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THE TOPOLOGICAL DEGREE OF A-PROPER MAPPING IN THE MENGER PN-SPACE (I)

HUANG XIAOQIN, WANG MIANSEN AND ZHU CHUANXI

In this paper, we introduce the concept of A-proper topological degree in Menger PN-space and study some of its properties. Utilising its properties, we obtain a new fixed point theorem.

1. INTRODUCTION

In 1940s, Menger advanced the concept of probabilistic metric space. In his theory, the distance between two points was represented by a distribution function. Obviously, compared with the structure of metric space, it further conformed to reality. Moreover, the ordinary metric space can be looked upon as its special cases. So, the study of probabilistic metric space has important practical significance. As everyone knows, the A-proper topological degree theory is a forceful tool in the research of operator theory in normed spaces. Then, how to establish and study the A-proper topological degree in probabilistic metric space? In this paper, we introduce the concept of A-proper topological degree in Menger PN- space and study some of its properties. Utilising its properties, we obtain a new fixed point theorem.

For the sake of convenience, we recall some definitions and properties of PN-space.

DEFINITION 1: (Chang [1].) A probabilistic normed space (shortly a PN-space) is an ordered pair (E,F), where E is a real linear space, F is a mapping of E into D (D is the set of all distribution functions. We shall denote the distribution function $F(x)$ by F_x, $F_x(t)$ denotes the value F_x for $t \in R$) satisfying the following conditions:

- (PN-1) $F_x(0) = 0$;
- (PN-2) $F_x(t) = H(t)$ for all $t \in$ R if and only if $x = \theta$, where H(t)=0 when $t \leqslant 0$, and H(t)=1 when $t > 0$;
- (PN-3) For all $\alpha \neq 0$, $F_{\alpha x}(t) = F_x\big(t/|\alpha|\big)$;
- (PN-4) For any $x, y \in E$ and $t_1, t_2 \in R$, if $F_x(t_1) = 1$ and $F_y(t_2) = 1$, then we have $F_{x+y}(t_1 + t_2) = 1$.

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Lemma 1. (Chang [1].) *Let (E, F, Δ) be a Menger PN-space with a continuous t-norm Δ, then $x_n \subset E$ is said to be convergent to $x \in E$ if for any $t > 0$, we have $\lim_{n\to\infty} F_{x_n - x}(t) = H(t)$.*

Let (E, F, Δ) be a Menger PN-space with a continuous t-norm Δ, then (E, F, Δ) with the induced family of neighbourhoods

$$\{U_y(\varepsilon, \lambda) : y \in E, \varepsilon > 0, \lambda > 0\} = \{y + U_\theta(\varepsilon, \lambda) : y \in E, \varepsilon > 0, \lambda > 0\}$$

is a Hausdorff linear topological space.

Sherwood has proved that every Menger space with a continuous t-norm must have a completion (See Chang [1].) Hence, without generalisation, for Menger space with a continuous t-norm, we always think that the space is complete.

We can refer to Chang [1, 2], Guo [3] and Petryshyn[4] for the properties of PN-space and A-proper mapping.

2. Main Results

Definition 2: (E, F, Δ) is said to be a projected complete Menger PN-space, where Δ is a continuous t-norm, if the following conditions are satisfied:

- (i) X_n is a sequence of finite dimensional subspace of E and $Q_n : E \to X_n$ is a linear bounded projection operator satisfying $Q_n(E) = X_n$, $Q_n^2 = Q_n$;
- (ii) For any $x \in E$, we have $\lim_{n\to\infty} F_{Q_n x - x}(t) = H(t)$, $\forall t > 0$;
- (iii) (E, F, Δ) is a Menger PN-space. In here, $\Gamma = \{X_n, Q_n\}$ is called a probabilistic metric approximation scheme of (E, F, Δ).

