A SPECTRAL MAPPING THEOREM FOR THE WEYL SPECTRUM

by WOO YOUNG LEE and SANG HOON LEE

(Received 15 August, 1994; revised 1 February, 1995)

Introduction. Suppose H is a Hilbert space and write $\mathscr{L}(H)$ for the set of all bounded linear operators on H. If $T \in \mathscr{L}(H)$ we write $\sigma(T)$ for the spectrum of T; $\pi_0(T)$ for the set of eigenvalues of T; and $\pi_{00}(T)$ for the isolated points of $\sigma(T)$ that are eigenvalues of finite multiplicity. If K is a subset of $\mathbf{C}$, we write iso K for the set of isolated points of K. An operator $T \in \mathscr{L}(H)$ is said to be *Fredholm* if both $T^{-1}(0)$ and $T(H)^{\perp}$ are finite dimensional. The *index* of a Fredholm operator T, denoted by index(T), is defined by

$$\operatorname{index}(T) = \dim T^{-1}(0) - \dim T(H)^{\perp}.$$

The *essential spectrum* of T, denoted by $\sigma_e(T)$, is defined by

$$\sigma_e(T) = \{\lambda \in \mathbf{C} : T - \lambda I \text{ is not Fredholm}\}$$

A Fredholm operator of index zero is called a *Weyl* operator (cf. [**7**], [**8**]). The *Weyl spectrum* of T, denoted by $w(T)$, is defined by

$$w(T) = \{\lambda \in \mathbf{C} : T - \lambda I \text{ is not Weyl}\}.$$

Recall ([**12**]) that an operator $T \in \mathscr{L}(H)$ is said to be *hyponormal* if

$$T^*T \geq TT^*. \tag{0.1}$$

If T is Fredholm then, by (0.1),

$$T \text{ hyponormal} \Rightarrow \operatorname{index}(T) \leq 0. \tag{0.2}$$

It was shown in [**10**] that the mapping $T \to w(T)$ is upper semi-continuous, but not continuous, at T and that if $T_n \to T$ with $T_nT = TT_n$ for all $n \in \mathbf{N}$ then

$$\lim w(T_n) = w(T). \tag{0.3}$$

It was also shown ([**5**, Theorem 2(b)]) that $w(T)$ satisfies the one-way spectral mapping theorem for analytic functions: if f is analytic on a neighborhood of $\sigma(T)$ then

$$w(f(T)) \subset f(w(T)). \tag{0.4}$$

The inclusion (0.4) may be proper (see [**2**, Example 3.3]). If T is normal then $\sigma_e(T)$ and $w(T)$ coincide. Thus if T is normal then, since $f(T)$ is also normal, it follows that $w(T)$ satisfies the spectral mapping theorem for analytic functions.

In this note we show that the Weyl spectrum of a hyponormal operator satisfies the spectral mapping theorem for analytic functions and then answer an old question of Oberai ([**11**]).

Our main result is as follows.

THEOREM 1. *If S and T are commuting hyponormal operators then*

$$S, T \text{ Weyl} \Leftrightarrow ST \text{ Weyl} \tag{1.1}$$

and hence if f is analytic on a neighborhood of $\sigma(T)$ then

$$w(f(T)) = f(w(T)). \tag{1.2}$$

Proof. The forward implication of (1.1) uses the Index Product Theorem ([**8**, Theorem 6.5.4]). For the backward implication of (1.1), we observe that if $ST = TS$ then

$$S^{-1}(0) \cup T^{-1}(0) \subseteq (ST)^{-1}(0) \quad \text{and} \quad (S^*)^{-1}(0) \cup (T^*)^{-1}(0) \subseteq ((ST)^*)^{-1}(0),$$

which yields

$$ST \text{ Fredholm} \Rightarrow S, T \text{ Fredholm},$$

which together with (0.2) gives

$$\text{index}(ST) = 0 \Rightarrow \text{index}(S) = \text{index}(T) = 0.$$

This gives (1.1). Now suppose p is any polynomial. Let

$$p(T) - \lambda I = a_0(T - \mu_1 I) \dots (T - \mu_n I).$$

If T is hyponormal then $T - \mu_i I$ $(i = 1, \dots, N)$ are commuting hyponormal operators. It thus follows from (1.1) that

$$\begin{aligned} \lambda \notin w(p(T)) &\Leftrightarrow a_0(T - \mu_1 I) \dots (T - \mu_n I) \text{ Weyl} \\ &\Leftrightarrow T - \mu_i I \text{ Weyl} \quad \text{for each} \quad i = 1, \dots, n \\ &\Leftrightarrow \mu_i \notin w(T) \quad \text{for each} \quad i = 1, \dots, n \\ &\Leftrightarrow \lambda \notin p(w(T)), \end{aligned}$$

which says that

$$w(p(T)) = p(w(T)). \tag{1.3}$$

Next suppose r is any rational function with no poles in $\sigma(T)$. Write $r = \frac{p}{q}$, where p and q are polynomials and q has no zeros in $\sigma(T)$. Then

$$r(T) - \lambda I = (p - \lambda q)(T)(q(T))^{-1}.$$

By (1.3)

$$(p - \lambda q)(T) \text{ Weyl} \Leftrightarrow p - \lambda q \quad \text{has no zeros in} \quad w(T).$$

