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Simplicity of the Lyapunov spectrum for classes of Anosov flows

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Abstract. For $k \geq 2$, we prove that in a C^1-open and C^k-dense set of some classes of C^k-Anosov flows, all Lyapunov exponents have multiplicity one with respect to appropriate measures. The classes are geodesic flows with equilibrium states of Holder-continuous potentials, volume-preserving flows, and all fiber-bunched Anosov flows with equilibrium states of Holder-continuous potentials.

Key words: Lyapunov exponents, geodesic flows, Anosov flows
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1. *Introduction*

The existence of a positive Lyapunov exponent and, more generally, the multiplicity of the Lyapunov exponents of a system are of essential interest due to their relation to other dynamical invariants and the geometry of the associated dynamical foliations. In this paper, we seek to address the question of how often simplicity (that is, all exponents of multiplicity one) of Lyapunov spectrum arises for some classes of hyperbolic flows. In the classical settings of random matrix products, criteria for simplicity of spectrum were first established in the seminal papers [**GM89**, **GR86**] and with a variety of different techniques simplicity has more recently been proved in a large variety of settings [**BP21**, **BPVL20**, **MMY15**, **PV18**].

In [**BV04**], Bonatti and Viana first established a criterion for the simplicity of the Lyapunov spectrum of a cocycle over a discrete symbolic base which holds in great generality with respect to a large class of measures. Applying a Markov partition construction, the authors also extended the results to cocycles over hyperbolic maps, which naturally leads to the question of whether the criterion generically holds for the derivative cocycle in the space of diffeomorphisms. Indeed, without any further restrictions, the arguments in [**BV04**] can be modified without much difficulty to show that such a result would be possible, for appropriate choices of measures.

Here we consider the question of genericity of simple spectrum in the continuous-time setting; in particular, in more restrictive classes (geodesic flows, conservative flows, etc.) of Anosov flows, which presents significant differences relative to the discrete-time scenario. We establish a method of constructing appropriate perturbations of the Lyapunov spectrum by perturbing the 1-jet of an appropriate Poincareé map within a given class.

We apply it in different settings to obtain the following results. Let X be a smooth closed manifold; precise definitions of the other terms below are given in §2.

THEOREM 1.1. (Geodesic flows) *For $3 \leq k \leq \infty$, we denote by $\mathcal{G}^k$ the set of C^k-Riemannian metrics on X with sectional curvatures $1 \leq -K < 4$.*

There exists a C^2-open and C^k-dense set in $\mathcal{G}^k$ of metrics such that with respect to the equilibrium state of any Hölder potential (e.g. Liouville measure, measure of maximal entropy (m.m.e.)) the derivative cocycle of the geodesic flow has simple Lyapunov spectrum, that is, all its Lyapunov exponents have multiplicity one.

THEOREM 1.2. (Conservative flows) *For a fixed smooth volume m and for $2 \leq k \leq \infty$, let $\mathfrak{X}^k_m(X)$ be the set of divergence-free (with respect to m) C^k vector fields on M that generate (strictly) $\frac{1}{2}$-bunched Anosov flows.*

Then flows in a C^1-open and C^k-dense set of $\mathfrak{X}^k_m(X)$ have simple Lyapunov spectrum with respect to m.

THEOREM 1.3. (All flows) *For $2 \leq k \leq \infty$, let $\mathfrak{X}^k_A(X)$ be the set of C^k vector fields on M which generate (strictly) $\frac{1}{2}$-bunched Anosov flows.*

Then flows in a C^1-open and C^k-dense set of $\mathfrak{X}^k_A(X)$ have simple Lyapunov spectrum with respect to the equilibrium state of any Hölder potential (e.g. Sinai–Ruelle–Bowen (SRB) measure, m.m.e.).

As indicated before, the proofs are accomplished by constructing a discrete symbolic system via a Markov partition to apply a simplicity criterion of Avila and Viana [**AV07**], which is itself an improvement of the criterion of Bonatti and Viana [**BV04**]. In each class, we prove or use a previously established perturbational result to obtain density in the theorems above.

One main difficulty particular to the setting of $\mathbb{R}$-cocycles, which was already present in [**BV04**], arises in attempting to perturb the norms of pairs of complex eigenvalues generically. In [**BV04**], through the introduction of rotation numbers that vary continuously with the perturbation for orbits near a periodic point, a small rotation on a periodic orbit is propagated to an arbitrarily large one for a homoclinic point, which can then be made to have real eigenvalues.

Although such rotation numbers are well-defined for the particular perturbation of the cocycle introduced in [**BV04**], a general construction which allows for perturbations of the base system has only been introduced recently in [**Go20**]. However, the constructions in [**Go20**] do not apply directly to flows, and so we introduce new ideas to control the eigenvalues of the cocycle in the continuous-time setting.

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As the class of geodesic flows is the substantially more difficult case, we carry out the proof of Theorem 1.1 in detail, and in §5 we prove the analogous results needed for Theorem 1.2.

1.1. *Outline.* In §2, we give the necessary background for the later sections; we summarize the main results of [**AV07**, **KT72**] and introduce rotation numbers. For a more basic introduction to Lyapunov exponents and cocycles, we refer the reader to [**Vi14**], and for background on geodesic flows, we refer the reader to [**Pa99**]. In §§3 and 4, we specialize to the setting of the geodesic flows, giving the main arguments to prove of Theorem 1.1. Finally, in §5, we prove a perturbational result for the volume-preserving class, which by direct adaptation of the arguments of the previous sections proves Theorems 1.2 and 1.3.

2. *Preliminaries*

2.1. *Lyapunov exponents and simplicity of spectrum.* Here we collect and fix the definitions and background results used in later sections. For a continuous flow $\Phi^t : X \to X$ on a compact metric space X preserving an ergodic measure μ, a continuous linear cocycle over Φ on a linear bundle $\pi : \mathcal{E} \to X$ is a continuous map $\mathcal{A} : \mathbb{R} \times \mathcal{E} \to \mathcal{E}$ such that $\Phi^t \circ \pi = \pi \circ \mathcal{A}^t$, where $\mathcal{A}^t := \mathcal{A}(t, \cdot)$. Moreover, we require that the maps $A^t_{\pi(v)} := \mathcal{A}(t, \cdot)|_{\mathcal{E}_{\pi(v)}}$ are linear isomorphisms $\mathcal{E}_{\pi(v)} \to \mathcal{E}_{\phi^t\pi(v)}$ and satisfy the *cocycle property* $A^{t+s}(x) = A^s(\Phi^s(x)) \circ A^t(x)$.

Suppose $\log^+ \|A^t(x)\| \in L^1(X, \mu)$ for all $t \in \mathbb{R}$. For some fixed choice of norm $\|\cdot\|$ on the fibers, the fundamental result describing asymptotic growth of vectors under $\mathcal{A}$ is Oseledets' theorem: there exists a set of numbers $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, with $\lambda_i \neq \lambda_j$ for $i \neq j$, a measurable splitting $\mathcal{E} = \mathcal{E}^1 \oplus \cdots \oplus \mathcal{E}^n$ and a set of full measure $Y \subseteq X$ such that, for all $x \in Y$ and $t \in \mathbb{R}$, we have $A^t_x \mathcal{E}^i_x = \mathcal{E}^i_{\Phi^t(x)}$ and, moreover, for $v \in \mathcal{E}^i_x$:

$$\lim_{t \to \pm\infty} \frac{1}{t} \log \|A^t_x v\| = \lambda_i.$$

The numbers λ_i are the *Lyapunov exponents* of $\mathcal{A}$ with respect to μ.

When all bundles $\mathcal{E}^i$ are one-dimensional, $\mathcal{A}$ is said to have *simple Lyapunov spectrum* with respect to μ. When X is a smooth manifold and Φ^t is C^1, the *dynamical cocycle* on $\mathcal{E} = TX$ is the derivative map $D\Phi^t$ of the flow; we often refer to its Lyapunov exponents as the Lyapunov exponents of Φ with respect to μ. Similarly, we say Φ has simple Lyapunov spectrum when the dynamical cocycle does.

The definitions above hold in the discrete-time setting of [**AV07**], with appropriate modifications, where the criterion for simplicity of Lyapunov spectrum we need is proved. Following their notation, we let $\hat{f}$ be the shift map on a subshift of finite type $\hat{\Sigma}$ and $\mathcal{A}$ be a measurable cocycle on $\hat{\Sigma} \times \mathbb{R}^d$ over $\hat{f}$, which alternatively can be equivalently described by some measurable $\hat{A} : \hat{\Sigma} \to \mathrm{GL}(d, \mathbb{R})$.

The theorems of Avila and Viana all require the additional bunching assumption.

Definition 2.1. (Domination/holonomies) We say that $\hat{A}$ is dominated if there exists a distance d in $\hat{\Sigma}$ and constants $\theta < 1$ and $\nu \in (0, 1]$ such that, up to replacing $\hat{A}$ by some power $\hat{A}^N$:

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(1) $d(\hat{f}(\hat{x}), \hat{f}(\hat{y})) \leq \theta d(\hat{x}, \hat{y})$ and $d(\hat{f}^{-1}(\hat{x}), \hat{f}^{-1}(\hat{y})) \leq \theta d(\hat{x}, \hat{y})$ for every $\hat{y} \in W^s_{\text{loc}}(\hat{x})$ and $\hat{z} \in W^u_{\text{loc}}(\hat{x})$;
(2) the map $\hat{x} \mapsto \hat{A}(\hat{x})$ is ν-Hölder continuous and $\|\hat{A}(\hat{x})\|\|\hat{A}^{-1}(\hat{x})\|\theta^\nu < 1$ for every $\hat{x} \in \hat{\Sigma}$.

If $\hat{A}$ is either dominated or constant on each cylinder, there exists a family of holonomies $\phi^u_{\hat{x},\hat{y}}$, that is, linear isomorphisms of $\mathbb{R}^d$ such that for each $\hat{x}, \hat{y}, \hat{z} \in \hat{\Sigma}$ in the same unstable manifold of $\hat{f}$ there exists $C_1 > 0$ such that:
(1) $\phi^u_{\hat{x},\hat{x}} = id$ and $\phi^u_{\hat{x},\hat{y}} = \phi^u_{\hat{x},\hat{z}} \circ \phi^u_{\hat{z},\hat{y}}$;
(2) $\hat{A}(\hat{f}^{-1}(\hat{y})) \circ \phi^u_{\hat{f}^{-1}(x),\hat{f}^{-1}(y)} \circ \hat{A}^{-1}(\hat{x}) = \phi^u_{\hat{x},\hat{y}}$;
(3) $\|\phi^u_{\hat{x},\hat{y}} - id\| \leq C_1 d(\hat{x}, \hat{y})^\nu$.

There is a family ϕ^s of holonomies over stable manifolds satisfying analogous properties.

For such cocycles, the holonomies allow the dynamics to be propagated over single periodic orbits to obtain data on the Lyapunov spectrum of certain measures. Thus, the adaptation of the original pinching and twisting conditions for a monoid of matrices can be adapted to these cocycles as follows.

Definition 2.2. (Simple cocycles) Suppose $\hat{A} : \hat{\Sigma} \to \text{GL}(d, \mathbb{R})$ is either dominated or constant on each cylinder of $\hat{\Sigma}$. We say that $\hat{A}$ is *simple* if there exists a periodic point $\hat{p}$ and a homoclinic point $\hat{z}$ associated with $\hat{p}$ such that:
(P) the eigenvalues of $\hat{A}$ on the orbit of $\hat{p}$ have multiplicity 1 and distinct norms; let $\omega_j \in \mathbb{R}P^{d-1}$ represent the eigenspaces, for $1 \leq j \leq d$; and
(T) $\{\psi_{\hat{p},\hat{z}}(\omega_i) : i \in I\} \cup \{\omega_j : j \in J\}$ is linearly independent, for all subsets I and J of $1, \ldots, d$ with $\#I + \#J \leq d$ where, denoting by ϕ^u and ϕ^s the holonomies as above,

$$\psi_{\hat{p},\hat{z}} = \phi^s_{\hat{z},\hat{p}} \circ \phi^u_{\hat{p},\hat{z}}.$$

An invariant probability measure $\hat{\mu}$ has local product structure if, for every cylinder $[0 : i]$,

$$\hat{\mu}|[0 : i] = \psi \cdot (\mu^+ \times \mu^-),$$

where $\psi : [0 : i] \to \mathbb{R}$ is continuous and μ^+ and μ^- are the projections of $\hat{\mu}|[0 : i]$ to spaces of one-sided sequences indexed by positive and negative indices, respectively. For instance, this property holds for every equilibrium state of $\hat{f}$ associated with a Hölder potential [**Bo75**].