Definition 3: Let (E, F, Δ) be a projected complete Menger PN-space; Δ be a continuous t-norm; Ω be a bounded open set of E and $f : \overline{\Omega} \longrightarrow E$ be a continuous bounded mapping. $\Omega_n = \Omega \cap X_n$ $(n = 1, 2, \ldots)$. f is said to be an A-proper mapping with respect to the probabilistic metric approximation scheme Γ if for any sequence $x_{n_k} \in \overline{\Omega}_{n_k}$ satisfying $\lim_{k\to\infty} F_{Q_{n_k} f(x_{n_k}) - Q_{n_k}(y)}(t) = H(t)$, $\forall t > 0$ (where $y \in E$), there exists a convergent subsequence $\{x_{n_{k_i}}\}$ of $\{x_{n_k}\}$, such that $x_{n_{k_i}} \longrightarrow x \in \overline{\Omega}$ and $f(x) = y$. When Γ is fixed, f is said to be A-proper.

Throughout this paper, we assume that Γ is fixed.

Definition 4: Let (E, F, Δ) be a projected complete Menger PN-space; Δ be a continuous t-norm; Ω be a bounded open set of E; and $f : \overline{\Omega} \longrightarrow E$ be an A-proper mapping, $p \in E \backslash f(\partial\Omega)$. Z denotes the set of integers. $Z^* = Z \cup \{-\infty, +\infty\}$. Generalised topological degree $\operatorname{Deg}(f, \Omega, p)$ is defined to be:

$$\operatorname{Deg}(f, \Omega, p) = \Big\{ z \in Z^* \mid \text{ there exists a subsequence } \{n_k\} \text{ of } \{n\} \text{ such that } \deg_R(Q_{n_k} f, \Omega_{n_k}, Q_{n_k}(p)) \longrightarrow z \Big\}.$$

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The degree $\deg_R(Q_nf, \Omega_n, Q_n(p))$ which is used in the above definition is the topological degree of the continuous mapping $Q_nf : \overline{\Omega}_n \longrightarrow X_n$ in a finite dimensional space. It is easy to see that when n is sufficiently large, we have $Q_n(p) \notin Q_nf(\partial\Omega_n)$ (otherwise, there exists an $x_n \in \partial\Omega_n \subset \partial\Omega$ such that $Q_nf(x_n) = Q_n(p)$, then we have

$$\lim_{n\to\infty} F_{Q_nf(x_n)-Q_n(p)}(t) = H(t).$$

Because f is an A-proper mapping, by the definition, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ satisfying $x_{n_k} \to x \in \partial\Omega$ and $f(x) = p$. This contradicts $p \notin f(\partial\Omega)\big)$. Thus, when n is sufficiently large, $\deg_R(Q_nf, \Omega_n, Q_n(p))$ is significant. Therefore $\mathrm{Deg}(f, \Omega, p)$ is a nonempty subset of Z^*.

THEOREM 1. *The generalised topological degree* $\mathrm{Deg}(f, \Omega, p)$ *has the following properties:*

(i) $\mathrm{Deg}(I, \Omega, p) = 1, \forall p \in \Omega$, *where I is an identity operator;*

(ii) *If* $\mathrm{Deg}(f, \Omega, p) \neq \{0\}$, *then the equation $f(x) = p$ has a solution in Ω;*

(iii) *If $L : [0,1]\times\overline{\Omega} \to E$ is continuous and for any fixed $t \in [0,1]$, $L(t,.) : \overline{\Omega} \to E$ is an A-proper mapping satisfying*

$$\lim_{t\to t_0} \inf_{x\in\overline{\Omega}} F_{L(t,x)-L(t_0,x)}(\varepsilon) = H(\varepsilon), \quad \forall\varepsilon > 0,$$

and $p \notin h_t(\partial\Omega)$, $0 \leqslant t \leqslant 1$, where $h_t(x) = L(t,x)$, then we have

$$\mathrm{Deg}(h_t, \Omega, p) = \mathrm{Deg}(h_0, \Omega, p), \ \forall\ 0 \leqslant t \leqslant 1;$$

