, we put $C_B = \cap_{m\in B} A_m \setminus (\cup_{m\notin B} A_m)$. It should be noticed that for $B \neq B'$, C_B and $C_{B'}$ are disjoint. So, it is clear that $A_j = \cup_{\{B\in\mathcal{A}: j\in B\}} C_B$, $j = 0, 1, \ldots, n_{N+1}$ and the sets which appear in these unions are disjoint. Hence, we can write $f_i(t) = \sum_{B\in\mathcal{A}} a_{i,B}\chi_{C_B}(t)$ for certain numbers $a_{i,B}$. From this decomposition, we get

$$\begin{aligned}\sum_{i,j=1}^{N} \int f_i(t)\overline{f_j(t)}\, d\mu_{i,j}(t) &= \sum_{i,j=1}^{N} \sum_{B\in\mathcal{A}} \int a_{i,B}(t)\overline{a_{j,B}(t)}\chi_{C_B}(t)\, d\mu_{i,j}(t) \\ &= \sum_{B\in\mathcal{A}} \sum_{i,j=1}^{N} a_{i,B}\overline{a_{j,B}}\mu_{i,j}(C_B).\end{aligned}$$

Since the matrix μ is positive definite, we get for every $B \in \mathcal{A}$ that $\sum_{i,j=1}^N a_{i,B}\bar{a}_{j,B}\mu_{i,j}(C_B) \geq 0$, and so (3.2) holds. ∎

Now, we can define the space $L^2(\mu)$ of vector valued functions

DEFINITION 3.3. Let $\mu = (\mu_{i,j})_{i,j=1}^N$ be a positive definite $N \times N$ matrix of measures. We say that a function $f\colon \mathbb{R} \longrightarrow \mathbb{C}^N$ belongs to the space $L^2(\mu)$ if there exists a sequence of simple functions $(\phi_n)_n$, $\phi_n = (\phi_{n,1}, \ldots, \phi_{n,N})$ which tends $\mu_{i,j}$ a.e. to f for $i, j = 1, \ldots, N$ and which is a Cauchy sequence with respect to μ, *i.e.*

$$\lim_{n,m\to\infty} \sum_{i,j=1}^{N} \int \big(\phi_{n,i}(t) - \phi_{m,i}(t)\big)\big(\overline{\phi_{n,j}(t) - \phi_{m,j}(t)}\big)\, d\mu_{i,j}(t) = 0$$

For $f, g \in L^2(\mu)$ and $(\phi_n)_n$, $(\psi_n)_n$ two Cauchy sequences with respect to μ which tends to f, g μ a.e. as n tends to infinity respectively, we put

$$\langle f, g\rangle = \lim_n \sum_{i,j=1}^{N} \int \phi_{n,i}(t)\overline{\psi_{n,j}(t)}\, d\mu_{i,j}(t)$$

From Lemma 3.2, it follows that $\langle\cdot,\cdot\rangle$ is an inner product on $L^2(\mu)$ possibly degenerate, *i.e.*, $\langle f,f\rangle \geq 0$ for all $f \in L^2(\mu)$, although $\langle f,f\rangle = 0$ does not imply $f = 0$.

REMARK 3.4. In the literature we can find other definitions for the space $L^2(\mu)$ (see [R]) which we recall here. If $\mu = (\mu_{i,j})_{i,j=1}^N$ is a positive definite matrix of measures, then the measures on the diagonal $\mu_{i,i}$, $i = 1, \ldots, N$ are positive; moreover, all the measures $\mu_{i,j}$ are absolutely continuous with respect to the positive *trace measure* $\tau(\mu) = \sum_{i=1}^N \mu_{i,i}$. We put $0 \leq \phi_{i,j} \leq 1$ for the Radon-Nikodym derivative of the measure $\mu_{i,j}$ with respect to the trace measure $\tau(\mu)$, then, a vector valued function $f = (f_i)_{i=1}^N$ belongs to $L^2(\mu)$ if

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and only if the functions $f_i\bar{f}_j\phi_{i,j}$ are integrable with respect to the trace measure $\tau(\mu)$. Put $\langle f,f\rangle = \sum_{i,j=1}^N \int f_i(t)\bar{f}_j(t)\phi_{i,j}(t)\,d\tau(\mu)(t)$, then this inner product defined on $L^2(\mu)$ coincides with the one defined above.

Of course, in $L^2(\mu)$, we identify f and g if $\langle f-g,f-g\rangle = 0$ which is equivalent (via the Cauchy-Schwarz inequality) to $\sum_{i=1}^N \big(f_i(t)-g_i(t)\big)\phi_{i,j}(t) = 0$ $\tau(\mu)$-a.e. for all $j=1,\dots,N$. The functions for which $\langle f,f\rangle = 0$ can be characterized in a simple way. Let $t \in \operatorname{supp}\big(\tau(\mu)\big)$; if $\det\big(\phi_{i,j}(t)\big)_{i,j=1}^N = 0$, we put $\psi_1(t),\dots,\psi_m(t) \in \mathbb{C}^N$ for a basis of the subspace of eigenvectors of the matrix $\big(\phi_{i,j}(t)\big)_{i,j=1}^N$ corresponding to the eigenvalue $\lambda = 0$, and $\psi_{m+1}(t) = \cdots = \psi_N(t) = (0,\dots,0)$. If $\det\big(\phi_{i,j}(t)\big)_{i,j=1}^N > 0$ or $t \notin \operatorname{supp}(\tau(\mu)$, we put $\psi_1(t) = \cdots = \psi_N(t) = (0,\dots,0)$. Then it is not hard to see that $\langle f,f\rangle = 0$ if and only if $f = \sum_{i=1}^N a_i(t)\psi_i(t)$ for certain measurable functions $a_i\colon \mathbb{R} \longrightarrow \mathbb{C}$.

For $x \in \mathbb{R}$ and h a real polynomial of degree N, it follows from the previous remark that the following conditions are equivalent:

i) The inner product (h,x)-defined by a positive definite $N\times N$ matrix of measures μ on the space of complex polynomials $\mathbb{P}$ by the expression (2.3) is a nondegenerate inner product, *i.e.*, $B_\mu(p,p) = 0$ if and only if $p \equiv 0$.

ii) Given any measurable functions $a_i\colon \mathbb{R} \longrightarrow \mathbb{C}$, $i = 1,\dots,N$, if the restriction of the components of the function $\sum_{i=1}^N a_i(t)\psi_i(t)$ to $\sup\big(\tau(\mu)\big)$ coincides with the restriction of certain polynomials r_j, $j = 1,\dots,N$ then $r_j = 0$, $j = 1,\dots,N$ (where ψ_i are the functions associated with μ defined above).

REMARK 3.5. (i) The degenerate inner product of discrete Sobolev type $B(p,q) = Mp^{(m)}(x)\overline{q^{(m)}(x)}$ can be represented by a positive definite matrix of measures. Let N be a non-negative integer greater than m and h a real polynomial of degree N for which x is a root of multiplicity $m+1$. For a polynomial $p \in \mathbb{P}$, we can write

$$p(t) = \sum_{\substack{0\le k\le N-1\\ n\in\mathbb{N}}} a_{k,n}(t-x)^k h^n(t)$$

and since x is a root of multiplicity $m+1$ for h, and $m < N$, we get $p^{(m)}(x) = m!a_{m,0}$. Then, if we put

$$\mu_{i,j} = \begin{cases} 0 & \text{if } 1 \le i,j \le N,\, i \text{ or } j \ne m+1 \\ Mm!^2\delta_x & \text{if } i,j = m+1 \end{cases}$$

from (2.2) and (2.3), it follows that the inner product (h,x)-defined by μ is $B(p,q) = Mp^{(m)}(x)\overline{q^{(m)}(x)}$.