THEOREM 2.3. [**AV07**, Theorem A] *If $\hat{A}$ is a simple cocycle then it has Lyapunov exponents of multiplicity one with respect to any $\hat{\mu}$ with local product structure.*

2.2. *Anosov flows.*

The continuous-time hyperbolic systems we study are as follows.

Definition 2.4. (C^k-Anosov flows) A C^k ($1 \leq k \leq \infty$) flow $\Phi^t : X \to X$ on a smooth manifold X is called *Anosov* if it preserves a splitting $E^u \oplus E^0 \oplus E^s$ of TX such that

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E^0 is the flow direction and there exist $\lambda > 0$ and $C > 1$ such that, for all $v \in E^u$ and $u \in E^s$,

$$\|D\Phi^t v\| \geq Ce^{\lambda t}\|v\|, \quad \|D\Phi^{-t} u\| \geq Ce^{\lambda t}\|u\|.$$

A significant class of cocycles over Anosov flows related to the theory of partially hyperbolic systems and to the class of dominated cocycles over shift maps is that of *fiber bunched cocycles*, whose expansion and contraction rates are dominated by the base dynamics.

Definition 2.5. (Fiber bunching) A β-Hölder continuous cocycle $\mathcal{A} : \mathcal{E} \times \mathbb{R} \to \mathcal{E}$ over an Anosov flow $\Phi^t : X \to X$ is said to be α-fiber bunched if $\alpha \leq \beta$ and there exists $T > 0$ such that for all $p \in M$ and $t \geq T$:

$$\|A^t_p\| \|(A^t_p)^{-1}\| \|D\Phi^t|_{E^s}\|^\alpha < 1, \quad \|A^t_p\| \|A^{-t}_p\| \|D\Phi^{-t}|_{E^u}\|^\alpha < 1.$$

When the cocycles $D\Phi^t|_{E^{i=u,s}}$ themselves satisfy the inequalities above in place of $\mathcal{A}$, the Anosov flow is said to be α-bunched.

Fiber bunching is a partial hyperbolicity condition on the projectivization of the fiber bundle. The strong stable and unstable manifold theorem for this partially hyperbolic system can be interpreted as defining holonomy maps between the fibers.

THEOREM 2.6. [**KS13**] *Suppose $\mathcal{A}$ is β-Hölder and fiber bunched over a base system as in Definition 2.5. Then, the cocycle admits holonomy maps h^u, that is, a continuous map $h^u : (x, y) \to h^u_{x,y}$, $x \in M$, $y \in W^u(x)$ the strong unstable manifold of x, such that:*

(1) *$h^u_{x,y}$ is a linear map $\mathcal{E}_x \to \mathcal{E}_y$;*
(2) *$h^u_{x,x} = Id$ and $h^u_{y,z} \circ h^u_{x,y} = h^u_{x,z}$;*
(3) *$h^u_{x,y} = (A^t_y)^{-1} \circ h^u_{\Phi^t(x),\Phi^t(y)} \circ A^t_x$ for every $t \in \mathbb{R}$.*

*Moreover, the holonomy maps are unique, and, fixing a system of linear identifications $I_{xy} : \mathcal{E}_x \to \mathcal{E}_y$, see [**KS13**], they satisfy*

$$\|h^u_{x,y} - I_{x,y}\| \leq Cd(x, y)^\beta.$$

Using property (3), for sufficiently small $r > 0$, one may extend these holonomies for all $y \in W^{cu}_r(x)$, the ball of radius r centered at x in the center-unstable manifold sometimes called the *local center-unstable manifold*, and such holonomies are denoted by h^{cu}. Namely, one lets

$$h^{cu}_{xy} = h^u_{\Phi^t(x),y} \circ A^t_x,$$

where $t \in \mathbb{R}$ is chosen to minimize $|t|$ among the times such that $\Phi^t(x) \in W^u(y)$. Observe that this is only well-defined locally and that the same construction holds over W^{cs}.

For the case where Φ is itself α-bunched, it is known that [**Ha94**] the bunching constant is directly related to the regularity of the Anosov splitting: for $\frac{1}{2}$-bunched Anosov flows, the weak stable and unstable bundles $E^{cu,cs} := E^0 \oplus E^{u,s}$ are of class C^1. Thus, we have the following.

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PROPOSITION 2.7. *For $\Phi^t : X \to X$ a $\frac{1}{2}$-bunched C^2-Anosov flow, the cocycle $\mathcal{A}^u$ (respectively, $\mathcal{A}^s$) on the bundle $Q^u := E^{cu}/E^0$ (respectively, $Q^s := E^{cs}/E^0$) given by the derivative $D\Phi^t$ is* 1*-bunched.*

Proof. The cocycle $D\Phi^t|_{E^{cu}/E^0}$ is C^1 by the regularity of the splitting mentioned previously and, by hypothesis $D\Phi^t|_{E^{cu}/E^0}$, satisfies the inequalities in the definition of fiber bunching with $\alpha = 1$. Same for E^{cs}. □

Finally, we describe the class of measures with respect to which we prove our results. Fix a topologically mixing C^2-Anosov flow Φ^t. Let $\rho : X \to \mathbb{R}$ be a Hölder-continuous function, which we refer to as a *potential*. Then an *equilibrium state* μ_ρ of ρ is an invariant measure satisfying the variational principle:

$$h_{\mu_\rho}(\Phi) + \int \rho \, d\mu_\rho = \sup_{\mu \in \mathcal{M}_\Phi(X)} h_\mu(\Phi) + \int \rho \, d\mu,$$

where $h_\mu(\Phi)$ is the measure-theoretic entropy of Φ with respect to μ and $\mathcal{M}_\Phi(X)$ is the set of invariant measures of Φ. The existence and uniqueness of the equilibrium state μ_ρ, is, in this setting, a foundational result in the theory of the thermodynamical formalism [**Bo75**].

Important examples of equilibrium states include the case $\rho = 0$, which gives the measure of maximal entropy as the equilibrium state, and $\rho(x) = -(d/dt) \log J^u(x, t)|_{t=0}$, where $J^u(x, t) = \det D_x\Phi^t|_{E^u}$, which gives the SRB measure. Moreover, the product structure property mentioned in the previous section is also a classical result for equilibrium states proved in [**Bo75**].

2.3. *Rotation numbers.* As indicated in the introduction, in order to perturb away complex eigenvalues by a small rotation, one needs the formalism of rotation numbers, which we introduce in complete form here. We roughly follow the discussion in §3 of [**Go20**].

As a brief introduction, recall that for an orientation-preserving homeomorphism of the circle $f : S^1 \to S^1$, the *Poincaré rotation number* $\rho(f) \in S^1 = \mathbb{R}/2\pi$ of f is defined as

$$\rho(f) = \lim_{n\to\infty} \frac{\tilde{f}^n(x) - x}{n} \pmod{2\pi},$$

for a lift $\tilde{f} : \mathbb{R} \to \mathbb{R}$ of f. This limit always exists and is independent of the choice of $x \in \mathbb{R}$ and the lift $\tilde{f}$. For an orientation-reversing homeomorphism we define $\rho(f) = 0$.

The Poincaré rotation number measures, on average, how much an element is rotated by an application of f and is a conjugation invariant, that is, $\rho(g^{-1} f g) = \rho(f)$, for g also a homeomorphism of S^1. In what follows, we extend this definition for cocycles on circle bundles.

Throughout this section, let X a compact metric space and Φ^t a continuous flow on X. Let $\mathcal{M}_\Phi(X)$ be the space of probability measures on X invariant under Φ^t with the weak-* topology.

For our purposes, it will suffice to work with trivial bundles $E = X \times S^1$, and a continuous cocycle $\mathcal{A} : \mathbb{R} \times E \to E$ over Φ^t. Then, for $(x, \theta) \in E$, the map $t \mapsto A_x^t(\theta)$

is a continuous map from $\mathbb{R} \to S^1$, so it may be lifted to some $w_{x,\theta} : \mathbb{R} \to \mathbb{R}$. Let $\tilde{w}_{x,\theta}(t) := w_{x,\theta}(t) - w_{x,\theta}(0)$, so that $\tilde{w}$ does not depend on the lift w.

Definition 2.8. (Pointwise rotation number) The average rotation number $\rho : X \to \mathbb{R}$ is defined by the limit:

$$\rho(x) = \lim_{t\to\infty} \frac{\tilde{w}_{x,\theta}(t)}{t},$$

whenever it exists, and is independent of the choice of θ.

Indeed, for any $\theta, \theta' \in S^1$, we have $|\tilde{w}_{x,\theta}(t) - \tilde{w}_{x,\theta'}(t)| < 2\pi$ for any t so the limit does not depend on choice of $\theta \in S^1$.

Now define $\sigma : X \times \mathbb{R} \to \mathbb{R}$ and $\tau : X \times \mathbb{R} \to \mathbb{R}$ by

$$\sigma^t(x) := \sigma(x, t) = \sup_{\theta\in S^1} \tilde{w}_{x,\theta}(t),$$

$$\tau^t(x) := \tau(x, t) = \inf_{\theta\in S^1} \tilde{w}_{x,\theta}(t),$$

which, by the continuity of $\mathcal{A}$, are evidently continuous in t and in x. Moreover, by the cocycle equation for $\mathcal{A}$, it is clear that the functions σ and τ are subadditive and superadditive in the $\mathbb{R}$ coordinate, respectively.

By Kingman's subadditive ergodic theorem for flows, for any $\mu \in \mathcal{M}_\Phi(X)$:

(1) the sequence $(1/t)\sigma^t$ converges μ-almost everywhere to a Φ invariant map, which agrees with ρ;

(2) we may compute the integral of ρ by

$$\rho_\mu := \int \rho \, d\mu = \inf_{t>0} \frac{1}{t} \int \sigma^t \, d\mu. \tag{2.1}$$

The discussion above then implies the following result.

THEOREM 2.9. *The map $\mathcal{M}_\Phi(X) \to \mathbb{R}$ given by $\mu \mapsto \rho_\mu$ is continuous.*

Proof. Note that by the compactness of X and continuity of $\sigma^t : X \to \mathbb{R}$, the map $\mu \to \int \sigma^t \, d\mu$ is continuous and, hence, by

$$\int \rho \, d\mu = \inf_{t>0} \frac{1}{t} \int \sigma^t \, d\mu$$

and the analogous equation for τ, we obtain upper and lower semicontinuity of $\mu \mapsto \rho_\mu$. □

Remark 2.10. When μ is supported on a periodic orbit $\mathcal{O}$, we often write $\rho_{\mathcal{O}}$ for ρ_μ.

Next, we consider perturbations of cocycles over a fixed base flow. The space of cocycles C_Φ over the same Φ has a C^0-topology of uniform convergence defined by the property that $\mathcal{A}_n \to \mathcal{A}$ if, for each $x \in X$ and $|t| < 1$, the maps $(A_n)^t_x \to A^t_x$ in $C^0(S^1, S^1)$ uniformly.

Associated with the cocycles $\mathcal{A}$ are rotation numbers $\rho_\mu(\mathcal{A})$ for invariant measures μ defined by (2.1). Then we have the following.

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PROPOSITION 2.11. *For a $\mu \in \mathcal{M}_\Phi(X)$, the map $C_\Phi \to \mathbb{R}$ given by*

$$\mathcal{A} \mapsto \rho_\mu(\mathcal{A})$$

is continuous.

Proof. The proof is nearly identical to that of Theorem 2.9. Namely, one uses continuity of $\mathcal{A} \mapsto \int \sigma^t(\mathcal{A})\, d\mu$ and the subadditive ergodic theorems. □

Now we specialize to the case where $X = O$ is a hyperbolic periodic orbit of a C^1 flow Φ_0 on a Riemannian manifold N, which will be $N = SM$ with the Sasaki metric in the setting of this paper. We are interested in how ρ_O varies as the flow Φ varies, for the derivative cocycle on certain circle bundles.

By structural stability of the hyperbolic set O, there exists $\mathcal{U}$ a C^1-neighborhood of Φ_0 and a continuous $h : \mathcal{U} \times O \to N$ such that the maps $h_\Phi(x) := h(\Phi, x)$ are C^1-diffeomorphisms onto their images, and $O_\Phi := h_\Phi(O)$ is a closed orbit of Φ. Moreover, because the maps h_Φ are C^1 there exists a continuous $\kappa : \mathcal{U} \times O \times \mathbb{R} \to \mathbb{R}$ such that $\kappa_\Phi(x, t) := \kappa(\Phi, x, t)$ is C^1 and the flow $\tilde{\Phi}$ (defined on O_Φ) given by

$$\tilde{\Phi}^t(x) = \Phi^{\kappa_\Phi(h_\Phi(x),t)}(x),$$

is, in fact, conjugated to Φ_0 by h_Φ, that is, $h_\Phi \circ \Phi_0 = \tilde{\Phi} \circ h_\Phi$.