(iv) *If Ω_0 is an open subset of Ω and $p \notin f(\overline{\Omega}\backslash\Omega_0)$, then we have* $\mathrm{Deg}(f, \Omega, p) = \mathrm{Deg}(f, \Omega_0, p)$*;*

(v) *If $\Omega_{(1)}$ and $\Omega_{(2)}$ are two disjoint open subsets of Ω and*

$$p \notin f\Big(\overline{\Omega} \setminus \big(\Omega_{(1)} \cup \Omega_{(2)}\big)\Big),$$

then

$$\mathrm{Deg}(f, \Omega, p) \subseteq \mathrm{Deg}(f, \Omega_{(1)}, p) + \mathrm{Deg}(f, \Omega_{(2)}, p).$$

If either $\mathrm{Deg}(f, \Omega_{(1)}, p)$ *or* $\mathrm{Deg}(f, \Omega_{(2)}, p)$ *is single-valued, then*

$$\mathrm{Deg}(f, \Omega, p) = \mathrm{Deg}(f, \Omega_{(1)}, p) + \mathrm{Deg}(f, \Omega_{(2)}, p);$$

(vi) *If $p \notin f(\partial\Omega)$, then* $\mathrm{Deg}(f, \Omega, p) = \mathrm{Deg}(f - p, \Omega, \theta)$*;*

(vii) *If p varies on every connected component of $E \setminus f(\partial\Omega)$, then* $\mathrm{Deg}(f, \Omega, p)$ *is a constant.*

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PROOF: (i) Because

$$\operatorname{Deg}(I,\Omega,p)=\Big\{z\in Z^* \mid \text{ there exists a subsequence } \{n_k\} \text{of } \{n\} \\ \text{ such that } \deg_R\big(Q_{n_k}I,\Omega_{n_k},Q_{n_k}(p)\big)\longrightarrow z\Big\}$$

and $\deg_R\big(Q_{n_k}I,\Omega_{n_k},\ Q_{n_k}(p)\big)=1$, $\forall p\in\Omega_{n_k}$, then we have $\operatorname{Deg}(I,\Omega,p)=1$, $\forall p\in\Omega$.

(ii) Because $\operatorname{Deg}(f,\Omega,p)\neq\{0\}$, there must exist a subsequence $\{n_k\}$ of $\{n\}$ such that $\deg_R\big(Q_{n_k}f,\Omega_{n_k},Q_{n_k}(p)\big)\neq 0$. Hence, there exists an $\{x_{n_k}\}\in\Omega_{n_k}\subset\Omega$ such that $Q_{n_k}f(x_{n_k})=Q_{n_k}(p)$ $(k=1,2,\ldots)$, and $\lim_{k\to\infty}F_{Q_{n_k}f(x_{n_k})-Q_{n_k}(p)}(t)=H(t)$, $\forall t>0$. By the A-proper property of f, there exists the convergent subsequence $\{x_{n_{k_i}}\}$ of $\{x_{n_k}\}$ satisfying $x_{n_{k_i}}\to x_0\in\overline{\Omega}$ and $f(x_0)=p$. By the definition of $\operatorname{Deg}(f,\Omega,p)$, we have $p\notin f(\partial\Omega)$. Hence $x_0\in\Omega$. Hence $f(x)=p$ has a solution in Ω.

(iii) For any $t_0\in[0,1]$, we prove that there exists a $\delta_0>0$ such that

$$Q_n(p)\notin Q_nh_s(\partial\Omega_n),\quad \forall s\in N(t_0,\delta_0).$$

Otherwise, for any $\delta_1>\delta_2>\cdots>\delta_k>\cdots\geqslant 0$, there exists an $s_{n_k}\in N(t_0,\delta_k)$ such that $Q_{n_k}(p)\in Q_{n_k}h_{s_{n_k}}(\partial\Omega_{n_k})$ $(k=1,2,\ldots)$. Therefore there exists an $x_{n_k}\in\partial\Omega_{n_k}\subset\partial\Omega$ such that $Q_{n_k}(p)=Q_{n_k}h_{s_{n_k}}(x_{n_k})$. So

$$\lim_{k\to\infty}F_{Q_{n_k}h_{s_{n_k}}(x_{n_k})-Q_{n_k}(p)}(\varepsilon)=H(\varepsilon)\qquad\forall\varepsilon>0$$