(ii) The case $m = 0$ *i.e.*, the Dirac delta δ_x can be represented using another different positive definite matrix of measures. Given $N \in \mathbb{N}$, $y \in \mathbb{R}$ and h a real polynomial of degree N, then the degenerate inner product $B(p,q) = p(x)\overline{q(x)}$ is (h,y)-defined by the positive definite matrix of measures $\mu = (\mu_{i,j})_{i,j=1}^N$ where $\mu_{i,j} = (x-y)^{i+j-2}\delta_{h(x)+y}$ for $i,j = 1,\dots,N$.

REMARK 3.6. Theorem 3.1 gives a positive definite $N \times N$ matrix of measures μ for which a sequence of polynomials $(p_n)_n$ satisfying a $(2N+1)$-term recurrence relation

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defined by a real polynomial of degree N, is (h,x)-orthonormal. Now we prove that the space of polynomials $\mathbb{P}^N$ is dense in the Hilbert space $L^2(\mu)$ for this matrix of measures μ which Theorem 3.1 provides. Indeed, let E be a resolution of the identity for the operator defined by $J + x\,\mathrm{Id}$, J being the N-Jacobi matrix. Since for every Borel set A the operator $E(A)$ is a projection, we have that the measure with density χ_A with respect to $E_{u,w}$ is just $E_{u,E(A)w}$ for any $u, w \in \ell^2$. For $k \in \mathbb{N}$ and by using the equality (3.1), we get that

$$e_k = \sum_{j=1}^{N} R_{h,x,j-1} p_k(J + x\,\mathrm{Id}) x_j, \tag{3.3}$$

where $x_j = T\big((t-x)^{j-1}\big)$ (as in the proof of Theorem 3.1). If $A_1, \ldots, A_N$ are Borel sets on $\mathbb{R}$, we put $f(t) = \big(\chi_{A_1}(t), \ldots, \chi_{A_N}(t)\big) \in L^2(\mu)$. The density of polynomials in $L^2(\mu)$ would follow if we prove the Parseval identity for the function f. Now, in ℓ^2 we have for $i,j = 1, \ldots, N$

$$\langle E(A_i \cap A_j)x_i, x_j\rangle = \langle E(A_i)x_i, E(A_j)x_j\rangle = \sum_{k\in\mathbb{N}} \langle E(A_i)x_i, e_k\rangle\langle e_k, E(A_j)x_j\rangle$$

for $i,j = 1, \ldots, N$. But from the definition of the matrix of measures μ, we have

$$\langle E(A_i \cap A_j)x_i, x_j\rangle = \int_{A_i\cap A_j} dE_{x_i,x_j} = \int \chi_{A_i}\chi_{A_j}\, d\mu_{i,j}.$$

Hence

$$B(f,f) = \sum_{i,j=1}^{N} \int \chi_{A_i}\chi_{A_j}\, d\mu_{i,j} = \sum_k \sum_{i,j=1}^{N} \langle E(A_i)x_i, e_k\rangle\langle e_k, E(A_j)x_j\rangle. \tag{3.4}$$

But from (3.3) and previous considerations, we have

$$\begin{aligned}
\langle E(A_i)x_i, e_k\rangle &= \langle E(A_i)x_i, \sum_{l=1}^{N} R_{h,x,l-1} p_k(J + x\,\mathrm{Id}) x_l\rangle \\
&= \sum_{l=1}^{N} \int_{A_i} \overline{R_{h,x,l-1} p_k(t)}\, dE_{x_i,x_l} \\
&= \sum_{l=1}^{N} \int \chi_{A_i} \overline{R_{h,x,l-1} p_k(t)}\, d\mu_{i,l}.
\end{aligned}$$

Hence (3.4) gives

$$\begin{aligned}
B(f,f) &= \sum_k \sum_{i,j=1}^{N} \Big(\sum_{l=1}^{N} \int \chi_{A_i} \overline{R_{h,x,l-1} p_k(t)}\, d\mu_{i,l}\Big) \\
&\qquad \times \Big(\sum_{l=1}^{N} \int R_{h,x,l-1} p_k(t) \chi_{A_j}\, d\mu_{l,j}\Big) \\
&= \sum_k \Big| \sum_{i,j=1}^{N} \int \chi_{A_i} \overline{R_{h,x,l-1} p_k(t)}\, d\mu_{i,j} \Big|^2 \\
&= \sum_k |B(f,p_k)|^2,
\end{aligned}$$

that is, the Parseval identity for the function f. ∎

The converse of this result is also true, *i.e.*, every positive definite $N \times N$ matrix of measures μ for which the polynomials are dense in the Hilbert space $L^2(\mu)$, can be reached from a resolution of the identity E for the operator defined by $J + x\,\mathrm{Id}$ (J being the N-Jacobi matrix) in the space ℓ^2. That means that the operators $E(A)$ are projections for any Borel set A if and only if the polynomials are dense in the Hilbert space $L^2(\mu)$ where μ is the positive definite $N \times N$ matrix of measures defined from E as in Theorem 3.1. Hence, we have that if the inner product defined by a positive definite $N \times N$ matrix of measures μ on the space $\mathbb{P}$ determines this matrix of measures (for $N = 1$ this means that the moment problem defined by the positive measure is determinate) then the polynomials are dense in the Hilbert space $L^2(\mu)$.

Indeed, given the positive definite $N \times N$ matrix of measures $\mu = (\mu_{i,j})_{i,j=1}^N$, for any Borel set A we define the operator $E(A)$ by

$$\langle E(A)Tp, Tq\rangle = \sum_{i,j=1}^{N} \int_A R_{h,x,i-1}(p)\overline{R_{h,x,j-1}(q)}\, d\mu_{i,j}$$

for all polynomials $p, q \in \mathbb{P}$, that is, if we put $f_{p,A}(t) = \big(R_{h,x,i-1}(p)(t)\chi_A(t)\big)_{i=1,\dots,N} \in L^2(\mu)$ for $p \in \mathbb{P}$, we have $\langle E(A)Tp, Tq\rangle = B(f_{p,A}, f_{q,A})$. Thus, we have defined the operator $E(A)$ on the subspace of finite sequences belonging to ℓ^2. Since

$$\begin{aligned} |\langle E(A)Tp, Tq\rangle| &= |B(f_{p,A}, f_{q,A})| \\ &\le |B(f_{p,A}, f_{p,A})|^{\frac{1}{2}} |B(f_{q,A}, f_{q,A})|^{\frac{1}{2}} \\ &\le \big(B(p,p)B(q,q)\big)^{\frac{1}{2}} = \|Tp\|_{\ell^2} \|T_q\|_{\ell^2}, \end{aligned}$$

we can extend the operator $E(A)$ to the whole space ℓ^2.

To prove that $E(A)$ is a projection, it will be enough to show that

$$\langle E(A)E(A)e_i, e_j\rangle = \langle E(A)e_i, e_j\rangle$$

for all $i, j \in \mathbb{N}$. But

$$\begin{aligned} \langle E(A)E(A)e_i, e_j\rangle &= \langle E(A)e_i, E(A)e_j\rangle \\ &= \sum_k \langle E(A)e_i, e_k\rangle\langle e_k, E(A)e_i\rangle \\ &= \sum_k \langle E(A)Tp_i, Tp_k\rangle\langle Tp_k, E(A)Tp_j\rangle \\ &= \sum_k B(f_{p_i,A}, p_k)B(p_k, f_{p_j,A}). \end{aligned}$$

But the polynomials are dense in $L^2(\mu)$, so using the Parseval equality, we get

$$\begin{aligned} \langle E(A)E(A)e_i, e_j\rangle &= B(f_{p_i,A}, f_{p_j,A}) \\ &= B(f_{p_i,A}, p_j) \\ &= \langle E(A)Tp_i, Tp_j\rangle = \langle E(A)e_i, e_j\rangle. \end{aligned}$$

4. **On positive semidefinite matrices.** In this section, we prove some results on positive semidefinite matrices, which we shall need to establish the behaviour of the kernel polynomials in the next section. They are interesting in their own right, although the reader in a first reading may skip over and return to them after Section 5.