For any bundle E, we write $\mathcal{P}E$ for its projectivization. Let F_0 be a two-dimensional trivial subbundle of $TN|_O$ which is part of a dominated splitting $E_0 \oplus_\le F_0 \oplus_\le G_0$ of $TN|_O$. The derivative cocycle $D\Phi$ on $\mathcal{P}F_0$ is then a cocycle on a trivial S^1 bundle, and it has a rotation number ρ_O as before.

Assuming $\mathcal{U}$ is taken sufficiently small, by persistence of dominated splittings for each $\Phi \in \mathcal{U}$, there is a splitting $TN|_{O_\Phi} = E_\Phi \oplus_\le F_\Phi \oplus_\le G_\Phi$ for Φ and the bundle F_Φ is trivial. Moreover, the splitting is also dominated for the flow $\tilde{\Phi}$, which is simply a time change of Φ. Hence, $D\tilde{\Phi}$ and $D\Phi$ on $\mathcal{P}F_\Phi$ also have well-defined rotation numbers $\rho_{O_{\tilde{\Phi}}}$, ρ_{O_Φ}, which satisfy the relation:

$$\rho_{O_{\tilde{\Phi}}}\ell(O_\Phi) = \rho_{O_\Phi}\ell(O),$$

as they differ by a time change, where ℓ denotes the period of a periodic orbit.

With all the objects defined, we now state the continuity with respect to the parameters as follows.

PROPOSITION 2.12. *The map $\mathcal{U} \to \mathbb{R}$ given by*

$$\Phi \mapsto \rho_{O_\Phi},$$

is continuous in some open $\mathcal{V} \subseteq \mathcal{U}$ containing Φ_0.

Proof. First, we would like to consider all cocycles $D\tilde{\Phi}$ constructed on F_Φ as existing on the same bundle over the same base map.

For $x \in O$, there exists a unique length-minimizing geodesic segment (from the Riemannian structure on N) from x to $h_\Phi(x)$, as long as h_Φ is close to the identity, which may be ensured by reducing the neighborhood $\mathcal{U}$ to some $\mathcal{V} \subseteq \mathcal{U}$ further if needed. By

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parallel transport of the bundle F_Φ over $\mathcal{O}_\Phi$ along such segments, one then obtains a two-dimensional trivial bundle F'_Φ over $\mathcal{O}$. By reducing $\mathcal{V}$ further if needed, the bundle F'_Φ obtained is a given by a graph over F_0 with respect to the fixed Riemannian metric on N and, hence, by orthogonal projection they may be identified.

As all maps above are continuous, the construction describes a continuous map $Th : \mathcal{U} \times F_0 \to TN$, so that $Th_\Phi(\cdot) := Th(\Phi, \cdot)$ are bundle isomorphisms $F_0 \to F_\Phi$ fibering over h_Φ. Hence, conjugating by Th_Φ, we may regard $D\tilde{\Phi}$ on F_Φ as a cocycle on F_0 over Φ_0. By continuous dependence on Φ, this defines a continuous map $\Phi \to D\tilde{\Phi}$, where $D\tilde{\Phi}$ are now regarded as elements of the space of cocycles over Φ_0 on F_0 with the C^0-topology.

As all rotation numbers $\rho_{\mathcal{O}_{\tilde{\Phi}}}$ defined previously are preserved by conjugation, it suffices to check continuity of the rotation numbers of the conjugated cocycles, which is given by Proposition 2.11. Thus, the map $\Phi \mapsto \rho_{\mathcal{O}_{\tilde{\Phi}}}$ is continuous and, finally, because

$$\rho_{\mathcal{O}_{\tilde{\Phi}}}\ell(\mathcal{O}_\Phi) = \rho_{\mathcal{O}_\Phi}\ell(\mathcal{O}),$$

and the periods vary continuously, the map $\Phi \mapsto \rho_{\mathcal{O}_\Phi}$ is continuous as well. □

2.4. *Geodesic flows.* Let M be a smooth closed manifold. As we consider varying Riemannian metrics, it is useful to work on the sphere bundle over M of oriented directions of the tangent space, which we denote by SM, rather than on the unit tangent bundle. When a metric g is fixed, $T^1_g M$ is canonically diffeomorphic to SM, and one can pullback the Sasaki metric from $T^1_g M$ to SM.

Recall that for $3 \le k \le \infty$ we denote by $\mathcal{G}^k$ the set of C^k-Riemannian metrics on M with sectional curvatures $1 \le -K < 4$. The geodesic flow on the unit tangent bundle of a negatively curved Riemannian manifold is an Anosov flow with the horospherical foliations corresponding to the stable and unstable foliations; moreover, under the pinching condition above it is a $\frac{1}{2}$-bunched (see §2.2) Anosov flow [**Kl82**, Theorem 3.2.17]. In particular, the bundles $E^{cu,cs}$ are C^1 and, because the flow is contact and the kernel of the C^{k-1} contact form equals $E^u \oplus E^s$, in fact $E^u \oplus E^0 \oplus E^s$ is a (at least) C^1 Anosov splitting.

We describe now the perturbational results of [**KT72**] that will be used to perturb the derivative cocycle by perturbing the metric. For a fixed embedded compact interval or closed loop $\gamma \subseteq SM$, the set of metrics for which γ is an orbit segment of the geodesic flow is denoted by $\mathcal{G}^k_\gamma \subseteq \mathcal{G}^k$. For a fixed $g_0 \in \mathcal{G}^k_\gamma$, pick local hypersurfaces Σ_0 and Σ_1 in SM that are transverse to $\dot{\gamma}(t) \in TSM$ at $t = 0$ and $t = 1$, respectively. This allows us to define a Poincaré map

$$P_{g_0} : \Sigma_0 \supseteq U \to \Sigma_1,$$

where U is a neighborhood of $\gamma(0)$, by mapping $\xi \in U$ to $\varphi^{t_1}_{g_0}(\xi)$, where t_1 is the smallest positive time such that $\varphi^{t_1}_{g_0}(\xi) \in \Sigma_1$ and φ_{g_0} is the geodesic flow of the metric g_0. By the implicit function theorem and the fact that $\varphi^t_{g_0}$ is C^{k-1}, the map P is C^{k-1}.

By projecting the tangent spaces of $\Sigma_{i=0,1}$ to $E^u \oplus E^s$ one may give $\Sigma_{i=0,1}$ a symplectic structure ω that is preserved by the Poincaré map, because the symplectic form is invariant by the geodesic flow [**KT72**]. With g_0 fixed, we let $\mathcal{G}^k_{g_0,\gamma} \subseteq \mathcal{G}^k_\gamma$ be the set of metrics such that $\pi(\gamma(0)), \pi(\gamma(1)) \notin \operatorname{supp}(g - g_0)$ ($\pi : SM \to M$ is the canonical projection map), that is, metrics unperturbed at the ends of the fixed geodesic segment γ relative to g_0.

We repeatedly use the main result on generic metrics established by Klingenberg and Takens in [**KT72**] to perturb the metric g_0.

THEOREM 2.13. [**KT72**, Theorem 2] *Suppose $g_0 \in \mathcal{G}^\infty_\gamma$, and let Q be some open dense subset of the space of $(k-1)$-jets of symplectic maps $(\Sigma_0, \gamma(0)) \to (\Sigma_1, \gamma(1))$.*

Then there is arbitrarily C^k-close to g_0 a $g' \in \mathcal{G}^k_{g_0,\gamma}$ such that $P_{g'} \in Q$, where $P_{g'} : (\Sigma_0, \gamma(0)) \to (\Sigma_1, \gamma(1))$ is the Poincaré map for the geodesic flow of g'.

Remark 2.14. The technical assumption that g_0 is C^∞ needed in [**KT72**] is virtually harmless, because by smooth approximation $\mathcal{G}^\infty \subseteq \mathcal{G}^k$ is dense for all k.

We need two additional facts about how these perturbations can be made, both of which follow directly from the proof of Theorem 2.13 in [**KT72**].

PROPOSITION 2.15. *Let $h := g' - g_0$, where g' and g_0 are given as in the statement of Theorem 2.13. For any tubular neighborhood V of γ, h can be taken to satisfy:*

(1) $\operatorname{supp}(h) \subseteq V$;

(2) *for a system of coordinates $\{x_0, \ldots, x_{2n-2}\}$ on V where ∂_{x_0} is parallel to the geodesic flow, the k-jets of h_{00} (where $h = h_{ij}\, dx_i\, dx_j$) vanish identically along $\{x_0 = 0\}$.*

In particular, this implies that the parametrization of γ by arc-length in g_0 is the same as that in g', that is, the geodesic flow for both metrics agree along γ.

Let J^{k-1}_s denote the Lie group of $(k-1)$-jets of C^{k-1} symplectic maps $(\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0)$ with the standard symplectic form $\sum_i dx^i \wedge dy^i$. If $\mathcal{O}$ is a closed orbit, we may take $v := \gamma(0) = \gamma(1) \in \mathcal{O}$ and fix $\Sigma := \Sigma_0 = \Sigma_1$, so by Darboux's theorem we may choose coordinates that identify the space of $(k-1)$-jets of C^{k-1} symplectic maps $(\Sigma, v) \to (\Sigma, v)$ with J^{k-1}_s.

COROLLARY 2.16. *If $\mathcal{O}$ is a closed geodesic for $g_0 \in \mathcal{G}^\infty_{\mathcal{O}}$ and $Q \subseteq J^{k-1}_s$ is an open dense invariant (Q satisfies $\sigma Q \sigma^{-1} = Q$ for any $\sigma \in J^{k-1}_s$) set, then there is arbitrarily C^k-close to g_0 a $g' \in \mathcal{G}^k_{\mathcal{O}}$ such that for any $v \in \mathcal{O}$ and any Σ a transverse at v, $P_{g'} \in Q$, where $P_{g'} = P(v, \Sigma)$ is the Poincaré return map for the geodesic flow of g'.*

Proof. The choice of a different section Σ or a different point v of the orbit changes $P_{g'}$ by conjugation, so the property that $P_{g'} \in Q$ needs only to be assured at one fixed point and one fixed section, which is done by Theorem 2.13. □

Remark 2.17. As the map $\pi^{k-1} : J^{k-1}_s \to J^1_s \cong \operatorname{Sp}(2n)$ is a submersion, for Q an open dense invariant subset of $\operatorname{Sp}(2n)$, $(\pi^{k-1})^{-1}(Q)$ is an open dense invariant subset of J^{k-1}_s, so in the statement of Corollary 2.16 we may take an open dense invariant $Q \subseteq \operatorname{Sp}(2n)$ instead, while the approximation is still in $\mathcal{G}^k$.

In the context of Theorem 2.13, the analogous observation holds; that is, one may take Q to be an open dense subset of 1-jets of symplectic maps $(\Sigma_0, \gamma(0)) \to (\Sigma_1, \gamma(1))$, and approximate in $\mathcal{G}^k$.

3. *Pinching and twisting for flows*

In this section, we present the main technical results of the paper, namely, the construction of perturbations of Anosov flows leading to an appropriate pinching and twisting condition. For the sake of simplicity, we specialize to the class of geodesic flows, but the main arguments here adapt to the proofs of the other theorems with adjustments which we describe in the last section. We define pinching and twisting for orbits of the geodesic flow in analogy with Definition 2.2, and use the results on generic metrics to show that these are C^1-open and C^k-dense.

We fix the following useful notation. For a metric g such that $\mathcal{O} \subseteq SM$ is a periodic orbit of its geodesic flow with period ℓ, let $v \in \mathcal{O}$ and let $\{\lambda_1, \ldots, \lambda_{2n}\}$ be the generalized eigenvalues of $D_v\varphi_g^\ell|_{E^u\oplus E^s}$, which do not depend on the choice of v, sorted so that $|\lambda_i| \geq |\lambda_j|$ whenever $i < j$. We write

$$\vec{\lambda}^u(\mathcal{O}, g) := (\lambda_1, \ldots, \lambda_n), \quad \vec{\lambda}^s(\mathcal{O}, g) := (\lambda_{n+1}, \ldots, \lambda_{2n}) \in \mathbb{C}^n,$$

$$\vec{\lambda}(\mathcal{O}, g) := (\lambda_1, \ldots, \lambda_{2n}) \in \mathbb{C}^{2n}.$$

The ith coordinates of the vectors above are written as $\vec{\lambda}_i^{u,s,\cdot}(\mathcal{O}, g)$ (where $\cdot$ means no superscript).

The following continuity lemma about these $\vec{\lambda}$ is the bread and butter of all 'openness' arguments which follow.