Hence

$$\begin{aligned}F_{Q_{n_k}h_{t_0}(x_{n_k})-Q_{n_k}(p)}(\varepsilon)&=F_{Q_{n_k}h_{t_0}(x_{n_k})-Q_{n_k}h_{s_{n_k}}(x_{n_k})+Q_{n_k}h_{s_{n_k}}(x_{n_k})-Q_{n_k}(p)}(\varepsilon)\\&\geqslant\Delta\left(F_{Q_{n_k}h_{t_0}(x_{n_k})-Q_{n_k}h_{s_{n_k}}(x_{n_k})}\left(\frac{\varepsilon}{2}\right),F_{Q_{n_k}h_{s_{n_k}}(x_{n_k})-Q_{n_k}(p)}\left(\frac{\varepsilon}{2}\right)\right)\end{aligned}$$

Because $\lim_{t\to t_0}\inf_{x\in\overline{\Omega}}F_{L(t,x)-L(t_0,x)}(\varepsilon)=H(\varepsilon)$ and $\inf_{x\in\overline{\Omega}}F_{L(t,x)-L(t_0,x)}(\varepsilon)\leqslant F_{L(t,x)-L(t_0,x)}(\varepsilon)$, we have $\lim_{t\to t_0}F_{L(t,x)-L(t_0,x)}(\varepsilon)=H(\varepsilon)$, $\forall x\in\overline{\Omega},\varepsilon>0$. Hence $h_t(x_{n_k})\to h_{t_0}(x_{n_k})$ $(t\to t_0)$. By the continuity of Q_{n_k}, we have $Q_{n_k}h_t(x_{n_k})\to Q_{n_k}h_{t_0}(x_{n_k})$ $(t\to t_0)$. Because $s_{n_k}\in N(t_0,\delta_k)$, we have $s_{n_k}\to t_0$ $(k\to\infty)$. Hence $Q_{n_k}h_{s_{n_k}}(x_{n_k})\to Q_{n_k}h_{t_0}(x_{n_k})$ $(k\to\infty)$, and

$$\lim_{k\to\infty}F_{Q_{n_k}h_{t_0}(x_{n_k})-Q_{n_k}h_{s_{n_k}}(x_{n_k})}\left(\frac{\varepsilon}{2}\right)=H(\varepsilon)$$

Therefore

$$\begin{aligned}&\lim_{k\to\infty}F_{Q_{n_k}h_{t_0}(x_{n_k})-Q_{n_k}(p)}(\varepsilon)\\&\quad\geqslant\Delta\left(\lim_{k\to\infty}F_{Q_{n_k}h_{t_0}(x_{n_k})-Q_{n_k}h_{s_{n_k}}(x_{n_k})}\left(\frac{\varepsilon}{2}\right),\lim_{k\to\infty}F_{Q_{n_k}h_{s_{n_k}}(x_{n_k})-Q_{n_k}(p)}\left(\frac{\varepsilon}{2}\right)\right)\\&\quad=\Delta\big(H(\varepsilon),H(\varepsilon)\big)\\&\quad=H(\varepsilon)\end{aligned}$$