We start with the following lemma concerning the determinant of certain matrices

LEMMA 4.1. *Let A, B, C be the finite matrices $A = (a_{i,j})_{i,j=1}^N$,*

$$B = \begin{pmatrix} 0 & x_1 & \cdots & x_N \\ y_1 & & & \\ \vdots & & A & \\ y_N & & & \end{pmatrix}. \tag{4.1}$$

and $C = (a_{i,j} - Mx_i y_j)_{i,j=1}^N$, where $a_{i,j}, x_i, y_i$ and M are complex numbers.

a) The following identity for the determinants of A, B, C holds

$$\det A = -M \det B + \det C.$$

b) If $y_i = \bar{x}_i$, $i = 1, \ldots, N$, $y_{i_0} \neq 0$ for certain i_0 and the matrix A is positive definite then $\det B < 0$.

PROOF. Part a) of this lemma is straightforward.

Now, we prove part b). Since the matrix A is obtained from B by deleting the first row and first column, the inclusion principle ([HJ, p. 189]) gives that

$$\beta_1 \geq \alpha_1 \geq \beta_2 \geq \alpha_2 \geq \cdots \geq \beta_N \geq \alpha_N \geq \beta_{N+1}$$

where the α's and β's are the eigenvalues of A and B respectively. Since the matrix A is positive definite, we get that the matrix B has at most one eigenvalue less than or equal to zero. If all the eigenvalues of B are non-negative, the matrix B would be positive semidefinite and since this is not the case ($y_{i_0} \neq 0$ for certain i_0), B has one eigenvalue negative and the other ones are positive, hence $\det B < 0$. ■

The following lemma establishes a criterion for positive semidefiniteness. Since we have not found this in the literature, we are including a proof

LEMMA 4.2. *Let $A = (a_{i,j})_{i,j=1}^N$ be a Hermitian matrix and A_m ($1 \leq m \leq N$) the matrix obtained from the first m rows and the first m columns of A. Let $r_m(\lambda) =$ $\det(A_m - \lambda I)$ be the characteristic polynomial of A_m, and let n_m be the smallest natural number for which $r_m^{(n_m)}(0) \neq 0$. Then A is positive semidefinite if and only if $(-1)^{n_m} r_m^{(n_m)}(0) > 0$ for every $m = 1, \ldots, N$.*

PROOF. If A is positive semidefinite then every A_m also is. Write

$$r_m(\lambda) = (-\lambda)^{n_m} \prod_i (\alpha_i - \lambda)$$

where $\alpha_i > 0$, then we get $(-1)^{n_m} r_m^{(n_m)}(0) = n_m! \prod_i \alpha_i > 0$.

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To prove the converse, we shall prove by induction that every A_m $(1 \leq m \leq N)$ is positive semidefinite and so $A = A_N$ also is. For $m = 1$ it is clear. Now, let us assume A_m to be positive semidefinite. Since A_m is obtained from A_{m+1} by deleting the last row and the last column, the inclusion principle gives that if $\beta_1 \geq \cdots \geq \beta_m, \gamma_1 \geq \cdots \geq \gamma_{m+1}$ are the eigenvalues for A_m, A_{m+1} respectively, then

$$\gamma_1 \geq \beta_1 \geq \gamma_2 \geq \beta_2 \geq \cdots \geq \beta_{m-1} \geq \gamma_m \geq \beta_m \geq \gamma_{m+1}.$$

Since A_m is positive semidefinite, it follows that $\beta_m \geq 0$, and so $\gamma_1 \geq \cdots \gamma_m \geq 0$. If $\gamma_{m+1} < 0$, we could write $r_{m+1}(\lambda) = (-\lambda)^{n_{m+1}} \prod_{i=1}^{m-n_{m+1}} (\gamma_i - \lambda)(\gamma_{m+1} - \lambda)$, then

$$(-1)^{n_{m+1}} r_{m+1}^{(n_{m+1})}(0) = n_{m+1}! \prod_{i=1}^{m-n_{m+1}} \gamma_i \gamma_{m+1} < 0.$$

But from the hypothesis $(-1)^{n_{m+1}} r_{m+1}^{(n_{m+1})}(0) > 0$, so $\gamma_{m+1} \geq 0$ and A_{m+1} is positive semidefinite. ∎

Finally, the next two lemmas establish two criteria to preserve positive semidefiniteness under certain perturbations on positive semidefinite matrices.

LEMMA 4.3. *Let $A = (a_{i,j})_{i,j=1}^n$ be a positive semidefinite matrix, and A_m $(1 \leq m \leq n)$, the matrix formed by deleting row m and column m from A, and let $r(\lambda)$, $r_m(\lambda)$ be the characteristic polynomials of the matrices A, A_m respectively. For a real number M, we define the matrix $B = (b_{i,j})_{i,j=1}^n$ by*

$$b_{i,j} = \begin{cases} a_{i,j} & \text{if } i \text{ or } j \neq m \\ a_{m,m} - M & \text{if } i = j = m. \end{cases}$$

Then the matrix B is positive semidefinite if and only if

$$M \leq \lim_{\lambda \to 0} \frac{r(\lambda)}{r_m(\lambda)}.$$

PROOF. Let us set $\alpha_1 \geq \cdots \geq \alpha_{n-1}$ for the eigenvalues of A_m and $\beta_1 \geq \cdots \geq \beta_n$ for the eigenvalues of A. The inclusion principle gives

$$\beta_1 \geq \alpha_1 \geq \beta_2 \geq \alpha_2 \geq \cdots \geq \beta_{n-1} \geq \alpha_{n-1} \geq \beta_n,$$

and since $\alpha_i, \beta_j \geq 0$ $(i = 1, \ldots, n-1$ and $j = 1, \ldots, n)$ (because both A, A_m are positive semidefinite), we get that if $r_m^{(j)}(0) = 0$ then $r^{(j)}(0) = 0$. Hence, if we put k for the smallest natural number for which $r_m^{(k)}(0) \neq 0$, we obtain

$$\lim_{\lambda \to 0} \frac{r(\lambda)}{r_m(\lambda)} = \frac{r^{(k)}(0)}{r_m^{(k)}(0)}.$$

Let us put $\gamma_1 \geq \cdots \geq \gamma_n$ for the eigenvalues of B, $s(\lambda)$ for its characteristic polynomial and l for the smallest natural number for which $s^{(l)}(0) \neq 0$. Since A_m is also the

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matrix obtained by deleting row m and column m from B, again by applying the inclusion principle we get that $|l-k| \leq 1$. And from the formula $s(\lambda) = r(\lambda) - Mr_m(\lambda)$, we conclude that $l = k$ or $l = k+1$. From Lemma 4.2, we get that B is positive semidefinite if and only if $(-1)^l s^{(l)}(0) > 0$. Let us assume $M \leq \lim_{\lambda \to 0} \frac{r(\lambda)}{r_m(\lambda)}$, that is $M \leq \frac{r^{(k)}(0)}{r_m^{(k)}(0)}$. If $l = k+1$, automatically, $\gamma_n = 0$ and so B is positive semidefinite. If $l = k$, then $(-1)^l s^{(l)}(0) = (-1)^k r^{(k)}(0) - (-1)^k M r_m^{(k)}(0)$. Since A_m is positive semidefinite we have $(-1)^k r_m^{(k)}(0) > 0$, hence $(-1)^l s^{(l)}(0) \geq (-1)^k r^{(k)}(0) - (-1)^k r^{(k)}(0) = 0$, and so, B is positive semidefinite.