LEMMA 3.1. *Fix $k \geq 2$. For a metric $g_0 \in \mathcal{G}^k$ there exists a neighborhood $\mathcal{U} \subseteq \mathcal{G}^k$ of g_0 such that for any $g \in \mathcal{U}$ any orbit $\mathcal{O}$ of the geodesic flow of g_0 has a hyperbolic continuation $\mathcal{O}_g$ for the geodesic flow of g, and the maps $\mathcal{U} \to \mathbb{C}^n$ given by*

$$g \mapsto \vec{\lambda}^{u,s}(\mathcal{O}_g, g)$$

are continuous with respect to the C^2-topology.

Proof. Let Σ be a smooth hypersurface parallel to $E^u \oplus E^s$ at v so that $\mathcal{O} \cap \Sigma =: \{v\}$. The return map for the geodesic flow φ_{g_0} then defines a map $P_{g_0} : U \to \Sigma$, where $U \subseteq \Sigma$ is some neighborhood of v, for which v is a hyperbolic fixed point.

For any g sufficiently close to g_0, we also obtain a map $P_g : U \to \Sigma$ given by the return map of φ_g, and by the standard hyperbolic theory, a fixed point v_g such that $g \mapsto v_g$ is continuous. The geodesic flow φ_g varies in a C^{k-1} fashion as g varies in $\mathcal{G}^k$, and by the implicit function theorem, so does P_g. Then, by fixing a coordinate system, because $k \geq 3$ the matrices $D_{v_g}P_g$ vary continuously, so their eigenvalues vary continuously as g varies in $\mathcal{G}^k$.

Finally, the eigenvalues of the matrices $D_{v_g}\varphi_g^{\ell_g}|_{E^u\oplus E^s}$ and $D_{v_g}P_g$ agree, so we obtain the desired result. □

3.1. *Pinching.*

Before moving to the definition of pinching, first we verify that generically there exists a periodic orbit with a dominated splitting of $E^u \oplus E^s$ into one-dimensional subspaces and two-dimensional subspaces corresponding to conjugate pairs of eigenvalues.

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PROPOSITION 3.2. *Let*

$$\mathcal{G}_d^k := \{g \in \mathcal{G}^k : \textit{there exists } \mathcal{O} : |\lambda_i| \neq |\lambda_j|, \textit{ unless } \lambda_i = \bar{\lambda}_j, \\ \textit{where } (\lambda_1, \ldots, \lambda_n) := \vec{\lambda}^u(\mathcal{O}, g)\}.$$

The set $\mathcal{G}_d^k$ is C^2-open and C^k-dense in $\mathcal{G}^k$.

Proof. Openness follows directly from Lemma 3.1, because by the continuity of $\vec{\lambda}^u$, the continuations of $\mathcal{O}$ will satisfy the same condition defining $\mathcal{G}_d^k$.

For density, we start by assuming that $g_0 \in \mathcal{G}_{\mathcal{O}}^\infty$, for some $\mathcal{O}$, which is possible by the density of $\mathcal{G}^\infty$ in $\mathcal{G}^k$. It remains to check that the property defining $\mathcal{G}_d^k$ is indeed an open dense in J_s^{k-1}, so that we may apply Corollary 2.16 to finish the proof. By Remark 2.17, it suffices to check that having eigenvalues distinct with distinct norms, apart from complex conjugate pairs, is an open and dense $\mathrm{Sp}(2n)$.

Openness is clear, because the eigenvalues depend continuously on the matrix entries. For density, we note that the condition of distinct eigenvalues is given by the complement of the equation $\Delta = 0$, where Δ is the discriminant of the characteristic polynomial, which is a non-empty Zariski open set in $\mathrm{Sp}(2n)$ and, thus, dense in the analytic topology. In particular, the set of diagonalizable matrices is dense. As diagonalizable matrices are symplectically diagonalizable, by the lemma following this proof, by a small perturbation on the norm of the diagonal blocks, we obtain the density of eigenvalues of distinct norms. □

We prove the linear algebra lemma used above, which is also useful in what follows.

LEMMA 3.3. *A matrix $A \in Sp(2n)$ with all eigenvalues distinct is symplectically diagonalizable in the sense that there exists $P \in Sp(2n)$ such that $P^{-1}AP$ is in real Jordan form (that is, given by diagonal blocks which are either trivial or 2×2 conformal).*

Proof. Recall that eigenvalues of $A \in \mathrm{Sp}(2n)$ appear in 4-tuples

$$\{\lambda, \bar{\lambda}, \lambda^{-1}, \bar{\lambda}^{-1}\}$$

for $\lambda \notin \mathbb{R}$ and in pairs $\{\lambda, \lambda^{-1}\}$ for $\lambda \in \mathbb{R}$. For each λ, we let $E_\lambda = E_{\lambda^{-1}}$ be the two-dimensional subspace spanned by the eigenspaces of λ and λ^{-1}.

Let ω be the canonical symplectic form on $\mathbb{R}^{2n}$ and extend ω and A to $\omega_{\mathbb{C}}$ and $A_{\mathbb{C}}$ in the complexification $\mathbb{C}^{2n} = \mathbb{R}^{2n} \otimes \mathbb{C}$. By definition, $A_{\mathbb{C}}$ and $\omega_{\mathbb{C}}$ agree with A and ω on $\mathbb{R}^{2n} \otimes 1$.

The identity for eigenvectors v_λ and v_η:

$$\omega_{\mathbb{C}}(v_\lambda, v_\eta) = \omega_{\mathbb{C}}(A_{\mathbb{C}} v_\lambda, A_{\mathbb{C}} v_\eta) = \lambda\eta\, \omega_{\mathbb{C}}(v_\lambda, v_\eta),$$

implies that, unless $\lambda\eta = 1$, we have $\omega_{\mathbb{C}}(v_\lambda, v_\eta) = 0$. Therefore, $E_\lambda \otimes \mathbb{C}$ is symplectically orthogonal to $E_\eta \otimes \mathbb{C}$ for any $\lambda \neq \eta, \eta^{-1}$.

In particular, this implies that the $E_\lambda \otimes 1$ are symplectic subspaces with respect to ω the real form, and symplectically orthogonal to each other. In each E_λ, A can be put in Jordan real form with respect to a symplectic basis. By orthogonality, we may construct a symplectic basis for $\mathbb{R}^{2n}$ by taking the union of symplectic bases for the E_λ. Then, let P be the matrix which sends the standard $\mathbb{R}^{2n}$ basis to the constructed symplectic basis. $\square$

The next step is to construct a metric with a periodic orbit with simple real spectrum with an arbitrarily small perturbation of the metric. Following [**BV04**], this is accomplished by slightly perturbing a periodic orbit $\mathcal{O}$ rotating a complex eigenspace, and propagating the perturbation to a periodic orbit which shadows a homoclinic orbit of $\mathcal{O}$ that spends a long time near $\mathcal{O}$.

Recall the following definitions: an *ε-pseudo-orbit* for a flow Φ on a space X is a (possibly discontinuous) function $\gamma : \mathbb{R} \to X$ such that

$$d(\gamma(t+\tau), \Phi^\tau(\gamma(t))) < \varepsilon \quad \text{for } t \in \mathbb{R} \quad \text{and} \quad |\tau| < 1.$$

For γ a ε-pseudo-orbit, γ is said to be *δ-shadowed* if there exists a point $p \in X$ and a homeomorphism $\alpha : \mathbb{R} \to \mathbb{R}$ such that $\alpha(t) - t$ has Lipschitz constant δ and $d(\gamma(t), \Phi^{\alpha(t)}(p)) \leq \delta$ for all $t \in \mathbb{R}$.

The classic closing lemma for Anosov flows we need is as follows.

THEOREM 3.4. ([**FH18**] Anosov closing lemma) *If Λ is a hyperbolic set for a flow Φ, then there are a neighborhood U of Λ and numbers $\varepsilon_0, L > 0$ such that for $\varepsilon \leq \varepsilon_0$ any compact ε-pseudo-orbit in U is $L\varepsilon$-shadowed by a unique compact orbit for Φ.*

We use it to prove the main result of this section.

PROPOSITION 3.5. *Let*

$$\mathcal{G}_p^k := \{g \in \mathcal{G}^k : \textit{there exists } \mathcal{O} : \lambda_i \neq \lambda_j, \lambda_i \in \mathbb{R}, \ \textit{where } (\lambda_1, \ldots, \lambda_n) := \vec{\lambda}^u(\mathcal{O}, g)\}.$$

In this situation, we say $\mathcal{O}$ has the pinching property for g. Then $\mathcal{G}_p^k$ is C^2-open and C^k dense in $\mathcal{G}^k$.

Proof. Fix a C^2-open set $\mathcal{U} \subseteq \mathcal{G}^k$. First, because $\mathcal{G}_d^k$ is C^2-open and dense and $\mathcal{G}^\infty$ is C^k-dense in $\mathcal{G}^k$, we may fix some $g_0 \in \mathcal{U} \cap \mathcal{G}_d^\infty$. Let $\mathcal{O}$ be as in Proposition 3.2.

Suppose that the vector $\vec{\lambda}^u(\mathcal{O}, g)$ has $2c$ entries in $\mathbb{C} \setminus \mathbb{R}$, for some $c > 0$. It suffices to show that there exists a metric g' in $\mathcal{U}$ that has a periodic orbit $\mathcal{O}'$ such that $\vec{\lambda}^u(\mathcal{O}', g')$ has $2(c-1)$ complex entries and all real entries distinct.

Along $\mathcal{O}$ there is a dominated splitting $E^u = E_1^- \oplus \cdots \oplus E_k^-$ such that each E_i is either one- or two-dimensional. Fix the smallest index $i \in \{1, \ldots, k\}$ such that $E_i^\pm$ is two-dimensional and let P_{g_0} denote the Poincaré return map of the geodesic flow for a fixed section Σ transverse to the flow small enough so that $\mathcal{O} \cap \Sigma =: \{v\}$. By shrinking $\mathcal{U}$ further if needed we may assume that $\mathcal{U} \subseteq \mathcal{G}_d^k$, that is, that the dominated splitting for φ_{g_0} along $\mathcal{O}$ persists for the continuation of $\mathcal{O}$ for all $g \in U$; thus, by Lemma 3.1, the map $g \mapsto \theta_g := |\arg(\lambda_g)|$ is well defined and continuous, where λ_g is an eigenvalue of $D\varphi_g$ on E_i^- on the continuation of $\mathcal{O}$.

LEMMA 3.6. *There exists $g_1 \in \mathcal{U} \cap \mathcal{G}_{\mathcal{O}}^k$ such that $\theta_{g_1} \neq \theta_{g_0}$.*

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Proof. The derivative of the Poincaré map is conjugate to $D\varphi_g|_{E^u \oplus E^s}$ over the closed orbit $\mathcal{O}$, so θ_{g_0} agrees with the argument of the eigenvalue of DP_{g_0} along the two-dimensional Jordan block $F^- \subseteq T_v\Sigma$ mapped to E_i^- under the conjugation aforementioned. Moreover, let F^+ be the Jordan block corresponding to E_i^+ in the same manner.

Identifying the space of symplectic maps $T_v\Sigma \to T_v\Sigma$ with $\mathrm{Sp}(2n)$, there exists some neighborhood $\mathcal{V} \subseteq \mathrm{Sp}(2n)$ of the original map DP_{g_0}, such that for $A \in \mathcal{V}$ the Jordan block F has a continuation for A, and we call the norm of the argument of the eigenvalue of A along this continuation θ_A. Let $\mathcal{W} \subseteq \mathcal{V}$ be the set of matrices A such that $\theta_A \neq \theta_{g_0}$. If $\mathcal{W}$ is open and dense in $\mathcal{V}$, then by Remark 2.17 we may apply Corollary 2.16 to $\mathcal{W} \cup ((\mathrm{Sp}(2n) \setminus \mathrm{Cl}(\mathcal{V}))$, which will be open and dense in $\mathrm{Sp}(2n)$ to find that the set of metrics that has $\theta_{g_1} \neq \theta_{g_0}$ is dense (and open) in $\mathcal{U}$.

It remains to check that $\mathcal{W}$ is open and dense in $\mathcal{V}$. Openness is clear by continuous dependence of eigenvalues on matrix entries. For density, let R_θ be given by rotation of any angle of $\theta > 0$ on the subspaces F^-, F^+ and the identity on the other subspaces, satisfies $R_\theta \Omega R_\theta^T = \Omega$, where Ω is the standard symplectic form. Then $R_\theta DP_{g_0}$ has $\theta_{R_\theta DP_{g_0}} \neq \theta_{g_0}$; because $\theta > 0$ can be made arbitrarily small, this finishes the proof. □

Let g_1 be given as in the lemma and, for $0 \leq s \leq 1$, we let $g_s = sg_1 + (1-s)g_0$, which, if g_1 is taken sufficiently close to g_0, also satisfies $\{g_s\} \subseteq \mathcal{U} \cap \mathcal{G}^k_{\mathcal{O}}$. Clearly, the map $[0, 1] \to \mathcal{G}^k$ given by $s \mapsto g_s$ is continuous. In addition, note that, by Proposition 2.15(2), $\mathcal{O}$ is not only a closed orbit of φ_{g_s} for all $s \in [0, 1]$, but it, in fact, has the same arc-length parametrization with respect to all g_s.