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Thus $\lim_{k\to\infty} F_{Q_{n_k}h_{t_0}(x_{n_k})-Q_{n_k}(p)}(\varepsilon) = H(\varepsilon)$, $\forall\varepsilon > 0$. By the A-proper property of h_{t_0}, there exists a convergent subsequence $\{x_{n_{k_i}}\}$ of $\{x_{n_k}\}$ satisfying $x_{n_{k_i}} \to x_0 \in \partial\Omega$ and $h_{t_0}(x_0) = p$. It contradicts $p \notin h_{t_0}(\partial\Omega)$. Hence $Q_n(p) \notin Q_n h_s(\partial\Omega_n)$. By the homotopy invariance property of topological degree in finite dimensional space, we have

$$\deg_R(Q_n h_t, \Omega_n, Q_n(p)) = \deg_R(Q_n h_{t_0}, \Omega_n, Q_n(p)), \quad \forall n > N(t).$$

So $\mathrm{Deg}(h_t, \Omega, p) = \mathrm{Deg}(h_{t_0}, \Omega, p)$, $\forall t \in N(t_0, \delta_0)$. Hence we prove that for any $t_0 \in [0,1]$, there exists a neighbourhood $N(t_0, \delta_0)$, when $t \in N(t_0, \delta_0)$, $\mathrm{Deg}(h_t, \Omega, p)$ is a constant. By the arbitrariness of t_0, we have $\mathrm{Deg}(h_t, \Omega, p) = \mathrm{Deg}(h_0, \Omega, p)$, $\forall t \in [0,1]$.

(iv) Let $\Omega_n^{(0)} = \Omega_0 \cap X_n$. There must exist a $N > 0$ satisfying $Q_n(p) \notin Q_n f(\overline{\Omega}_n \setminus \Omega_n^{(0)}), \forall n > N$. Otherwise, there exist a subsequence $\{n_k\}$ of $\{n\}$ and an $x_{n_k} \in \overline{\Omega}_{n_k} \setminus \Omega_{n_k}^{(0)} \subset \overline{\Omega} \setminus \Omega_0$ such that $Q_{n_k} f(x_{n_k}) = Q_{n_k}(p)$, $(k = 1, 2, \ldots)$, hence $\lim_{k\to\infty} F_{Q_{n_k} f(x_{n_k})-Q_{n_k}(p)}(\varepsilon) = H(\varepsilon)$. Because f is an A-proper mapping, then we have $x_{n_{k_i}} \to x_0$ and $f(x_0) = p$. Because $\overline{\Omega} \setminus \Omega_0$ is a closed set, we have $x_0 \in \overline{\Omega} \setminus \Omega_0$. It contradicts $p \notin f(\overline{\Omega} \setminus \Omega_0)$. Hence $Q_n(p) \notin Q_n f(\overline{\Omega}_n \setminus \Omega_n^{(0)})$. By the properties of topological degree in finite dimensional space, we have

$$\deg_R(Q_n f, \Omega_n, Q_n(p)) = \deg_R(Q_n f, \Omega_n^{(0)}, Q_n(p)), \ \forall n > N.$$

Hence

$$\mathrm{Deg}(f, \Omega, p) = \mathrm{Deg}(f, \Omega_0, p).$$

(v) Let $\Omega_n^{(1)} = \Omega_{(1)} \cap X_n$ and $\Omega_n^{(2)} = \Omega_{(2)} \cap X_n$, then $\Omega_n^{(1)}$ and $\Omega_n^{(2)}$ are two disjoint open subsets of $\Omega_n = \Omega \cap X_n$. In the following, we prove that when n is sufficiently large, $Q_n(p) \notin Q_n f(\overline{\Omega}_n \setminus (\Omega_n^{(1)} \cup \Omega_n^{(2)}))$. Otherwise, there exist a subsequence $\{n_k\}$ of $\{n\}$ and an $\{x_{n_k}\} \in \overline{\Omega}_{n_k} \setminus (\Omega_{n_k}^{(1)} \cup \Omega_{n_k}^{(2)}) \subset \overline{\Omega} \setminus (\Omega_{(1)} \cup \Omega_{(2)})$ such that $Q_{n_k} f(x_{n_k}) = Q_{n_k}(p) (k = 1, 2, \ldots)$. Hence $\lim_{k\to\infty} F_{Q_{n_k} f(x_{n_k})-Q_{n_k}(p)}(\varepsilon) = H(\varepsilon)$, $\forall\varepsilon > 0$. Because f is an A-proper mapping, then we have $x_{n_{k_i}} \to x_0$ and $f(x_0) = p$. Because $\overline{\Omega} \setminus (\Omega_{(1)} \cup \Omega_{(2)})$ is a closed set, we have $x_0 \in \overline{\Omega} \setminus (\Omega_{(1)} \cup \Omega_{(2)})$. This contradicts $p \notin f(\overline{\Omega} \setminus (\Omega_{(1)} \cup \Omega_{(2)}))$. Hence $Q_n(p) \notin Q_n f(\overline{\Omega}_n \setminus (\Omega_n^{(1)} \cup \Omega_n^{(2)}))$, $\forall n > N$. Hence $\deg_R(Q_n f, \Omega_n, Q_n(p)) = \deg_R(Q_n f, \Omega_n^{(1)}, Q_n(p)) + \deg_R(Q_n f, \Omega_n^{(2)}, Q_n(p))$.