Conversely, assume B to be positive semidefinite. If $l = k+1$, then $s^{(k)}(0) = 0 = r^{(k)}(0) - Mr_m^{(k)}(0)$, and so

$$M = \frac{r^{(k)}(0)}{r_m^{(k)}(0)} = \lim_{\lambda \to 0} \frac{r(\lambda)}{r_m(\lambda)}.$$

If $l = k$, $(-1)^k s^{(k)}(0) > 0$ since B is positive semidefinite *i.e.*,

$$(-1)^k r^{(k)}(0) - (-1)^k M r_m^{(k)}(0) > 0,$$

and again using that $(-1)^k r_m^{(k)}(0) > 0$, we get

$$M < \frac{r^{(k)}(0)}{r_m^{(k)}(0)} = \lim_{\lambda \to 0} \frac{r(\lambda)}{r_m(\lambda)}$$

and the lemma is proved. ∎

LEMMA 4.4. *Let $A = (a_{i,j})_{i,j=1}^N$ be a positive semidefinite matrix. For $x_1, \ldots, x_N \in \mathbb{C}$, $x_1 \neq 0$, we consider the matrix*

$$B = \begin{pmatrix} 0 & \bar{x}_1 & \cdots & \bar{x}_N \\ x_1 & & & \\ \vdots & & A & \\ x_N & & & \end{pmatrix}.$$

For m, $1 \leq m \leq N$, we set $r_m(\lambda)$ for the characteristic polynomial of the matrix $A_m = (a_{i,j})_{i,j=1}^m$ and $s_m(\lambda)$ for the polynomial

$$s_m(\lambda) = \begin{vmatrix} 0 & \bar{x}_1 & \cdots & \bar{x}_m \\ x_1 & & & \\ \vdots & & A_m - \lambda I & \\ x_m & & & \end{vmatrix}.$$

Then, for a real number M, the matrix $C = (a_{i,j} - Mx_i\bar{x}_j)_{i,j=1}^N$ is positive semidefinite if and only if

$$M \leq -\min\left\{ -\lim_{\lambda \to 0} \frac{r_m(\lambda)}{s_m(\lambda)} : m = 1, \ldots, N \right\}.$$

PROOF. We prove that the conditions

$$(4.2) \quad s_m(0) = s_m'(0) = \cdots = s_m^{(l)}(0) = 0 \quad \text{imply} \quad r_m(0) = r_m'(0) = \cdots = r_m^{(l)}(0) = 0.$$

We proceed by induction on l. Let us set B_m for the matrix obtained from the first m rows and the first m columns of B, and let p_m be the characteristic polynomial of this matrix. We have that $s_m(\lambda) = p_m(\lambda) + \lambda r_m(\lambda)$. Since the matrix A_m is obtained from B_m by deleting the first row and first column, the inclusion principle gives

$$\beta_1 \geq \alpha_1 \geq \beta_2 \geq \alpha_2 \geq \cdots \geq \beta_m \geq \alpha_m \geq \beta_{m+1}, \tag{4.3}$$

where the α's and β's are the eigenvalues of A_m and B_m respectively. Since A_m is positive semidefinite and B_m is not positive semidefinite ($x_1 \neq 0$), we get that $\beta_{m+1} < 0$. Hence, if $r_m(0) > 0$, (4.3) gives $p_m(0) < 0$ and so $s_m(0) < 0$. This proves the case $l = 0$.

Let us assume $s_m(0) = s'_m(0) = \cdots = s_m^{(l)}(0) = 0$. The induction hypothesis gives $r_m(0) = r'_m(0) = \cdots = r_m^{(l-1)}(0) = 0$. Hence, since $s_m(\lambda) = p_m(\lambda) + \lambda r_m(\lambda)$, we get $p_m^{(j)}(0) = 0, j = 0, \ldots, l$ and so from (4.2) (because $\beta_{m+1} < 0$), we get $r_m^{(l)}(0) = 0$.

Now, if we put l_m for the smallest natural number for which $s_m^{(l_m)}(0) \neq 0$, we get

$$\lim_{\lambda \to 0} \frac{r_m(\lambda)}{s_m(\lambda)} = \frac{r_m^{(l_m)}(0)}{s_m^{(l_m)}(0)}. \tag{4.4}$$

Let us put $k_m(\lambda)$ for the characteristic polynomial of the matrix $(a_{i,j} - Mx_i\bar{x}_j)_{i,j=1}^m$, and n_m for the smallest natural number for which $k_m^{(n_m)}(0) \neq 0$. From Lemma 4.1a), we get that $k_m(\lambda) = r_m(\lambda) + Ms_m(\lambda)$. So from (4.2), it follows that

$$l_m \leq n_m. \tag{4.5}$$

Applying Lemma 4.1b) several times, we have that if $\lambda < 0$ then $(-1)^l s_m^{(l)}(\lambda) < 0$ for $0 \leq l \leq m-1$. Hence

$$(-1)^l s_m^{(l)}(0) \leq 0 \quad \text{if } 0 \leq l \leq m-1. \tag{4.6}$$

From Lemma 4.2, the matrix C is positive semidefinite if and only if $(-1)^{n_m} k_m^{(n_m)}(0) > 0$ for $m = 1, \ldots, N$. Let us assume C to be positive semidefinite. From (4.5), we have

$$0 \leq (-1)^{l_m} k_m^{(l_m)}(0) = (-1)^{l_m} r_m^{(l_m)}(0) + M(-1)^{l_m} s_m^{(l_m)}(0).$$

So from (4.6) and (4.4), we have

$$M \leq -\frac{r_m^{(l_m)}(0)}{s_m^{(l_m)}(0)} = -\lim_{\lambda \to 0} \frac{r_m(\lambda)}{s_m(\lambda)}.$$

Conversely, from the continuous dependence of eigenvalues on matrices, it will be enough to assume $M < -\lim_{\lambda \to 0} \frac{r_m(\lambda)}{s_m(\lambda)}$ for $m = 1, \ldots, N$. Since

$$(-1)^{l_m} k_m^{(l_m)}(0) = (-1)^{l_m} r_m^{(l_m)}(0) + M(-1)^{l_m} s_m^{(l_m)}(0),$$

from (4.4) and (4.6), we get $(-1)^{l_m} k_m^{(l_m)}(0) > 0$. So, (4.4) gives $l_m = n_m$ and the lemma is proved. ■

5. **The behaviour of kernel polynomials and their derivatives.** Given an inner product B defined on the linear space of polynomials $\mathbb{P}$, and its sequence of orthonormal polynomials $(p_n)_n$, the kernel polynomials are defined by

$$k_n(z,w) = \sum_{k=0}^{n} p_k(z)\overline{p_k(w)} \quad \text{for } z, w \in \mathbb{C}.$$

If we put $a_{i,j} = B(t^i, t^j)$, and $\Delta_n = \det\big((a_{i,j})_{i,j=0}^n\big)$ we obtain the following expression for these kernel polynomials:

$$k_n(z,w) = -\frac{1}{\Delta_n}\begin{vmatrix} 0 & 1 & z & \cdots & z^n \\ 1 & a_{0,0} & a_{0,1} & \cdots & a_{0,n} \\ \bar{w} & a_{1,0} & a_{1,1} & \cdots & a_{1,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{w}^n & a_{n,0} & a_{n,1} & \cdots & a_{n,n} \end{vmatrix}. \tag{5.1}$$

In this section we are going to study asymptotic properties for the kernel polynomials associated to an inner product defined by a positive definite matrix of measures μ.

For an inner product defined by a positive measure ρ, these kernel polynomials have an interesting asymptotic property: the number $\frac{1}{\sum_n |p_n(x)|^2}$ for $x \in \mathbb{R}$, is just the maximum mass that can be concentrated at the point x in a distribution of mass on the axis for which the polynomials $(p_n)_n$ are orthonormal (see [A], p. 63).

It is also well-known that if there exists more than one measure for which $(p_n)_n$ are orthonormal, then

$$\frac{1}{\sum_n |p_n(x)|^2} > 0 \quad \text{for all } x \in \mathbb{C}.$$

But, a precise interpretation for this series when $x \notin \mathbb{R}$ does not seem to be known. Similar properties for the derivatives $\frac{1}{\sum_n |p_n^{(m)}(x)|^2}$ are also not known.