For the geodesic flow of g_0, fix w a transverse homoclinic point of v, that is, $w \in W^u(v) \cap W^{cs}(v)$. Fix some $\varepsilon > 0$ so that the geodesic flow has local product structure at scale 2ε. Then there exists $t_1, t_2 > 0$ such that $\varphi_{g_0}^{-t_2}(w) \in W^u_\varepsilon(v)$, $\varphi_{g_0}^{t_1}(w) \in W^s_\varepsilon(v)$ and also a $C > 0$ such that, for all $t > 0$,

$$d(\varphi_{g_0}^{-(t_2+t)}(w), \varphi_{g_0}^{-t}(v)) < C\varepsilon e^{-t},$$

$$d(\varphi_{g_0}^{t_1+t}(w), \varphi_{g_0}^{t}(v)) < C\varepsilon e^{-t}.$$

Hence, for $n \in \mathbb{N}$ the $\gamma_n : \mathbb{R} \to SM$ given by

$$\gamma_n(t) = \varphi_{g_0}^{\tilde{t}-(t_2+n\ell)}(w) \quad \text{where } \tilde{t} = t \bmod (t_2 + t_1 + 2n\ell)$$

are ε_n-pseudo-orbits where $\varepsilon_n < 2C\varepsilon e^{-n\ell}$, by the fact that the minimal expansion of the geodesic flow is $\tau = 1$ by the assumption on curvature.

For n sufficiently large, there exist unique periodic w_n which $L\varepsilon_n$-shadow γ_n. Let $w_{n,s}$ be continuations of w_n for $0 \leq s \leq 1$ (where $w_{n,0} = w_n$, by definition). Let w_s be the hyperbolic continuations of w. By uniqueness of shadowing, note that the $w_{n,s}$ can also be constructed by shadowing segments of the orbit of w_s. The following lemma shows we can extend the dominated splitting of $\mathcal{O}$ to the new orbits we defined.

LEMMA 3.7. *There exists N large so that for each $0 < s < 1$, the compact invariant set*

$$K_{N,s} = \bigcup_{n \geq N} \mathcal{O}(w_{n,s}) \cup \mathcal{O}(w_s) \cup \mathcal{O},$$

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for the geodesic flow φ_{g_s} of g_s admits a dominated splitting for the bundle $E^u = E^-_{s,1} \oplus \cdots \oplus E^-_{s,k}$ over $K_{m,s}$ coinciding with the dominated splitting of E^u over $\mathcal{O}$, and similarly for $E^s = E^+_{s,1} \oplus \cdots \oplus E^+_{s,k}$.

Proof sketch (see [**BV04**]*, Lemma 9.2).* We sketch the proof for $s = 0$, which is almost identical to the result cited. Then, because dominated splittings over compact invariant sets persists under C^1-small perturbations by an invariant cone argument, this shows the result for all $s \in [0, 1]$. Consider the case of E^u.

As $w \in W^u(v) \cap W^{cs}(v)$, one can extend the dominated splitting of $\mathcal{O}$ to $\mathcal{O}(w)$ as follows. Consider the bundles over $\mathcal{O}$ given by $F^i = E^-_1 \oplus \cdots \oplus E^-_{j+1}$, and $G^i = E^-_j \oplus \cdots \oplus E^-_k$ for $i, j = 1, \ldots, k-1$. Then we define

$$E^j(w) := \phi^{cs}_{v,w} F^j(v) \cap \phi^{cu}_{v,w} G^j(v)$$

and extend the E^j bundles to $\mathcal{O}(w)$ by the derivative of the flow. Proof of continuity and domination of this splitting follows closely that in [**BV04**].

For N sufficiently large we observe that $K_{N,0}$ is contained in an arbitrarily small neighborhood of $\mathcal{O} \cup \mathcal{O}(w)$, so the dominated splitting extends by continuity. □

For each n, we let $\theta_n : [0, 1] \to S^1 = \mathbb{R}/2\pi\mathbb{Z}$ be defined by setting $\theta_n(s)$ to be the argument of the eigenvalue of $D\varphi_{g_s}$ along $E^-_{s,i}$ on the closed orbit $w_{n,s}$. By Lemma 3.1, the θ_n are continuous so for each n they may be lifted to some $\tilde{\theta}_n : [0, 1] \to \mathbb{R}$.

The main result about these rotation numbers, whose proof is postponed to the next section due to its length, is:

LEMMA 3.8. *There exists $n \in \mathbb{N}$ so that $|\tilde{\theta}_n(1) - \tilde{\theta}_n(0)| > 2\pi$.*

By continuity, one then finds n, s such that $\tilde{\theta}_n(s)$ is an integer multiple of 2π, that is, such that the eigenvalues in $\vec{\lambda}^u(\mathcal{O}(w_{n,s}), g_s)$ corresponding to the subspace $E^-_{i,s}$ are real. By another perturbation using Corollary 2.16, there exists a metric such that these eigenvalues become distinct. Then, by induction on the other eigenspaces with complex eigenvalues, all eigenvalues are real and distinct.

To finish the proof, openness follows again by Lemma 3.1, because the requirements on the products of the eigenvalues is an open condition. □

3.2. *Proof of Lemma 3.8.* We apply the notions introduced in §2.3 to give a proof of Lemma 3.8.

Proof of Lemma 3.8. Fix N large enough so that $K_{N,s}$ satisfies the conclusion of Lemma 3.7. We begin with the following result.

PROPOSITION 3.9. *There exists $N' > N$, which we denote by N after this proposition, such that the bundles with total spaces E_s defined by the fibers $E_s(x) := E^-_{s,i}(x)$ over $x \in K_{N',s}$ are continuously trivializable, for each $s \in [0, 1]$ and where i is some index as set in the proof of Proposition 3.5.*

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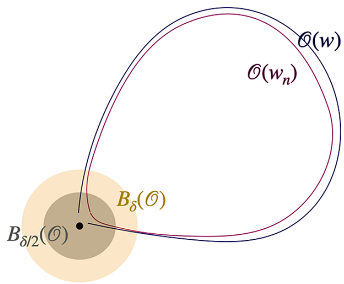

FIGURE 1. Proof of Proposition 3.9. The closed orbit $\mathcal{O}$ is schematically represented by the black dot.

Proof. First, note that it suffices to prove that E_s is trivializable for $s = 0$, because the bundles E_s vary continuously in the ambient space TSM as s varies.

We construct a non-vanishing section of the frame bundle F associated with E_0 over some $K_{N',0}$ for N' large, which is equivalent to a continuous choice of basis for E_0, proving triviality of the bundle.

For $\delta > 0$, let $B_\delta(\mathcal{O})$ a δ-tubular neighborhood of $\mathcal{O}$. If δ is sufficiently small relative to the scale of local product structure of the Anosov flow, for all $n \geq N'$, and N' sufficiently large, $\mathcal{O}(w_n)_\delta := B_\delta(\mathcal{O}) \cap \mathcal{O}(w_n)$ consists of a connected segment of the embedded circle $\mathcal{O}(w_n)$ and, moreover, $\mathcal{O}(w)_\delta := B_\delta(\mathcal{O}) \cap \mathcal{O}(w)$ consists of the complement of a connected closed interval in $\mathcal{O}(w)$, that is, two immersed connected components (see Figure 1).

Note that we may assume that the map $D\varphi_{g_0}^{\ell(w_n)}$ preserves orientation on E_0 over any periodic orbit $\mathcal{O}(w_n)$, because otherwise it would have real eigenvalues (any $A \in \mathrm{GL}(2, \mathbb{R})$ with negative determinant has real eigenvalues) and we would obtain a proof of Lemma 3.8. Hence, the bundle E_0 is trivializable over any $\mathcal{O}(w_n)$. It is also clearly so over $\mathcal{O}(w)$, because it is an immersed real line, and we may assume this is also true for $\mathcal{O}$, because otherwise, again, we would have real eigenvalues.

By shrinking δ further if necessary, there exists a well-defined closest point projection $p : B_\delta(\mathcal{O}) \to \mathcal{O}$ which is a surjective submersion. Fix a trivialization of E_0 over $\mathcal{O}$, that is, a non-vanishing section $S : \mathcal{O} \to F$, which is possible by the previous paragraph.

For $x \in B_\delta(\mathcal{O}) \cap K_{N',0} =: K_\delta$, again shrinking δ further if necessary, there exists a unique length-minimizing geodesic segment between x and $p(x)$, and by parallel transporting $E_0(p(x))$ along such segments and then projecting orthogonally onto $E_0(x)$ one obtains a continuous bundle map $E_0|_{K_\delta} \to E_0|_{\mathcal{O}}$ which is an isomorphism on fibers. This map induces a map $F|_{K_\delta} \to F_{\mathcal{O}}$ and so by pulling back the non-vanishing section $S : \mathcal{O} \to F$ we obtain a non-vanishing section, which we now denote by $S : K_\delta \to F$ because its restriction to $\mathcal{O}$ agrees with the previous S, of F over K_δ.

Recall that $\mathcal{O}(w)_\delta$ consists of two connected immersed components homeomorphic to $\mathbb{R}$. As $\mathcal{O}(w)$ is contractible, it is possible to define a determinant on $F|_{\mathcal{O}(w)}$; up to scalar it is unique and, hence, there is a well-defined continuous sign function on each fiber. Then we claim that $S|_{\mathcal{O}(w)_\delta}$ has the same determinant sign on both components, so that it may be extended to a continuous section $\mathcal{O}(w) \to F|_{\mathcal{O}(w)}$. Suppose this is not the case for a contradiction.

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Define the line bundle $L := \bigwedge^2 E^0$ over K, which restricted to individual orbits is trivial since E_0 is. At each point $x \in K$ there is a natural map $F(x) \to L(x)$ given by $(e_1, e_2) \mapsto e_1 \wedge e_2$, which extends to a continuous global map $W : F \to L$. Considering the image of $S|_{\mathcal{O}(w)_\delta}$ under W, we obtain a section $\mathcal{O}(w)_\delta \to L$, which has opposite signs in the two connected components. Let $B : \mathcal{O}(w) \to L$ be any extension of this section to all of $\mathcal{O}(w)$; by the previous remark, B must have an odd number of zeros.

By continuity,

$$\mathcal{O}(w_n) \setminus B_{\delta/2}(\mathcal{O}) \to \mathcal{O}(w) \setminus B_{\delta/2}(\mathcal{O})$$

as $n \to \infty$, so by continuity of the bundle for n sufficiently large, we can parallel transport the section B on $\mathcal{O}(w) \setminus B_{\delta/2}(\mathcal{O})$ to $\mathcal{O}(w_n) \setminus B_{\delta/2}(\mathcal{O})$ to obtain a section B_n on $\mathcal{O}(w_n) \setminus B_{\delta/2}(\mathcal{O})$ that has the same number of zeros as B on $\mathcal{O}(w) \setminus B_{\delta/2}(\mathcal{O})$, that is, oddly many.

On the other hand, as $n \to \infty$,

$$B_n|_{\mathcal{O}(w_n)_\delta \setminus B_{\delta/2}(\mathcal{O})} \to (W \circ S)|_{\mathcal{O}(w_n)_\delta \setminus B_{\delta/2}(\mathcal{O})}$$

and, hence, for n large enough, B_n has constant sign on $\mathcal{O}(w_n)_\delta \setminus B_{\delta/2}(\mathcal{O})$. Thus, B_n extends to $\mathcal{O}(w_n)_\delta$ without any zeros. Hence, we obtain a global section B_n on $\mathcal{O}(w_n)$ with an odd number of zeros, contradicting the triviality of L over $\mathcal{O}(w_n)$.

Hence, we may extend $S|_{\mathcal{O}(w)_\delta}$ continuously to all of $\mathcal{O}(w)$. Since in K_δ the section S is continuous, and again $\mathcal{O}(w_n) \setminus B_{\delta/2}(\mathcal{O}) \to \mathcal{O}(w) \setminus B_{\delta/2}(\mathcal{O})$ as $n \to \infty$, we can then continuously extend $S|_{\mathcal{O}(w) \setminus B_{\delta/2}(\mathcal{O})}$ to $\mathcal{O}(w_n) \setminus B_{\delta/2}(\mathcal{O})$ while agreeing with S in K_δ. As $S|_{\mathcal{O}(w)}$ is non-vanishing, the section obtained in this way is also globally non-vanishing. □

The projectivization $\mathcal{P}E_s$ of the bundle E_s then defines a trivial circle bundle over $K_{N,s}$, and we fix a trivializing bundle isomorphism $\phi_s : \mathcal{P}E_s \to K_{N,s} \times S^1$. By conjugating with ϕ_s, the derivative of the geodesic flow then defines a continuous cocycle $\mathcal{A}_s$ on $K_{N,s} \times S^1$ over the geodesic flow, so we may apply the results of §2.3 for $\mathcal{A}_s$.