For any $z \in \mathrm{Deg}(f, \Omega, p)$, there exists a subsequence $\{n_k\}$ of $\{n\}$ such that

$$\deg_R(Q_{n_k} f, \Omega_{n_k}, Q_{n_k}(p)) \to z.$$

Obviously, there exists a subsequence $\{n_{k_i}\}$ of $\{n_k\}$ such that

$$\deg_R(Q_{n_{k_i}} f, \Omega_{n_{k_i}}^{(1)}, Q_{n_{k_i}}(p)) \to z_1 \in \mathrm{Deg}(f, \Omega_{(1)}, p).$$

There exists a subsequence $\{n_{k_{i_j}}\}$ of $\{n_{k_i}\}$ such that

$$\deg_R(Q_{n_{k_{i_j}}} f, \Omega_{n_{k_{i_j}}}^{(2)}, Q_{n_{k_{i_j}}}(p)) \to z_2 \in \mathrm{Deg}(f, \Omega_{(2)}, p).$$

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Hence $z = z_1 + z_2 \in \mathrm{Deg}(f, \Omega_{(1)}, p) + \mathrm{Deg}(f, \Omega_{(2)}, p)$. By the arbitrariness of z, we have $\mathrm{Deg}(f, \Omega, p) \subseteq \mathrm{Deg}(f, \Omega_{(1)}, p) + \mathrm{Deg}(f, \Omega_{(2)}, p)$.

On the other hand, if either $\mathrm{Deg}(f, \Omega_{(1)}, p)$ or $\mathrm{Deg}(f, \Omega_{(2)}, p)$ is single-valued, for example, $\mathrm{Deg}(f, \Omega_{(1)}, p) = \{\alpha\}$, then we have

$$\deg_R\big(Q_n f, \Omega_n^{(1)}, Q_n(p)\big) \to \alpha \ (n \to \infty).$$

If $z \in \mathrm{Deg}(f, \Omega_{(1)}, p) + \mathrm{Deg}(f, \Omega_{(2)}, p)$, then we have $z = \alpha + z_2$ and the subsequence $\{n_j\}$ of $\{n\}$ such that

$$\deg_R\big(Q_{n_j} f, \Omega_{n_j}^{(2)}, Q_{n_j}(p)\big) \to z_2, \ (j \to \infty).$$

Hence

$$\deg_R\big(Q_{n_j} f, \Omega_{n_j}, Q_{n_j}(p)\big) \to \alpha + z_2 = z,$$

then $z \in \mathrm{Deg}(f, \Omega, p)$, and

$$\mathrm{Deg}(f, \Omega_{(1)}, p) + \mathrm{Deg}(f, \Omega_{(2)}, p) \subseteq \mathrm{Deg}(f, \Omega, p).$$

So

$$\mathrm{Deg}(f, \Omega_{(1)}, p) + \mathrm{Deg}(f, \Omega_{(2)}, p) = \mathrm{Deg}(f, \Omega, p).$$