Thus we have

$$\begin{aligned} \lambda \notin w(r(T)) &\Leftrightarrow (p - \lambda q)(T) \text{ Weyl} \\ &\Leftrightarrow p - \lambda q \quad \text{has no zeros in} \quad w(T) \\ &\Leftrightarrow ((p - \lambda q)(x))q(x)^{-1} \neq 0 \quad \text{for any} \quad x \in w(T) \\ &\Leftrightarrow \lambda \notin r(w(T)), \end{aligned}$$

which says that $w(r(T)) = r(w(T))$. If f is an analytic function on a neighborhood of $\sigma(T)$ then by Runge's theorem (cf. [**3**]), there is a sequence (r_n) of rational functions with no

https://doi.org/10.1017/S0017089500031268 Published online by Cambridge University Press

poles in $\sigma(T)$ such that $r_n \to f$ uniformly on $\sigma(T)$. Since $r_n(T)$ commutes with $f(T)$, (1.2) follows from (0.3).

If the "hyponormal" condition is dropped in (1.1) then the backward implication may fail even though T_1 and T_2 commute: for example, if U is the unilateral shift on ℓ_2, consider the following operators on $\ell_2 \oplus \ell_2$: $T_1 = U \oplus I$ and $T_2 = I \oplus U^*$.

We say ([**1**], [**4**], [**10**]) that *Weyl's theorem holds for T* if

$$w(T) = \sigma(T) - \pi_{00}(T).$$

There are several classes of operators including hyponormal operators (cf. [**1**], [**2**], [**4**], [**9**], [**11**]) for which Weyl's theorem holds. Oberai ([**11**]) has raised the following question: does there exist a hyponormal operator T such that Weyl's theorem does not hold for T^2? Note that T^2 may not be hyponormal even if T is hyponormal ([**6**, Problem 209]). We will show that Weyl's theorem holds for $f(T)$ when T is hyponormal.

Recall ([**2**], [**10**]) that $T \in \mathcal{L}(H)$ is said to be *isoloid* if iso $(\sigma(T)) \subseteq \pi_0(T)$.

We have a modification of [**11**, Lemma 1 and Proposition 1].

LEMMA. *If $T \in \mathcal{L}(H)$ is isoloid and if f is analytic on a neighborhood of $\sigma(T)$, then*

$$f(\sigma(T) - \pi_{00}(T)) = \sigma(f(T)) - \pi_{00}(f(T)). \tag{2.1}$$

Proof. The proof of (2.1) is taken straight from a slight modification of the proofs of [**11**, Lemma 1 and Proposition 1], which work with polynomials.

We conclude with the following result.

THEOREM 2. *If $T \in \mathcal{L}(H)$ is hyponormal, then for any function f analytic on a neighborhood of $\sigma(T)$, Weyl's theorem holds for $f(T)$.*

Proof. Since hyponormal operators are isoloid ([**12**]) and Weyl's theorem holds for hyponormal operators, it follows from (1.2) and (2.1) that

$$\sigma(f(T)) - \pi_{00}(f(T)) = f(\sigma(T) - \pi_{00}(T)) = f(w(T)) = w(f(T)).$$

ACKNOWLEDGEMENTS. The present study was partially supported by the Basic Science Research Institute Program, Ministry of Education, 1994, Project No. BSRI-94-1420 and KOSEF Grant No. 94-1400-02-01-3.

REFERENCES

1. S. K. Berberian, An extension of Weyl's theorem to a class of not necessary normal operators, *Michigan Math. J.* **16** (1969), 273–279.

2. S. K. Berberian, The Weyl spectrum of an operator, *Indiana Univ. Math. J.* **20** (1970, 71), 529–544.

3. J. B. Conway, *Subnormal operators* (Pitman, Boston, 1981).

4. L. A. Coburn, Weyl's theorem for nonnormal operators, *Michigan Math. J.* **13** (1966), 285–288.

5. G. Gramsch and D. Lay, Spectral mapping theorems for essential spectra, *Math. Ann.* **192** (1971), 17–32.

6. P. R. Halmos, *A Hilbert space problem book* (Springer-Verlag, 1984).

7. R. E. Harte, Fredholm, Weyl and Browder theory, *Proc. Roy. Irish Acad. Sect. A.* **85** (1985), 151–176.

8. R. E. Harte, *Invertibility and singularity for bounded linear operators* (Marcel Dekker, 1988).

9. W. Y. Lee and H. Y. Lee, On Weyl's theorem, *Math. Japon* **39** (1994), 545–548.

10. K. K. Oberai, On the Weyl spectrum, *Illinois J. Math.* **18** (1974), 208–212.

11. K. K. Oberai, On the Weyl spectrum (II), *Illinois J. Math.* **21** (1977), 84–90.

12. J. G. Stampfli, Hyponormal operators, *Pacific J. Math.* **12** (1962), 1453–1458.

Department of Mathematics,
Sung Kyun Kwan University,
Suwon 440-746,
Korea
E-mail address: wylee@yurim.skku.ac.kr