Then the rotation numbers have the following characterization over periodic orbits.

LEMMA 3.10. *For a closed orbit $\mathcal{O}(u)$ of a point $u \in K_{N,s}$, the argument $\theta(u)$ of the eigenvalue of the return map of the geodesic flow on $E^-_{s,i}$ satisfies*

$$\theta(u) = \ell(u) \cdot \rho_{\mathcal{O}(u)} \pmod{2\pi},$$

where $\ell(u)$ is the period of u, and $\rho_{\mathcal{O}(u)}$ is as in Remark 2.10 for the cocycle $\mathcal{A}_s$ defined above.

Proof. On the one hand, it follows from the definition of ρ that $\ell(u) \cdot \rho_{\mathcal{O}(u)}$ agrees mod 2π with the Poincaré rotation number for the map

$$(A_s)_u^{\ell(u)} : S^1 \to S^1.$$

On the other hand, the projectivization of the derivative of the flow also defines on the fiber a homeomorphism $S^1 \to S^1$ with Poincaré rotation number equal to the argument of the eigenvalue of the derivative.

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As these two statements differ by a conjugation given by $\pi_2 \circ \phi_s(u, \cdot) : S^1 \to S^1$, where $\pi_2 : K_{N,s} \times S^1$ is the natural projection, by invariance we obtain the result. $\square$

Applying Lemma 3.10 to the $\theta_n(s)$, we obtain, for $0 \leq s \leq 1$,

$$\theta_n(s) = \ell(w_{n,s})\rho_{O(w_{n,s})} \pmod{2\pi}.$$

By continuity of the functions θ_n, we may lift them to $\tilde{\theta}_n : [0, 1] \to \mathbb{R}$ satisfying $\tilde{\theta}_n(0) = \ell(w_{n,0})\rho_{O(w_{n,0})}$. By the continuity of $\rho_{O(w_{n,s})}$ in s, given by Proposition 2.12, our choice of lift then implies

$$\tilde{\theta}_n(s) = \ell(w_{n,s})\rho_{O(w_{n,s})} \quad \text{for } 0 \leq s \leq 1.$$

Let $\theta(s)$ be the argument of the eigenvalue of the $D\varphi_{g_s}$ on $E^-_{i,s}$ on the periodic orbit O (recall O is a closed geodesic for all g_s with $\ell(O)$ fixed), and repeat the constructions above to obtain $\tilde{\theta}(s)$ as well satisfying

$$\tilde{\theta}(s) = \ell(O)\rho_O(s), \tag{3.1}$$

where $\rho_O(s)$ is ρ_O of the geodesic flow of g_s.

As $\mu_{O(w_{n,s})} \to \mu_O$ (where μ_O is the invariant probability measure supported on the closed orbit O) we have $\rho_{O(w_{n,s})} \to \rho_O(s)$ as $n \to \infty$ by Theorem 2.9. By hypothesis, $\theta(1) \neq \theta(0)$ and because $\ell(O)$ is constant as s varies, equation (3.1) gives that $\rho_O(1) - \rho_O(0) \neq 0$. Hence, for n large enough, there exists some $\delta > 0$ such that $|\rho_{O(w_{n,1})} - \rho_{O(w_{n,0})}| \geq \delta$.

Finally, let $\delta_n = |\ell(w_{n,1}) - \ell(w_{n,0})|$. Again, we defer the proof of the following final proposition we need.

PROPOSITION 3.11. *There exists $M_2 > 0$ such that $\delta_n < M_2$ for all $n \in \mathbb{N}$.*

With Lemma 3.11, we complete the proof of Lemma 3.8:

$$\begin{aligned}
|\tilde{\theta}_n(1) - \tilde{\theta}_n(0)| &= |\ell(w_{n,1})\tilde{\rho}_{O(w_{n,1})} - \ell(w_{n,0})\tilde{\rho}_{O(w_{n,0})}| \\
&\geq |\ell(w_{n,1})(\tilde{\rho}_{O(w_{n,1})} - \tilde{\rho}_{O(w_{n,0})})| \\
&\quad - |(\ell(w_{n,1}) - \ell(w_{n,0}))\tilde{\rho}_{O(w_{n,0})}| \\
&\geq \delta\ell(w_{n,1}) - \delta_n|\tilde{\rho}_{O(w_{n,0})}| \\
&> \delta\ell(w_{n,1}) - M_1M_2 > 2\pi
\end{aligned}$$

for all n sufficiently large, because $\ell(w_{n,1}) \to \infty$. $\square$

Finally, we prove Proposition 3.11.

Proof of Proposition 3.11. To bound the variations δ_n, we use exponential shadowing and Hölder continuity of the geodesic stretch, defined in the following. As the geodesic flow is unperturbed on O and the orbits $O(w_{n,s})$ approximate O, the two mentioned properties give us the bound on δ_n.

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Recall that the w_n are constructed by shadowing $\gamma_n : \mathbb{R} \to SM$ given by

$$\gamma_n(t) = \varphi_{g_0}^{\tilde{t}-(t_2+n\ell)}(w), \quad \text{where } \tilde{t} = t \bmod (t_2 + t_1 + 2n\ell),$$

which is a ε_n-pseudo-orbit, where t_1 (respectively, t_2) is such that $\varphi_{g_0}^{t_1}(w)$ (respectively, $\phi_{g_0}^{-t_2}(w)$) is in $W_\varepsilon^s(v)$ (respectively, $W_\varepsilon^u(v)$) and $\varepsilon_n < 2C\varepsilon e^{-n\ell}$.

The following well-known theorem is an adaptation for flows of the usual 'exponential' shadowing theorem, which uses the Bowen bracket in its proof. The statement gives a sharper estimate on how well shadowing orbits approximate pseudo-orbits.

THEOREM 3.12. [**FH18**, Theorem 6.2.4] *For a hyperbolic set Λ of a flow Φ on a closed manifold, there exists $c, \eta > 0$ such that for all $\varepsilon > 0$, there exists $\delta > 0$ so that if $x, y \in \Lambda$, $s : \mathbb{R} \to \mathbb{R}$ continuous, $s(0) = 0$ and $d(\Phi^t(x), \Phi^{s(t)}(y)) < \delta$ for all $|t| \le T$, then:*

(1) $|t - s(t)| < 3\varepsilon$ *for all* $|t| \le T$;

(2) *there exists $t(x, y)$ with $|t(x, y)| < \varepsilon$ so that the ε-stable manifold $\Phi^{t(x,y)}(x)$ intersects uniquely the ε-unstable manifold of y and*

$$d(\Phi^t(y), \Phi^t(\Phi^{t(x,y)}(x))) < ce^{\eta(T-|t|)} \quad \text{for } |t| < T.$$

In the context of the current proof, we apply the above theorem as follows.

Let $T_n = \ell(w_n)$, $x = \varphi_{g_0}^{T_n/2}(w_n)$, and $y = \varphi_{g_0}^{\tau_n}(w)$ where $\tau_n := T_n/2 - (t_2 + n\ell)$. For n sufficiently large, $d(\varphi_{g_0}^t(x), \varphi_{g_0}^{s(t)}(y)) < \delta$ is satisfied, by the statement of shadowing, for $|t| < T_n/2$ and δ given by the theorem for the $\varepsilon > 0$ fixed before. Then the theorem gives a $t_n \in \mathbb{R}$ such that

$$d(\varphi_{g_0}^{t_n+t}(w_n), \varphi_{g_0}^{\tau_n+t}(w)) < ce^{\eta(T_n/2-|t|)} \quad \text{for } |t| < T_n/2.$$

Now we turn to computing the period of $w_{n,1}$ using the facts established previously. By structural stability, there exists $h : SM \to SM$ which conjugates the orbits of φ_{g_0} to those of φ_{g_1}. This conjugacy can be taken to be Hölder continuous and C^1 along the flow direction. Thus, there exists some $a : SM \to \mathbb{R}$ that is Hölder continuous with some exponent $1 \ge \beta > 0$, such that for $u \in SM$:

$$dh(u)X_{g_0}(u) = a(u)X_g(h(u)),$$

where X_g (respectively, X_{g_0}) is the vector field generating the geodesic flow for g (respectively, g_0). The function a is referred to as the *geodesic stretch*, and the proof of the facts above can be found, for instance, in [**GKL19**, pp. 12–13]

The period of $w_{n,1}$ is given by the formula

$$\ell(w_{n,1}) = \int_0^{T_n} a(\varphi_{g_0}^t(w_n))\, dt$$

By Proposition 2.15(2), because $\mathcal{O}$ is a closed geodesic, with same arclength parametrization for g_0 and g_1, it is clear that $a|_{\mathcal{O}} \equiv 1$. Therefore, we may compute the

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difference $\delta_n = |\ell(w_{n,1}) - \ell(w_{n,0})|$ as follows:

$$|\ell(w_{n,1}) - \ell(w_{n,0})| \leq \int_{-T_n/2}^{T_n/2} |a(\varphi_{g_0}^t(w_n)) - 1|\, dt$$
$$\leq M \int_{-T_n/2}^{T_n/2} d(\varphi_{g_0}^{t_n+t}(w_n), \mathcal{O})^\beta\, dt,$$

because a is β-Hölder continuous and the distance between a point and a compact set is well defined. To estimate the distance, note

$$d(\varphi_{g_0}^{t_n+t}(w_n), \mathcal{O}) \leq d(\varphi_{g_0}^{t_n+t}(w_n), \varphi_{g_0}^{\tau_n+t}(w)) + d(\varphi_{g_0}^{\tau_n+t}(w), \mathcal{O})$$
$$\leq c(e^{\eta(T_n/2-|t|)} + e^{-|t|}) \quad \text{for } |t| < T_n/2,$$

because w is a homoclinic point of $\mathcal{O}$ so $d(\varphi_{g_0}^{\tau_n+t}(w), \mathcal{O}) \leq ce^{-|t|}$ for some $c > 0$ which we assume, by taking the maximum if necessary, is the same as the previous c. Substituting this inequality into the previous integral, we obtain

$$|\ell(w_{n,1}) - \ell(w_{n,0})| \leq M \int_{-T_n/2}^{T_n/2} (e^{\eta(T_n/2-|t|)} + e^{-|t|})^\beta\, dt < M_2 < \infty,$$

for M_2 independent of n, as an easy calculus exercise shows. □

3.3. *Twisting.* Following the previous section, we fix a metric $g_0 \in \mathcal{G}_p^k$. Let $\mathcal{O}$ be the orbit with the pinching property, $v \in \mathcal{O}$, and let l be the period of $\mathcal{O}$. We fix an arbitrary $w \in W_{g_0}^s(v) \cap (W_r^{cu})_{g_0}(v)$ a transverse homoclinic point of the orbit of v, and consider the following composition of cocycle holonomy maps for the unstable bundle E^u:

$$\psi_{v,w}^{g_0} = h_{w,v}^s \circ h_{v,w}^{cu}$$

given by Theorem 2.6 and Proposition 2.7. Existence of w satisfying the above properties is given by the existence of homoclinic points and the fact that, by considering $\varphi^t(w)$ if needed, we may always choose w to lie in the *local* center-unstable manifold of v, so that the center-unstable holonomy is well defined (cf. Theorem 2.6 again) and, moreover, so that $w \in W_{g_0}^s(v)$ simultaneously.

Recall that $\vec{\lambda}^u(\mathcal{O}, g)$ consists of distinct real numbers, so let $\{e_i\}$ be an (non-generalized, real) eigenbasis for E^u. For all $1 \leq j \leq n$, the alternating powers $\Lambda^k E^u(v)$ have a basis obtained as exterior products of the e_i. We write $e_I^j := e_{i_1} \wedge \cdots \wedge e_{i_k}$, where $I = \{i_1, \ldots, i_j\}$.

PROPOSITION 3.13. *For $g_0 \in \mathcal{G}_p^k$ as above, we say g_0 has the twisting property for $w \in SM$ with respect to v, and we write $g_0 \in \mathcal{G}_{p,t}^k$, if*

$$\textit{for all } e_I^j, e_{I'}^l, \quad j + l = n : (\wedge^k \psi_{v,w}^{g_0})(e_I^j) \wedge e_{I'}^l \neq 0,$$

which is to say that the image of any direct sums of eigenspaces intersects any direct sum of eigenspaces of complementary dimension only at the origin.