Here, we shall show such formulas. For N a non-negative integer the value of the series $\frac{1}{\sum_n |p_n^{(N)}(x)|^2}$, and $\frac{1}{\sum_n |p_n(xw)|^2}$ where w satisfies $w^N = -1$, will depend on the positive definite $N \times N$ matrices of measures for which the polynomials $(p_n)_n$ are orthonormal. We extend these results for the kernel polynomials associated to an inner product defined by a positive definite matrix of measures μ.

Our starting point is the following lemma:

LEMMA 5.1. *Let B be an inner product defined on $\mathbb{P}$. For m a non-negative integer and z a complex number, we consider the set $\mathcal{B}_{m,z}$ of real numbers M for which the bilinear form B_M defined by $B_M(p,q) = B(p,q) - Mp^{(m)}(z)\bar{q}^{(m)}(z)$ is an inner product (possibly degenerate) on $\mathbb{P}$, i.e., $B_M(p,p) \geq 0$ for $p \in \mathbb{P}$. Then*

$$\frac{1}{\sum_n |p_n^{(m)}(z)|^2} = \sup \mathcal{B}_{m,z} \tag{5.2}$$

where $(p_n)_n$ are the orthonormal polynomials with respect to B.

PROOF. Let M be a real number in $\mathcal{B}_{m,z}$, for which $M < \sup \mathcal{B}_{m,z}$. Then the expression $B_M(p,q) = B(p,q) - Mp^{(m)}(z)\bar{q}^{(m)}(z)$ defines a nondegenerate inner product on the space of polynomials $\mathbb{P}$. So, the matrix $\big(B_M(t^i,t^j)\big)_{i,j=1}^{\infty}$ is positive definite, *i.e.*,

$$\det\big(B_M(t^i,t^j)\big)_{i,j=1}^{n} > 0$$

for all $n \in \mathbb{N}$. But

$$B_M(t^i,t^j) = B(t^i,t^j) - Mi(i-1)\cdots(i-m+1)j(j-1)\cdots(j-m+1)z^{i-m}\bar{z}^{j-m}.$$

Putting $\Delta_n = \det\big(B(t^i,t^j)\big)_{i,j=1}^{n})$ for $n > N$, we get from Lemma 4.1 that $M < -\frac{\Delta_n}{\det Y_n}$ for all $n > N$, where the matrices Y_n are defined by (4.1), for $a_{i,j} = B(t^i,t^j)$ and $x_i = \bar{y}_i = i(i-1)\cdots(i-m+1)z^{i-m}$. Now, from the definition of the kernel polynomials, it follows that

$$\sum_{k=0}^{n} |p_k^{(m)}(z)|^2 = \frac{\partial^m}{\partial w_1^m}\overline{\frac{\partial^m}{\partial w_2^m}}\big(k_n(w_1,w_2)\big)_{|w_1=w_2=z}.$$

Then, from (5.1) we get

$$\sum_{k=0}^{n} |p_k^{(m)}(z)|^2 = -\frac{\det Y_n}{\Delta_n}.$$

So,

$$\sup \mathcal{B}_{m,z} \le \inf_n \frac{1}{\sum_{k=0}^{n} |p_k^{(m)}(z)|^2} = \frac{1}{\sum_{k=0}^{\infty} |p_k^{(m)}(z)|^2}$$

To prove the converse, let $M_0 = \frac{1}{\sum_{k=0}^{\infty} |p_k^{(m)}(z)|^2}$ be. From Lemma 4.1, it follows that

$$\det\big(B_{M_0}(t^i,t^j)\big)_{i,j=1}^{n} = \Delta_n + M_0 \det Y_n$$

Since $\frac{1}{M_0} > \sum_{k=0}^{n} |p_k^{(m)}(z)|^2 = -\frac{\det Y_n}{\Delta_n}$, we get that

$$\det\big(B_{M_0}(t^i,t^j)\big)_{i,j=1}^{n} > 0 \quad \text{for } n > N$$

and so B_{M_0} defines an inner product, *i.e.*, $M_0 \in \mathcal{B}_{m,z}$. Then, the proof follows. ∎

Now we are ready to state an asymptotic property for the derivatives of the kernel polynomials associated with a positive definite matrix of measures:

THEOREM 5.2. *Let h be a real polynomial of degree N, $(p_n)_n$ a sequence of polynomials for which* $\operatorname{dgr}(p_n) = n$ *which satisfies a $(2N+1)$-term recurrence relation defined by the polynomial h. For $x \in \mathbb{R}$ and $m \in \mathbb{N}$, let g be the polynomial $g(t) = \big(h(t)-h(x)\big)^{m+1}$. Define the class $\mathcal{A}_{m,x}$ by*

$$\mathcal{A}_{m,x} = \{\mu : \mu \text{ is a positive definite } (m+1)N\times(m+1)N \text{ matrix of measures with respect to which the polynomials } (p_n)_n \text{ are } (g,x)\text{-orthonormal}\}.$$

For every matrix of measures $\mu = (\mu_{i,j})_{i,j=1}^{(m+1)N} \in \mathcal{A}_{m,x}$, let $r_{x,\mu}(\lambda)$, $r_{x,\mu,m}(\lambda)$ be the characteristic polynomials of the numerical matrices $\big(\mu_{i,j}(\{x\})\big)_{i,j=1}^{(m+1)N}$ and $\big(\mu_{i,j}(\{x\})\big)_{\substack{i,j=1\\ i,j\neq m+1}}^{(m+1)N}$, respectively. Then

$$\frac{1}{\sum_n |p_n^{(m)}(x)|^2} = \sup\left\{\frac{1}{m!^2}\lim_{\lambda\to 0}\frac{r_{x,\mu}(\lambda)}{r_{x,\mu,m}(\lambda)} : \mu \in \mathcal{A}_{m,x}\right\}$$

PROOF. Let B be the inner product with respect to which the polynomials $(p_n)_n$ are orthonormal. For $m \in \mathbb{N}$ and $x \in \mathbb{R}$, let us consider the set of real numbers $\mathcal{B}_{m,x}$ as in the Lemma 5.1, then (5.2) shows that

$$\frac{1}{\sum_n |p_n^{(m)}(x)|^2} = \sup \mathcal{B}_{m,x}.$$

For $M \in \mathcal{B}_{m,x}$, let us consider the inner product $B_M(p,q) = B(p,q) - Mp^{(m)}\overline{q^{(m)}(x)}$. Since the polynomials $(p_n)_n$ satisfy a $(2N+1)$-term recurrence relation defined by the polynomial h, the operator of multiplication by h is symmetric for B, and so the operator of multiplication by g also is. Since $g^{(k)}(x) = 0$, $k = 0, \ldots, m$, we get that the operator of multiplication by g is symmetric for B_M. But that is equivalent to the fact that the sequence of orthonormal polynomials with respect to B_M satisfies a $[2(m+1)N+1]$-term recurrence relation defined by g. Hence, by Theorem 3.1, there exists a positive definite $(m+1)N \times (m+1)N$ matrix of measures ν such that B_M is (g,x)-defined by ν. Since $B(p,q) = B_M(p,q) + Mp^{(m)}\overline{q^{(m)}(x)}$, and from Remark 3.5(i), it follows that there exists $\mu \in \mathcal{A}_{m,x}$ such that

$$\mu_{i,j} = \begin{cases} \nu_{i,j} & \text{if } i \text{ or } j \neq m+1, \\ \nu_{m,m} + Mm!^2\delta_x & \text{if } i = j = m+1\,. \end{cases}$$

Since the matrix of measures ν is positive definite, it follows that the numerical matrix $(b_{i,j})_{i,j=1}^{(m+1)N}$ defined by

$$b_{i,j} = \begin{cases} \mu_{i,j}(\{x\}) & \text{if } i \text{ or } j \neq m+1, \\ \mu_{m,m}(\{x\}) - Mm!^2 & \text{if } i = j = m+1, \end{cases}$$

must be positive semidefinite.