(vi) $\mathrm{Deg}(f, \Omega, p) = \Big\{z \in Z^* \mid$ there exists the subsequence $\{n_k\}$ of $\{n\}$ such that $\deg_R\big(Q_{n_k} f, \Omega_{n_k}, Q_{n_k}(p)\big) \to z\Big\}$, $\mathrm{Deg}(f - p, \Omega, \theta) = \Big\{z \in Z^* \mid$ there exists the subsequence $\{n_k\}$ of $\{n\}$ such that $\deg_R\big(Q_{n_k}(f - p), \Omega_{n_k}, Q_{n_k}(\theta)\big) \to z\Big\}$. By the property of topological degree in finite dimensional space and Q_{n_k} is continuous and linear, we have

$$\begin{aligned}\deg_R\big(Q_{n_k}(f - p), \Omega_{n_k}, Q_{n_k}(\theta)\big) &= \deg_R(Q_{n_k} f - Q_{n_k}(p), \Omega_{n_k}, \theta) \\ &= \deg_R\big(Q_{n_k} f, \Omega_{n_k}, Q_{n_k}(p)\big).\end{aligned}$$

Hence $\mathrm{Deg}(f, \Omega, p) = \mathrm{Deg}(f - p, \Omega, \theta)$.

(vii) We assume that V is a connected region of $E \setminus f(\partial\Omega)$ and $p \in V$. Then there must exist a neighbourhood $u(\varepsilon_0, \lambda_0)$ of θ such that $\big(p + u(\varepsilon_0, \lambda_0)\big) \cap f(\partial\Omega) = \phi$. We take $q \in \big(p + u(\varepsilon_0, \lambda_0)\big)$ and denote $h_t(x) = f(x) - t(q - p)$, $t \in [0, 1]$, $x \in \overline{\Omega}$. Obviously, h_t is continuous. If there exist $t_0 \in [0, 1]$ and $x_0 \in \partial\Omega$ such that $f(x_0) - t_0(q - p) = p$, then we have $F_{f(x_0)-p}(\varepsilon_0) = F_{t_0(q-p)}(\varepsilon_0) > 1 - \lambda_0$. This contradicts $p \notin f(\partial\Omega)$. Hence $p \notin h_t(\partial\Omega)$, $\forall t \in [0, 1]$. If for any subsequence $\{x_n\}$ satisfying $\lim_{n\to\infty} F_{Q_n(f(x_n) - t(q-p)) - Q_n(w)}(\varepsilon) = H(\varepsilon)$, that is, $\lim_{n\to\infty} F_{Q_n f(x_n) - Q_n(t(q-p)+w)}(\varepsilon) = H(\varepsilon)$, then by the A-proper property of f, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x_0$ and $f(x_0) = t(q - p) + w$. Hence $f(x_0) - t(q - p) = w$, and h_t is an A-proper mapping. Because

$$\begin{aligned}\liminf_{t\to t_0} \inf_{x\in\overline{\Omega}} F_{L(t,x)-L(t_0,x)}\ (\varepsilon) &= \lim_{t\to t_0} \inf_{x\in\overline{\Omega}} F_{f(x)-t(q-p)-(f(x)-t_0(q-p))}(\varepsilon) \\ &= \lim_{t\to t_0} F_{(t_0-t)(q-p)}(\varepsilon) \\ &= H(\varepsilon), \quad \forall \varepsilon > 0,\end{aligned}$$

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by Theorem 1 (iii), we have

$$\mathrm{Deg}(f,\Omega,p)=\mathrm{Deg}(f-(q-p),\Omega,p)=\mathrm{Deg}(f-q,\Omega,\theta)=\mathrm{Deg}(f,\Omega,q).$$

This implies that the mapping $\Psi : p \to \mathrm{Deg}(f,\Omega,p)$ is a continuous mapping on V. Because V is connected, then $\Psi(V)$ is a connected set in R. Since Ψ is an integer-valued function, then $\mathrm{Deg}(f,\Omega,p)$ is the same when $p \in V$. ∎