The set $\mathcal{G}_{p,t}^k$ is C^2-open and C^k-dense in $\mathcal{G}^k$.

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Proof. Again, by the density of $\mathcal{G}^\infty \subseteq \mathcal{G}^k$ and openness of $\mathcal{G}^k_p$ we may assume that $g_0 \in \mathcal{G}^\infty$ so we can apply Theorem 2.13. For some small $\varepsilon > 0$, consider the geodesic segment $\gamma = \varphi^{g_0}_{[0,\varepsilon]}(w)$. Note that because $\mathcal{O}(w)$ accumulates as $|t| \to \infty$ on the compact set $\mathcal{O}$, if we take $\varepsilon > 0$ small enough we may take $\pi(\gamma)$ to be disjoint from $\pi(\mathcal{O}(w) \setminus \gamma) \cup \pi(\mathcal{O})$, where $\pi : SM \to M$ is the projection map.

Then we apply Theorem 2.13 to $\gamma' \subseteq \gamma$, where $\gamma' = \varphi^{g_0}_{[\delta,\varepsilon-\delta]}(w)$ for $\delta > 0$ small, to perturb $D_w\varphi^{\varepsilon}_{g_0}$ by perturbing the metric only on a tubular neighborhood $V_{\gamma'}$ of γ' small enough (possible by Proposition 2.15(1)) so that

$$V_{\gamma'} \cap \mathrm{Cl}(\pi(\mathcal{O}) \cup \pi(\mathcal{O}(w) \setminus \gamma)) = \varnothing,$$

where Cl denotes closure.

By equivariance of holonomies, the map $\psi^{g_0}_{v,w}$ can be rewritten as

$$\psi^{g_0}_{v,w} = h^s_{\varphi^{\varepsilon}_{g_0}(w),v} \circ D_w\varphi^{\varepsilon}_{g_0}|_{E_u} \circ h^{cu}_{v,w}.$$

Then observe that perturbations to the metric of the form described in the previous paragraph affect only the $D_w\varphi^{\varepsilon}_{g_0}|_{E_u}$ term in the composition above. Indeed, we recall that $h^{cu}_{w,v}$ depends only on the values of the cocycle on a neighborhood of the $(-\infty, 0]$ part of the orbit $\varphi^t_{g_0}(w)$, and $h^s_{\varphi^{\varepsilon}_{g_0}(w),v}$ on a neighborhood of the $[\varepsilon, \infty)$ part of the orbit $\varphi^t_{g_0}(w)$ and on the cocycle along $\mathcal{O}$. By construction of $V_{\gamma'}$, the cocycle is not perturbed in any of these sets.

It remains to check that for an open and dense set of 1-jets of symplectic maps P from a small transversal to the flow at w to a small transversal section to the flow at $\varphi^{\varepsilon}_{g_0}(w)$, the map $\psi^{g_0}_{v,w}$ has the twisting property (we assume both transversals to be tangent to E^u at w and at $\varphi^{\varepsilon}_{g_0}(w)$, respectively), if we replace $D_w\varphi^{\varepsilon}_{g_0}|_{E_u}$ by $DP|_{E^u}$. This implies, by Theorem 2.13, that we can construct such a small perturbation in the space of metrics, completing the proof.

As both holonomy maps in the composition defining $\psi^{g_0}_{v,w}$ as above are symplectic isomorphisms, an open and dense subset of $\mathrm{Sp}(E^u(v) \oplus E^s(v))$ is mapped under composition with the holonomies to an open dense set of the 1-jets of symplectic maps P as above, so it suffices to check that twisting holds when the map $\psi^{g_0}_{v,w}$ takes value in an open and dense subset of $\mathrm{Sp}(E^u(v) \oplus E^s(v))$.

Again, observe that the condition defining twisting is given by a Zariski open subset of the matrices $\mathrm{Sp}(E^u(v) \oplus E^s(v))$. Hence, as long this set is non-empty the twisting set must also be open and dense in the analytic topology. Then, by the paragraph above, this translates to an open and dense condition in 1-jets of symplectic maps P, and as there is no condition imposed on higher jets, we obtain the desired result by Remark 2.17.

To finish the proof, it thus suffices to check that the Zariski open set defining twisting is non-empty in the symplectic group, which is done in the following. □

LEMMA 3.14. *There exists a matrix $A \in Sp(2n)$, where $\mathbb{R}^{2n}$ is taken with standard symplectic basis $\{e_i, f_i\}$ such that A preserves $E^u := span\ \{e_i\}_{i=1}^n$ and*

$$\textit{for all } e^j_I, e^l_{I'}, \quad j + l = n : (\wedge^k A)(e^j_I) \wedge e^l_{I'} \neq 0.$$

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Proof. Note that for fixed $e^k_I, e^j_{I'}$, the property that $(\wedge^j A)(e^j_I) \wedge e^l_{I'} \neq 0$ is open in $\mathrm{Sp}(2n)$. Thus, by induction, it suffices to show that for any $e^j_I, e^l_{I'}$ one can arrange so that $(\wedge^j A)(e^j_I) \wedge e^l_{I'} \neq 0$ and, moreover, A still preserves E^u, by an arbitrarily small perturbation of $A \in \mathrm{Sp}(2n)$. Then, by induction, the proof is completed by performing successively small perturbations over all pairs I, I'.

To prove the claim, suppose $(\wedge^j A)(e^j_I) \wedge e^l_{I'} = 0$, and write $(\wedge^j A)(e^j_I) = \sum_J a_J e^j_J$. As A is invertible, there exists J_0 such that $a_{J_0} \neq 0$ and such that $|J_0 \cap I'|$ is minimal. As $|J_0| + |I'| = n$, we have $|J_0 \cap I'| = \{1, \ldots, n\} \setminus (J_0 \cup I')$, so we take an arbitrarily chosen bijection $r \mapsto s_r$ from $J_0 \cap I$ to $\{1, \ldots, n\} \setminus (J_0 \cup I')$.

For $\theta > 0$, and $r, s \in \{1, \ldots, n\}$ let $R^{r,s}_\theta$ be given by rotating the (oriented) planes $\mathrm{span}(e_r, e_s)$ and $\mathrm{span}(f_r, f_s)$ by θ and preserving the other basis elements. Let A' be obtained by composing A with each of R^{r,s_r}_θ for $r \in J_0 \cap I$ (in any order, because the rotation matrices commute). One checks directly that $R^{r,s}_\theta \Omega (R^{r,s}_\theta)^T = \Omega$, where Ω is the standard symplectic form, so $R^{r,s}_\theta$ preserves E^u so A' is symplectic and preserves E^u. Writing

$$\prod_{r \in J_0 \cap I'} (\wedge^k R^{r,s_r}_\theta) e^k_J = \sum_L b_L e^j_L,$$

by a direct computation, one can check that $b_{\{1,\ldots,n\}\setminus I'} \neq 0$ if and only if $J = J_0$, which implies that $(\wedge^j A')(e^j_I) \wedge e^l_{I'} \neq 0$. □

4. *Proof of Theorem 1.1*

We finish the proof of Theorem 1.1. In what follows, let $\sigma : \Sigma \to \Sigma$ be the shift map of an invertible subshift of finite type Σ. The suspension of Σ under a continuous $f : \Sigma \to \mathbb{R}^+$ is the compact metric space,

$$\Sigma_f := (\Sigma \times \mathbb{R})/((x, s) \sim \alpha^n(x, s),\ n \in \mathbb{Z}),$$

where $\alpha(x, s) := (\sigma(x), s - f(x))$. The shift σ lifts to a continuous-time system $\sigma^t_f : \Sigma_f \to \Sigma_f$ given by $\sigma^t_f(x, s) = (x, s + t)$ for $t \in \mathbb{R}$.

First, we need to represent Anosov flows by the suspension of a shift. The following is the standard statement of the construction of a Markov partition for an Anosov flow.

THEOREM 4.1. [**FH18**, Theorem 6.6.5] *Let $\Phi : M \to M$ be a C^1 Anosov flow. There is a semiconjugacy from a hyperbolic symbolic flow to Φ that is finite-to-one and one-to-one on a residual set of points, where the roof function for the subshift of finite type corresponds to the travel times between the local sections for the smooth system.*

Finally, we prove Theorem 1.1.

Proof of Theorem 1.1. We prove that the statement holds for all $g \in \mathcal{G}^k_{p,t}$, so the theorem is proved by Proposition 3.13. Fix some such $g_0 \in \mathcal{G}^k_{p,t}$ and let $v, w \in SM$ be the vectors along whose orbits pinching and twisting hold, respectively.

Let Σ_f be a suspension of a subshift of finite type and $P : \Sigma_f \to SM$ be the semi-conjugacy map to the geodesic flow of g_0 given by Theorem 4.1. Following the proof

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of Theorem 4.1 in [**FH18**], we see that it is possible to construct the Markov partition so that $v \in SM$ has a unique lift (p, t) to the suspension of the shift space Σ_f by enlarging the Markov rectangles by an arbitrarily small amount so that the orbit of v only intersects their interiors, where $f : \Sigma \to \mathbb{R}$ is some roof function. Then, by [**FH18**, Claim 6.6.9 and Corollary 6.6.12], there is also a unique (q, s) that lifts the homoclinic point with twisting w.

We write $\mathcal{E} \to \Sigma_f$ for the pullback of the bundle $E^u \to SM$ to Σ_f under P, and by $A^t : \mathcal{E} \to \mathcal{E}$, the pullback of the derivative cocycle. By using the return map of A^t to the 0 section of Σ_f, the cocycle A^t determines a discrete time cocycle A on $\mathcal{E} \to \Sigma$ identified with $\Sigma \times \{0\} \subseteq \Sigma_f$. Following, the propositions in section 2.1 of [**BGV03**], there exists a distance on Σ which makes the cocycle A dominated, so that it admits holonomies H^s and H^u.

Recall that the local stable and unstable manifolds $W^s_{\mathrm{loc}}(\overline{x})$ (respectively, $W^u_{\mathrm{loc}}(\overline{x})$) for the shift space Σ are defined as the sequences $\overline{y}$ such that $(\overline{y})_i = (\overline{x})_i$ for all $i \geq 0$ (respectively, ≤ 0), where the subscript i denotes the ith entry of $\overline{x}$ and $\overline{y}$ regarded as a sequence in the shift space Σ. By reducing the size of the rectangles in the original construction if necessary, it is possible to ensure that points in the same local stable/unstable manifold in the shift Σ are mapped to the same local center stable/unstable manifold in SM by the semi-conjugacy P. Then we can prove the following lemma which verifies agreement of holonomies of the geodesic flow and its symbolic discrete representation:

LEMMA 4.2. *Let $x = \overline{x} \times \{0\} \in \Sigma_f$, and $y = \overline{y} \times \{0\} \in \Sigma_f$, where $\overline{y}$ is in the local stable manifold. Let $v = P(x)$ and $w = P(y)$ which, by the previous paragraph, lie in the same local center-unstable manifold in SM. Then $h^{cs}_{vw} = H^s_{\overline{x}\overline{y}}$ and the analogous result holds for unstable holonomies.*

Proof. By the proof of existence of holonomies as in [**BV04**], one obtains the holonomy map as a limit:

$$H^s_{\overline{x},\overline{y}} = \lim_{n\to\infty} ((A^n)_{\overline{x}})^{-1} \circ I_{\sigma^n\overline{x}\sigma^n\overline{y}} \circ (A^n)_{\overline{y}}.$$

As $n \to \infty$, note that $(\sigma^n\overline{x}) \times \{0\}$ and $(\sigma^n\overline{y}) \times \{0\}$ converge to the same stable manifold in Σ_f. Hence, if we let $T_n := \sum_{i=0}^{n-1} f(\sigma^i\overline{x})$ so that $(A^n)_{\overline{x}} = (A^{T_n})_{\overline{x}\times\{0\}}$, then $\sum_{i=0}^{n-1} f(\sigma^i\overline{y}) - (T_n + r) \to 0$, as $n \to \infty$, where $r \in \mathbb{R}$ is such that $\sigma^r_f(y) \in W^s(x)$.