If we apply the Lemma 4.3 to the matrix $(b_{i,j})_{i,j=1}^{(m+1)N}$, we get

$$M \leq \frac{1}{m!^2} \lim_{\lambda \to 0} \frac{r_{x,\mu}(\lambda)}{r_{x,\mu,m}(\lambda)},$$

and hence

$$\frac{1}{\sum_n |p_n^{(m)}(x)|^2} = \sup \mathcal{B}_{m,z} \leq \sup\left\{ \frac{1}{m!^2} \lim_{\lambda \to 0} \frac{r_{x,\mu}(\lambda)}{r_{x,\mu,m}(\lambda)} : \mu \in \mathcal{A}_{m,x} \right\}.$$

Conversely, for $\mu \in \mathcal{A}_{m,x}$, let us put $M = \frac{1}{m!^2} \lim_{\lambda \to 0} \frac{r_{x,\mu}(\lambda)}{r_{x,\mu,m}(\lambda)}$. We shall prove that $M \in \mathcal{B}_{m,x}$. It will be enough to prove that the matrix of measures defined by

$$\nu_{i,j} = \begin{cases} \mu_{i,j} & \text{if } i \text{ or } j \neq m+1, \\ \mu_{m,m} - Mm!^2\delta_x & \text{if } i = j = m+1, \end{cases}$$

is positive definite. That is, the numerical matrix $\left(\nu_{i,j}(\{x\})\right)_{i,j=1}^{(m+1)N}$ is positive semidefinite. For this, it is enough to apply Lemma 4.3 again. ∎

For orthonormal polynomials with respect to a positive measure, we then have:

COROLLARY 5.3. *Let ρ be a positive measure and $(p_n)_n$ its sequence of orthonormal polynomials. For $m \in \mathbb{N}$ and $x \in \mathbb{R}$, we put $\mathcal{A}_{m,x}$ for the class*

$$\mathcal{A}_{m,x} = \{\mu : \mu \text{ is a positive definite } (m+1)\times(m+1) \text{ matrix of measures with respect to which the polynomials } (p_n)_n \text{ are } x\text{-orthonormal}\}.$$

For every matrix of measures $\mu = (\mu_{i,j})_{i,j=1}^{(m+1)} \in \mathcal{A}_{m,x}$, we put $r_{x,\mu}(\lambda)$, $r_{x,\mu,m}(\lambda)$ for the characteristic polynomials of the numerical matrices $\big(\mu_{i,j}(\{x\})\big)_{i,j=1}^{(m+1)}$ and $\big(\mu_{i,j}(\{x\})\big)_{i,j=1}^{m}$, respectively. Then

$$\frac{1}{\sum_n |p_n^{(m)}(x)|^2} = \sup\left\{\frac{1}{m!^2}\lim_{\lambda\to 0}\frac{r_{x,\mu}(\lambda)}{r_{x,\mu,m}(\lambda)} : \mu \in \mathcal{A}_{m,x}\right\}. \qquad \blacksquare$$

For $m = 0$, and $(p_n)_n$ a sequence of orthonormal polynomials with respect to a matrix of measures, another expression for $\frac{1}{\sum_n |p_n(x)|^2}$ can be given.

THEOREM 5.4. *Let h be a real polynomial of degree N, and let $(p_n)_n$ be a sequence of polynomials for which $\operatorname{dgr}(p_n) = n$ and which satisfies a $(2N+1)$-term recurrence relation defined by h. Let $\mathcal{A}$ be for the class:*

$$\mathcal{A} = \{\mu : \mu \text{ is a positive definite } N\times N \text{ matrix of measures with respect to which the polynomials } (p_n)_n \text{ are } (h,0)\text{-orthonormal}\}.$$

For every matrix of measures $\mu \in \mathcal{A}$ and m with $1 \le m \le N$, we put $r_{x,\mu,m}(\lambda)$ for the characteristic polynomials of the matrix $\Big(\mu_{i,j}\big(\{h(x)\}\big)\Big)_{i,j=1}^{m}$ and $s_{x,\mu,m}(\lambda)$ for the polynomials

$$s_{x,\mu,m}(\lambda) = \begin{vmatrix} 0 & 1 \quad x \quad \cdots \quad x^{m-1} \\ 1 & \\ x & \\ \vdots & \Big(\mu_{i,j}\big(\{h(x)\}\big)\Big)_{i,j=1}^{m} - \lambda I \\ x^{m-1} & \end{vmatrix}.$$

Then we have

$$\frac{1}{\sum_n |p_n(x)|^2} = \sup\left\{\min_{1\le m\le N} -\lim_{\lambda\to 0}\frac{r_{x,\mu,m}(\lambda)}{s_{x,\mu,m}(\lambda)} : \mu \in \mathcal{A}\right\}.$$

PROOF. Let B be the inner product for which the polynomials $(p_n)_n$ are orthonormal. For $x \in \mathbb{R}$, let us put $\mathcal{B}_{0,x}$ for the set of real numbers which appears in Lemma 5.1, then (5.2) shows that

$$\frac{1}{\sum_n |p_n(x)|^2} = \sup \mathcal{B}_{0,x}$$

For $M \in \mathcal{B}_{0,x}$ we consider the inner product $B_M(p,q) = B(p,q) - Mp(x)\overline{q(x)}$. Since the operator of multiplication by h is symmetric for the inner product B, it follows that

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this operator also is symmetric for the inner product B_M. Hence there exists a positive definite $N \times N$ matrix of measures ν such that B_M is $(h,0)$-defined by ν. Since $B(p,q) = B_M(p,q) + Mp(x)\overline{q(x)}$, from Remark 3.5(ii), it follows that there exists a matrix of measures $\mu \in \mathcal{A}$ such that

$$\mu_{i,j} = \nu_{i,j} + Mx^{i+j-2}\delta_{h(x)} \quad \text{for } i,j = 1, \ldots, N.$$

Since the matrix of measures ν is positive definite, it follows that the numerical matrix $\left(\mu_{i,j}(\{h(x)\}) - Mx^{i+j-2}\right)_{i,j=1}^{N}$ must be positive semidefinite. Hence, the Lemma 4.4 gives

$$M \leq \sup\left\{ \min_{1 \leq m \leq N} - \lim_{\lambda \to 0} \frac{r_{x,\mu,m}(\lambda)}{s_{x,\mu,m}(\lambda)} : \mu \in \mathcal{A} \right\},$$

and so

$$\frac{1}{\sum_n |p_n(x)|^2} = \sup \mathcal{B}_{0,x} \leq \sup\left\{ \min_{1 \leq m \leq N} - \lim_{\lambda \to 0} \frac{r_{x,\mu,m}(\lambda)}{s_{x,\mu,m}(\lambda)} : \mu \in \mathcal{A} \right\}.$$

Conversely, for $\mu \in \mathcal{A}$, let us put $M = \min_{1 \leq m \leq N} - \lim_{\lambda \to 0} \frac{r_{x,\mu,m}(\lambda)}{s_{x,\mu,m}(\lambda)}$. We shall prove that $M \in \mathcal{B}_{0,x}$. It will be enough to prove that the matrix of measures defined by

$$\nu_{i,j} = \mu_{i,j} - Mx^{i+j-2}\delta_{h(x)} \quad \text{for } i,j = 1, \ldots, N$$

is positive definite. That is, the numerical matrix $\left(\mu_{i,j}(\{h(x)\} - Mx^{i+j-2})\right)_{i,j=1,\ldots,N}$ is positive semidefinite. For this, it is enough to apply Lemma 4.4 again. ■

Finally, we consider $\frac{1}{\sum_n |p_n(xw)|^2}$ where w satisfies $w^N = -1$ and $(p_n)_n$ are the orthonormal polynomials with respect to a positive measure.