3. Application

Theorem 2. *Let (E,F,Δ) be a projected complete Menger PN-space, Δ be a continuous t-norm, and Ω be a bounded open set of E, $\theta \in \Omega$, and $A : \overline{\Omega} \to E$ be a continuous bounded mapping. For any $\lambda \in [0,1]$, $I-\lambda A$ is an A-proper mapping. Moreover A satisfies the following condition:*

(1) $$F_{Ax}(t) > F_x(t), \quad \forall x \in \partial\Omega, \quad t>0$$

then A must have a fixed point in Ω.

Proof: By condition (1), A does not have a fixed point on $\partial\Omega$, that is, $Ax \neq x$, $\forall x \in \partial\Omega$. Let $h_s(x) = x - sAx$, $\forall s \in [0,1]$, $\forall x \in \overline{\Omega}$. In the following, we prove that $\theta \notin h_s(\partial\Omega)$, $\forall s \in [0,1]$. In fact, if $\theta \in h_s(\partial\Omega)$, then there exist an $s_0 \in [0,1]$ and an $x_1 \in \partial\Omega$ such that $\theta = x_1 - s_0Ax_1$. We have $s_0 \neq 0$ (If $s_0 = 0$, then we have $\theta = x_1$, that is, $\theta \in \partial\Omega$. It contradicts $\theta \in \Omega$) and $s_0 \neq 1$ (If $s_0 = 1$, then we have $\theta = x_1 - Ax_1$, that is, $x_1 = Ax_1$. It contradicts $Ax \neq x$, $\forall x \in \partial\Omega$). Hence $s_0 \in (0,1)$. By $\theta = x_1 - s_0Ax_1$, we have

(1^0) $$Ax_1 = \frac{1}{s_0}x_1$$

By (1), we have $F_{(1/s_0)x_1}(t) > F_{x_1}(t)$, $\forall t > 0$, that is, $F_{x_1}(ts_0) > F_{x_1}(t)$. By the nondecreasing property of F_{x_1}, we have $s_0t > t$. So $s_0 > 1$. This contradicts $s_0 \in (0,1)$. Hence $\theta \notin h_s(\partial\Omega)$. When $t \to t_0$, we have $x - tAx \to x - t_0Ax$. Thus $\lim_{t\to t_0} F_{x-tAx-(x-t_0Ax)}(\varepsilon) = H(\varepsilon)$, $\forall\varepsilon > 0$, $\forall x \in \overline{\Omega}$. Thus for any $\lambda > 0$, we have $F_{x-tAx-(x-t_0Ax)}(\varepsilon) > 1-\lambda$ $(t \to t_0)$. It is easy to prove that

$$\inf_{x\in\overline{\Omega}} F_{x-tAx-(x-t_0Ax)}(\varepsilon) \geqslant 1-\lambda \quad (t \to t_0).$$

By the arbitrariness of λ, we have

$$\lim_{t\to t_0}\inf_{x\in\overline{\Omega}} F_{x-tAx-(x-t_0Ax)}(\varepsilon) = H(\varepsilon), \forall\varepsilon > 0, \forall x \in \overline{\Omega}.$$

Because $h_t(x) = x - tAx$ is an A-proper mapping, by Theorem 1(iii), we have

$$\mathrm{Deg}(I-A,\Omega,\theta) = \mathrm{Deg}(I,\Omega,\theta) = 1.$$

Therefore, A has a fixed point x^* in Ω such that $Ax^* = x^*$. ∎

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Department of Mathematics
Xi'an Jiaotong University
Xi'an 710049
China
e-mail: : sjzhxq@163.com

Department of Mathematics
Xi'an Jiaotong University
Xi'an 710049
China

Institute of Mathematics
Nanchang University
Nanchang 330047
China
e-mail: zhuchuanxi@sina.com

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