On the other hand, using the formula defining the holonomies and the definition of A^t as a pullback cocycle of $D\varphi^t_{g_0}|_{E^u}$:

$$\begin{aligned} h^{cs}_{vw} &= \lim_{T\to\infty} (D\varphi_{g_0}|^T_{E^u})^{-1}_v \circ I_{\varphi^T_{g_0}(v),\varphi^{T+r}_g(w)} \circ (D\varphi_{g_0}|^T_{E^u})_{\varphi^r_{g_0}(w)} \circ (D\varphi_{g_0}|^r_{E^u})_w, \\ &= \lim_{T\to\infty} (A^T)^{-1}_x \circ I_{\sigma^T_f y,\sigma^{T+r}_f y} \circ (A^T)_{\sigma^r_f y} \circ (A^r)_y, \end{aligned}$$

so letting $T = T_n$ we conclude that $H^s_{\overline{x},\overline{y}} = h^{cs}_{vw}$. □

With the above proposition it is straightforward to verify using the equivariance of holonomies (with respect to A) that the cocycle A over Σ is simple. Let $\rho : SM \to \mathbb{R}$

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be a Hölder potential and μ_ρ its associated equilibrium state for the geodesic flow of g_0. Let $\tilde{\rho}$ be the Hölder continuous potential on Σ_f given by $\tilde{\rho} = \rho \circ P$, and $\tilde{\mu}_\rho$ its associated equilibrium state for $\sigma_f^t : \Sigma_f \to \Sigma_f$.

It is a well-known fact (see, e.g., [BR75]) that P is, in fact, a measurable isomorphism between $(\Sigma_f, \tilde{\mu}_\rho)$ and (SM, μ_ρ). Hence, the Lyapunov spectrum of A^t with respect to $\tilde{\mu}_\rho$ agrees with that of $D\varphi_{g_0}^t$ with respect to μ_ρ, and it suffices to show simplicity of the spectrum of the former.

As $f : \Sigma \to \mathbb{R}$ is Hölder, identifying Σ with $\Sigma \times \{0\} \subseteq \Sigma_f$, the Hölder continuous function:

$$\left(\int_0^{f(x)} \tilde{\rho}(x, t)\, dt \right) - P(\sigma_f, \tilde{\rho}) f_g(x),$$

where $P(\sigma_f^t, \tilde{\rho})$ is the pressure of σ_f^t with respect to $\tilde{\rho}$, defines a potential on $\Sigma = \Sigma \times \{0\} \subseteq \Sigma_{f_g}$ and has a unique equilibrium state μ which satisfies, for $F \in C^0(\Sigma_f)$,

$$\int_{\Sigma_f} F\, d\tilde{\mu}_\rho = \frac{\int_\Sigma \left(\int_0^{f(x)} F(x, t)\, dt \right) d\mu}{\int_\Sigma f(x)\, d\mu}$$

by [FH18, Proposition 4.3.17]. In particular, because μ is an equilibrium state it has local product structure.

The product $\mu \times dt$ defines a measure for the suspension flow σ_1^t on Σ_1 (where 1 is the constant function 1) which has the same Lyapunov spectrum as μ. As $\mu \times dt$ and $\tilde{\mu}_\rho$ are related by a time change, the Lyapunov spectrum of A^t with respect to μ_ρ and the Lyapunov spectrum of A with respect to μ differ by a scalar, see, e.g., [Bu17, Proposition 2.15]. Hence, applying Theorem 2.3 to the simple cocycle A for the measure μ we obtain simplicity of the Lyapunov spectrum for μ_ρ. □

5. *Proof of Theorems 1.2 and 1.3*

In this section, we explain the modifications needed to the previous sections to give the proofs of Theorems 1.2 and 1.3.

Proof of Theorems 1.2 and 1.3. For $\frac{1}{2}$-bunched Anosov flows, the splitting $E^u \oplus E^0 \oplus E^s$ may not be C^1, so instead we consider the derivative cocycle on the C^1-bundles $Q^u := E^{cu}/E^0$ and $Q^s := E^{cs}/E^0$, which we have shown to be 1-bunched in Proposition 2.7. In what follows, we prove simplicity for the spectrum on Q^u and Q^s implies the desired result because $D\Phi$ on $Q^{u,s}$ has the same spectrum as $D\Phi$ on $E^{u,s}$.

For Theorem 1.3, recall that topological mixing is C^1-open and C^k-dense in the space of Anosov flows. Then we follow the propositions in §3 to construct orbits with pinching and twisting for the cocycle on Q^u by a C^k-small perturbation, which in this case is achievable because the analog of Theorem 2.13 is clear in the space of all vector fields and $\mathfrak{X}_A^k(X)$ is open by structural stability in the space of all vector fields and, moreover, the linear algebra lemmas (Lemmas 3.3 and 3.14) needed for the case of $\mathrm{Sp}(2n)$ are immediate for

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GL(n). The C^1-openness of the conditions also is proved similarly. Then, by a symmetric argument, it is clear that pinching and twisting for both Q^u and Q^s is C^1-open and C^k-dense. The proof then follows the same outline in §4.

The proof of Theorem 1.2 is similar, in that the linear algebra lemmas (Lemmas 3.3 and 3.14) needed for the case of Sp($2n$) are still immediate for SL(n). Moreover, topological mixing is known for all C^2-volume-preserving Anosov flows. Finally, it remains to prove an analog of Theorem 2.13 for the conservative class, which we do in the next section. With that in hand, the proof also follows the same outline as Theorem 1.1. □

5.1. *Conservative perturbations.* In this section, we prove the analog of Theorem 2.13 in the volume-preserving category. To the best of the author's knowledge, the result is not found anywhere in the literature so the complete proof is included here. Throughout, we let $X \in \mathfrak{X}^\infty_m(M)$ be a non-vanishing vector field generating the flow φ_X on the smooth manifold M which preserves the smooth volume m. Fix an embedded segment of a flow orbit $l : [0, \varepsilon] \to M$ parametrized by the time parameter and a small transversal smooth hypersurface $\Sigma(0)$ to X at $l(0)$.

For $t \in [0, \varepsilon]$, set $\Sigma(t) = \varphi^t_X(\Sigma(0))$ so that $\iota_X m$ is a volume form on the hypersurfaces $\Sigma(t)$. The following result, whose proof is elementary except for an application of the conservative pasting lemma, shows that it is possible to perturb the k-jets in the conservative setting generically by C^k-small perturbations.

THEOREM 5.1. *Let Q be some dense subset of the space of k-jets of volume-preserving maps $(\Sigma(0), \iota_X m, l(0)) \to (\Sigma(\varepsilon), \iota_X m, l(\varepsilon))$.*

Then there is arbitrarily C^k-close to X an m-preserving X' such that:

(a) *$Y := X' - X$ is supported in an arbitrarily small tubular neighborhood B of $l([\delta, \varepsilon - \delta])$, for some $0 < \delta < \varepsilon$;*

(b) *$Y = 0$ on $l([0, 1])$ and Y is tangent to the hypersurfaces $\Sigma(t)$;*

(c) *the flow of X' generates a map $(\Sigma(0), l(0)) \to (\Sigma(\varepsilon), l(\varepsilon))$ with k-jets in Q.*

Proof. If B is sufficiently small, we may assume that it is foliated by the transversals $\Sigma(t)$ and, moreover, by passing to a further neighborhood we may assume that the transverse sections are mapped diffeomorphically onto each other by the flow X, that is, we may construct the perturbation in a flow box with transversals given by the $\Sigma(t)$.

In the flowbox, a classic application of Moser's trick allows us to assume that the flow is in normal coordinates $\varphi^t_X(x_1, \ldots, x_{n-1}, s) \mapsto (x_1, \ldots, x_{n-1}, s + t)$, where the image of l is contained in $\{x_1 = \cdots = x_n = 0\}$ and $m = dx^1 \wedge \cdots \wedge dx^n$. In these coordinates, we may regard $B \cong U \times [0, \varepsilon]$, where $U \subseteq \mathbb{R}^{n-1}$ is a domain and so $Q \subseteq J^k_m(n - 1, \mathbb{R})$, where $J^k_m(n - 1, \mathbb{R})$ is the Lie group of k-jets of volume-preserving maps fixing the origin.

Using the flow to identify the fibers of $U \times \mathbb{R} \to \mathbb{R}$, the problem is thus reduced to the construction, for each $\delta > 0$, of a time-dependent vector field $\{Y_t\}_{t\in[0,\varepsilon]}$ on $\mathbb{R}^{n-1}$ with the following properties:

(a) $Y_t(0) = 0$ and $\mathrm{supp}(Y_t) \subseteq U$ for $t \in [0, \varepsilon]$;

(b) $Y_t \equiv 0$ on $[0, \delta]$ and $Y_t \equiv 0$ on $[\varepsilon - \delta, \varepsilon]$;

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(c) Y_t is divergence free for all $t \in [0, \varepsilon]$;
(d) the time-ε map $f : (\mathbb{R}^{n-1}, 0) \to (\mathbb{R}^{n-1}, 0)$ of Y_t has derivative at 0 in Q;
(e) $\|Y_t\|_{C^\infty} < \delta$ for all $t \in [0, \varepsilon]$;

The construction is given by first specifying the time-ε map f and then finding an appropriate isotopy within the volume-preserving category to the identity.

Fix some $\theta \in Q$ sufficiently close to the k-jets of I (the identity map) and a map $F : (\mathbb{R}^{n-1}, 0) \to (\mathbb{R}^{n-1}, 0)$ whose k-jet at the origin is given by θ. Take some C^∞ bump function $\rho : \mathbb{R}^{n-1} \to \mathbb{R}$ which interpolates between the constant function 1 in $B(0, \eta/2)$ to the constant function 0 outside of $B(0, \eta)$ for some η small. Let $F' = \rho F$; if θ is sufficiently close to 0, then $\|F' - I\|_{C^k}$ is small so in particular $F' \in \mathrm{Diff}(\mathbb{R}^{n-1})$. Applying Moser's trick, we can find an $f \in \mathrm{Diff}_m(\mathbb{R}^{n-1})$, that is preserving m, which is C^k-close to the identity and which agrees with F' where it is conservative, namely, everywhere except $B(0, \eta) \setminus B(0, \eta/2)$. In particular, the k-jet of f at the origin equals $\theta \in Q$.

To obtain such an f, we construct a family $s \mapsto h_s \in \mathrm{Diff}(\mathbb{R}^{n-1})$ such that $h_1 = F$ and $h_0 =: f$ is conservative. Let $r : B \to \mathbb{R}$ be the smooth function C^k close to 1 satisfying $F_*\mu = r\mu$. Then, for $s \in [0, 1]$, we solve $(h_s)_*\mu = r^s\mu$, namely div $Z_s = r^s \log r^s$, where $Z_s = \partial_s h_s$. Moreover, the proof of the Poincaré lemma shows that we can take Z_s to be constant equal to 0 outside of $B(0, \eta)$. By the conservative pasting lemma [Te20], there exists W_s which agrees with Z_s on a neighborhood of $\mathbb{R}^{n-1} \setminus (B(0, \eta) \setminus B(0, \eta/2))$ and is divergence-free. Then $Z'_s = Z_s - W_s$ also satisfies div $Z'_s = r^s \log r^s$ and it is identically 0 where $r = 1$, so that $h_s(x) = F(x)$ on $B(0, \eta) \setminus B(0, \eta/2)$, where now $\partial_s h_s = Z'_s$. In particular, $f := h_0$ is the identity outside of $B(0, \eta)$ and its k-jet at the origin is given by θ. This constructs the desired f.

Now let $\alpha : [0, \varepsilon] \to [0, \varepsilon]$ be a C^∞ function such that $\alpha \equiv 0$ on $[0, \delta]$ and $\alpha \equiv \varepsilon$ on $[\varepsilon - \delta, \varepsilon]$. If $\|f - I\|_{C^k}$ is sufficiently small (which is ensured by taking θ closer to the jets of the identity), the maps $g_t := \alpha(t) f + (1 - \alpha(t)) I$ are all diffeomorphisms and $t \mapsto \partial_t g_t$ is a time-dependent vector field that satisfies all the desired properties except for being divergence-free.

To repair that, again Moser's trick constructs a family $s \mapsto g_{t,s}$ such that $g_{t,0} = g_t$ and $g_{t,1}$ is conservative as follows. Let $r_t : B \to \mathbb{R}$ be the smooth one-parameter family of smooth functions C^k close to 1 satisfying $(g_t)_*\mu = r_t\mu$. Then, for $s \in [0, 1]$, we solve $(g_{t,s})_*\mu = r_t^s\mu$, namely div $Z_{t,s} = r_t^s \log r_t^s$, where $Z_{t,s} = \partial_s g_{t,s}$. It is an easy consequence of the proof of the Poincaré lemma that the family $Z_{t,s}$ may be taken to be smooth in t with small t derivatives, because $t \mapsto r_t$ as a one-parameter family has the same properties. Moreover, we can take supp $Z_{t,s} \subseteq$ supp $(r_t - 1)$. The C^k norm of the $Z_{t,s}$ is a continuous function of the C^k norm of $r_t^s \log r_t^s$, so that taking $Y_t = \partial_t g_{t,1}$ finishes the proof. □

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