COROLLARY 4.8. *Let ρ be a positive measure and $(p_n)_n$ the sequence of orthonormal polynomials with respect to ρ. Let w be satisfying $w^N = -1$. We set*

$$\mathcal{A} = \{\mu : \mu \text{ is a positive definite } N \times N \text{ matrix of measures with respect to which the polynomials } (p_n)_n \text{ are orthonormal}\}.$$

For every matrix of measures $\mu \in \mathcal{A}$ and m with $1 \leq m \leq N$, we put $r_{x,\mu,m}(\lambda)$ for the characteristic polynomials of the matrix $\left(\mu_{i,j}(\{-x^N\})\right)_{i,j=1}^{m}$ and $s_{x,\mu,m,w}(\lambda)$ for the polynomials

$$s_{x,\mu,m,w}(\lambda) = \begin{vmatrix} 0 & 1 \quad\quad x \quad\quad \cdots \quad x^{m-1} \\ 1 & \\ x & \\ \vdots & \left(\bar{w}^{i-1}w^{j-1}\mu_{i,j}(\{-x^N\})\right)_{i,j=1}^{m} - \lambda I \\ x^{m-1} & \end{vmatrix}.$$

Then

$$\frac{1}{\sum_n |p_n(xw)|^2} = \sup\left\{ \min_{1 \leq m \leq N} - \lim_{\lambda \to 0} \frac{r_{x,\mu,m}(\lambda)}{s_{x,\mu,m,w}(\lambda)} : \mu \in \mathcal{A} \right\}.$$

PROOF. For w satisfying $w^N = -1$, we define the polynomials $q_n(t) = p_n(tw)$ for $n \in \mathbb{N}$. Since $(p_n)_n$ satisfies a three term recurrence relation, they also satisfy a $(2k+1)$-term recurrence relation for all $k \geq 1$. Since $w^N = -1$, we get that the polynomials $(q_n)_n$

also satisfy a $(2N+1)$-term recurrence relation. Hence, from the previous theorem, it follows that

$$\frac{1}{\sum_n |p_n(xw)|^2} = \frac{1}{\sum_n |q_n(x)|^2} = \sup\left\{\min_{1\le m\le N} -\lim_{\lambda\to 0}\frac{r_{x,\mu,m}(\lambda)}{s_{x,\mu,m,w}(\lambda)} : \mu \in \mathcal{A}_N\big((q_n)_n\big)\right\},$$

where

$$\mathcal{A}_N((q_n)_n) = \{\mu : \mu \text{ is a positive definite } N\times N \text{ matrix of measures with respect to which the polynomials } (q_n)_n \text{ are orthonormal}\}.$$

But we have the following expression for the polynomials q_n

$$q_n(t) = \sum_{k=0}^{N-1} w^k\left(\sum_m (-1)^m \frac{p^{(mN+k)}(0)}{(mN+k)!} t^{mN+k}\right),$$

and so the mapping

$$T_N\colon \mathcal{A}_N\big((p_n)_n\big) \longrightarrow \mathcal{A}_N\big((q_n)_n\big)$$

defined by $\big(T_N(\mu)_{i,j}\big)_{i,j=1}^N = (\bar{w}^{i-1}w^{j-1}\check{\mu}_{i,j})_{i,j=1}^N$ (where $\check{\mu}$ is the measure defined by $\check{\mu}(A) = \mu(-A)$) is a bijection from $\mathcal{A}_N\big((p_n)_n\big)$ to $\mathcal{A}_N\big((q_n)_n\big)$. Since

$$\det{}^{(k)}\Big[\big(\mu_{i,j}(\{-x^N\})\big)_{i,j=1}^N - \lambda I\Big]_{|\lambda=0} = \det{}^{(k)}\Big[\big(\bar{w}^{i-1}w^{j-1}\mu_{i,j}(\{-x^N\})\big)_{i,j=1}^N - \lambda I\Big]_{|\lambda=0}$$

for all $k \in \mathbb{N}$, the proof follows. ∎

6. **Examples.** In this section, we shall give some examples of orthogonal polynomials with respect to a positive definite matrix of measures. All these examples are close relatives of orthogonal polynomials with respect to a positive measure. Indeed, let ρ be a positive measure, its sequence of orthonormal polynomials $(p_n)_n$ will satisfy a three term recurrence relation ($p_{-1}(t) = 0$):

$$tp_n(t) = a_{n+1}p_{n+1}(t) + b_np_n(t) + a_np_{n-1}(t).$$

We set J for its Jacobi matrix

$$J = \begin{pmatrix} b_0 & a_1 & 0 & 0 & 0 & \cdots \\ a_1 & b_1 & a_2 & 0 & 0 & \cdots \\ 0 & a_2 & b_2 & a_3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

A) DEFINITION. Let N be a non-negative integer, and consider the matrix $J^N = \overbrace{J\cdots J}^{N \text{ times}}$ which is a $(2N+1)$-banded matrix

$$J^N = (j_{n,m})_{i,j=1}^\infty \quad \text{where } j_{n,m} = j_{m,n} \text{ and } j_{n,m} = 0 \text{ if } |n-m| > N.$$

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Given a finite sequence $(q_n)_{n=0}^{N-1}$ of polynomials for which $\mathrm{dgr}(q_n) \leq n$, we define the sequence of polynomials $(q_n)_{n=0}^{\infty}$ by using the $(2N+1)$-term recurrence relation defined by the matrix J^N:

$$t^N q_n(t) = \sum_{l=-N}^{N} j_{n,n+l} q_{n+l}(t), \tag{6.1}$$

with the initial conditions $(q_n)_{n=0}^{N-1}$ and $j_{n,m} = 0$, $q_m(t) = 0$ if $m < 0$.

It is clear that for $q_n(t) = p_n(t)$, $n = 0, \ldots, N-1$, we get the orthonormal polynomials $(p_n)_n$.

Now, we give an expression for the polynomials $(q_n)_n$ in terms of the polynomials $(p_n)_n$. Indeed, if w is a primitive N-th root of the unity, it is clear that the sequence of polynomials $\big(p_n(tw)\big)_n$ satisfies the $(2N+1)$-term recurrence relation. Hence, the sequence of polynomials $(q_{k,m,n})_n$ defined by

$$q_{k,m,n}(t) = t^k \frac{1}{Nt^m} \sum_{l=0}^{N-1} (w^{-m})^l p_n(w^l t),$$

where $m = k, \ldots, N-1$ and $k = 0, \ldots, N-1$ also satisfies the recurrence relation. It is not hard to see that

$$q_{k,m,n}(t) = t^k R_{N,m}(p_n)(t^N),$$

where $R_{N,m}$ is the operator defined in (1.1). Since the vector valued polynomials

$$\Big(t^k R_{N,m}(p_0)(t^N), \ldots, t^k R_{N,m}\big(p_{N-1}(t^N)\big)\Big)$$

for $m = k, \ldots, N-1$ and $k = 0, \ldots, N-1$ are a basis of $\mathbb{P}_0 \times \mathbb{P}_1 \times \cdots \times \mathbb{P}_{N-1}$, and every sequence $\big(t^k R_{N,m}(p_n)(t)\big)_n$ where $m = k, \ldots, N-1$ and $k = 0, \ldots, N-1$ satisfies the $(2N+1)$-term recurrence relation, we find the following expression for any sequence of polynomials $(q_n)_n$ satisfying the $(2N+1)$-term recurrence relation:

$$q_n(t) = \sum_{k=0}^{N-1} \sum_{m=k}^{N-1} t^k \gamma_{k,m} R_{N,m}(p_n)(t^N), \tag{6.2}$$

for certain complex numbers $\gamma_{k,m}$, $k = 0, \ldots, N-1$ and $m = k, \ldots, N-1$.

B) ORTHOGONALITY. Only when the initial conditions $(q_n)_{n=0}^{N-1}$ satisfy $\mathrm{dgr}(q_n) = n$, the sequence of polynomials $(q_n)_n$ defined by (6.1) is orthonormal with respect to a positive definite $N \times N$ matrix of measures. Here, we give an explicit expression for that matrix of measures. From the positive measure ρ, we define a positive definite $N \times N$ matrix of measures as we pointed out in the introduction, *i.e.*, we set $\rho_{i,j}$ for the measures defined by: $\rho_{i,j} = \rho$ if N is odd, and $\rho_{i,j}$ is the measure with support in $[0, +\infty)$ defined by $\rho_{i,j}(A) = \rho(A) + (-1)^{i+j}\rho(-A)$ when N is even. Now, we let $\mu_{\rho,i,j}$ be the measure with density $t^{\frac{i+j}{N}}$ with respect to the image measure $\rho_{i,j}^{\psi} = \rho_{i,j}\psi^{-1}$ where $\psi(t) = t^{\frac{1}{N}}$.

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Let us consider the expansion of t^k, $(k = 0, \ldots, N-1)$ in terms of the initial conditions $(q_n)_{n=0}^{N-1}$, that is

$$t^k = \sum_{l=0}^{k} \alpha_{l,k} q_l(t).$$

Finally, we write $p_k(t) = \sum_{l=0}^{k} \beta_{l,k} t^l$, $k = 0, \ldots, N-1$.

The polynomials $(p_n)_n$ and $(q_n)_n$ have the same N-Jacobi matrix. So, if we look at the proof of Theorem 3.1 and since the polynomials $(p_n)_n$ are orthonormal with respect to the matrix of measures $(\mu_{\rho,i,j})_{i,j=1}^{N}$, we get that the polynomials $(q_n)_n$ are orthonormal with respect to the positive definite matrix of measures

$$AB \begin{pmatrix} \mu_{\rho,1,1} & \cdots & \mu_{\rho,1,N} \\ \vdots & \ddots & \vdots \\ \mu_{\rho,N,1} & \cdots & \mu_{\rho,N,N} \end{pmatrix} B^* A^*$$

where A, B are the numerical matrices

$$A = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \alpha_{0,1} & \alpha_{1,1} & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ \alpha_{0,N-2} & & \cdots & \alpha_{N-2,N-2} & 0 \\ \alpha_{0,N-1} & & \cdots & & \alpha_{N-1,N-1} \end{pmatrix}$$

and

$$B = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \beta_{0,1} & \beta_{1,1} & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ \beta_{0,N-2} & & \cdots & \beta_{N-2,N-2} & 0 \\ \beta_{0,N-1} & & \cdots & & \beta_{N-1,N-1} \end{pmatrix}.$$

C) GENERATING FORMULA. From the formula (6.2), it is straightforward to find the generating formula for the polynomials $(q_n)_n$, assuming we know the generating function for the polynomials $(p_n)_n$. Thus, if we put $A(t,\lambda)$ for this function, *i.e.*, $A(t,\lambda) = \sum_n a_n p_n(t)\lambda^n$ where $(a_n)_n$ is a sequence of complex numbers without null terms, and assuming that

$$q_n(t) = \sum_{k=0}^{N-1} \sum_{m=k}^{N-1} t^k \gamma_{k,m} R_{N,m}(p_n)(t^N),$$

we have

$$\sum_n a_n q_n(t)\lambda^n = \sum_{k=0}^{N-1} \sum_{m=k}^{N-1} t^k \gamma_{k,m} \frac{1}{Nt^m} \sum_{l=0}^{N-1} (w^{-m})^l A(w^l t, \lambda). \tag{6.3}$$

It should be noticed that if the polynomials $(p_n)_n$ are of Brenke type, that is $A(t,\lambda) = B(\lambda)C(t\lambda)$, then, if we put $\gamma_{k,m} = \gamma_m \delta_{k,m}$ in (6.3), we get that

$$\sum_n a_n q_n(t)\lambda^n = B(\lambda) \sum_{m=0}^{N-1} \gamma_m \frac{1}{N} \sum_{l=0}^{N-1} (w^{-m})^l C(w^l t\lambda) = B(\lambda)D(t\lambda).$$

This provides some examples of polynomials $(q_n)_n$ which are of Brenke type, are not orthogonal with respect to any measure but are orthonormal with respect to a positive definite $N \times N$ matrix of measures.

D) A PARTICULAR CASE: LAGUERRE POLYNOMIALS. Let us take the Laguerre polynomials $\big(L_n(t)\big)_n$ and their three term recurrence relation

$$tL_n(t) = -(n+1)L_{n+1}(t) + (2n+1)L_n(t) - nL_{n-1}(t)$$

which gives the following five term recurrence relation

$$\begin{aligned} t^2 q_n(t) &= (n+1)(n+2)q_{n+2}(t) - 4(n+1)^2 q_{n+1} \\ &\quad + 2(3n^2+3n+1)q_n(t) - 4n^2 q_{n-1}(t) + n(n-1)q_{n-2}(t). \end{aligned}$$

Hence, for the initial conditions $q_0(t) = 1$ and $q_1(t) = at + b$ $(a \neq 0)$ we get the polynomials

$$\begin{aligned} q_n(t) &= \Big(\frac{1-a}{2}\Big)L_n(t) + \Big(\frac{1+a}{2}\Big)L_n(-t) + (1-b)\frac{L_n(t) - L_n(-t)}{2t} \\ &= \frac{1}{n!}\bigg(\Big(\Big(\frac{1-a}{2}\Big)(-1)^n + \frac{1+a}{2}\Big)t^n + \sum_{k=0}^{n-1}\binom{n}{k}\frac{n-k}{k!} \\ &\quad \Big(\Big(\frac{1-a}{2}(-1)^k + \frac{1+a}{2}\Big)\frac{1}{n-k} + \frac{1}{(k+1)^2}\frac{1-b}{2}\big((-1)^k - 1\big)\Big)t^k. \end{aligned}$$

The polynomials $(q_n)_n$ are orthonormal with respect to the positive definite matrix of measures

$$\begin{pmatrix} \frac{e^{-\sqrt{t}}}{2\sqrt{t}}\chi_{[0,+\infty)} & \frac{e^{-\sqrt{t}}}{2a\sqrt{t}}(1-b-\sqrt{t})\chi_{[0,+\infty)} \\ \frac{e^{-\sqrt{t}}}{2a\sqrt{t}}(1-b-\sqrt{t})\chi_{[0,+\infty)} & \frac{e^{-\sqrt{t}}}{2a^2\sqrt{t}}(1-b-\sqrt{t})^2\chi_{[0,+\infty)} \end{pmatrix}.$$

The generating function for $(q_n)_n$ is

$$\sum_n q_n(t)\lambda^n = \frac{1}{1-\lambda}\bigg(\Big(\frac{1-a}{2}\Big)e^{-\frac{t\lambda}{1-\lambda}} + \Big(\frac{1+a}{2}\Big)e^{\frac{t\lambda}{1-\lambda}} + (1-b)\frac{e^{-\frac{t\lambda}{1-\lambda}} - e^{\frac{t\lambda}{1-\lambda}}}{2t}\bigg).$$

For $b = 1$, we have an interesting case. If we put $r_n(t) = \frac{q_n(t)}{n!}$, we find the following Brenke type generating formula

$$\sum_n r_n(t)\lambda^n = e^{\lambda}\bigg(\Big(\frac{1-a}{2}\Big)J_0(2\sqrt{t\lambda}) + \Big(\frac{1-a}{2}\Big)J_0(2\sqrt{-t\lambda})\bigg).$$

However, straightforward calculation gives that except for $b = 1$ and $a = \pm 1$ (*i.e.* $q_n(t) = L_n(\pm t)$) the polynomials $(r_n)_n$ are not orthogonal with respect to any positive measure (they do not satisfy a three term recurrence relation). So, they show that the class of Brenke type polynomials which are orthogonal with respect to a positive definite 2x2 matrix of measures is wider than that of Brenke type polynomials orthogonal with respect to a positive measure.

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