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Strongly Summable Ultrafilters, Union Ultrafilters, and the Trivial Sums Property

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Abstract. We answer two questions of Hindman, Steprāns, and Strauss; namely, we prove that every strongly summable ultrafilter on an abelian group is sparse and has the trivial sums property. Moreover, we show that in most cases the sparseness of the given ultrafilter is a consequence of its being isomorphic to a union ultrafilter. However, this does not happen in all cases; we also construct (assuming Martin's Axiom for countable partial orders, *i.e.*, $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$), a strongly summable ultrafilter on the Boolean group that is not additively isomorphic to any union ultrafilter.

1 Introduction

The concept of a strongly summable ultrafilter originated from N. Hindman's efforts for proving the theorem that now bears his name (which at the time was known as the Graham–Rothschild conjecture), though later on it was realized that such ultrafilters have a rich algebraic structure in terms of the algebra in the Čech–Stone compactification, which in turn sheds light on the aforementioned theorem by providing an elegant proof of it. We conceive of the Čech–Stone compactification of an abelian group G (equipped with the discrete topology) as the set βG of all ultrafilters on G, where the basic open sets are those of the form $\overline{A} = \{p \in \beta G \mid A \in p\}$, for $A \subseteq G$. As it turns out, these sets are actually clopen. If we identify each point $x \in G$ with the principal ultrafilter $\{A \subseteq G \mid x \in A\}$, then G is a dense subset of βG, and what we denoted by $\overline{A}$ is really the closure in βG of the set A. The group operation $+$ from G is also extended by means of the formula

$$p + q = \Big\{ A \subseteq G \,\Big|\, \{x \in G \mid A - x \in q\} \in p \Big\},$$

which turns βG into a right topological semigroup. This means that for each $p \in \beta G$, the mapping $(q \mapsto q + p)\colon \beta G \to \beta G$ is continuous, although βG is not a group (nonprincipal ultrafilters have no inverse). Moreover, the extended operation $+$ is not commutative in βG, even though its restriction to G is, but elements $x \in G$ satisfy that $x + p = p + x$ for every $p \in \beta G$. The closed subsemigroup $G^* = \beta G \setminus G$ consisting of all nonprincipal ultrafilters will be of special importance. The book [10] is the standard reference on this topic.

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We reserve the lowercase roman letters p, q, r, u, v for ultrafilters, and the uppercase roman letters A, B, C, D, W, X, Y, Z, with or without subscripts, will always denote subsets of the abelian group at hand. Lowercase letters w, x, y, z will typically denote elements of the abelian group that is being dealt with, and the "vector" notation will be used for sequences of elements of the group, *e.g.*, $\vec{x} = \langle x_n \mid n < \omega \rangle$. When the sequences are finite, we use the symbol $\frown$ to denote their concatenation, as in $\vec{x} \frown \vec{y}$. If G is an abelian group and $x \in G$, the symbol $o(x)$ will denote the *order* of x, *i.e.*, the least natural number n such that $nx = 0$. We make liberal use of the von Neumann ordinals, usually denoted by Greek letters $\alpha, \beta, \gamma, \zeta, \eta, \xi$; thus, for two ordinals α, β, the expressions $\alpha < \beta$ and $\alpha \in \beta$ are interchangeable. In particular, a natural number n is conceived as the set $\{0, \dots, n-1\}$ of its predecessors, with 0 being equal to the empty set $\varnothing$, and ω denotes the set of finite ordinals, *i.e.,* the set $\mathbb{N} \cup \{0\}$. The lowercase roman letters i, j, k, l, m, n, with or without subscript, will be reserved to denote elements of ω. The letters M and N, with or without subscripts, will, in general, be reserved for denoting subsets of ω (finite or infinite). Given a subset $M \subseteq \omega$, $[M]^n$ will denote the set of subsets of M with n elements, $[M]^{<\omega} = \bigcup_{n<\omega} [M]^n$ will denote the set of finite subsets of M, and $[M]^\omega$ denotes the set of infinite subsets of M. The lowercase roman letters a, b, c, d, with or without subscripts, will stand for elements of $[\omega]^{<\omega}$, *i.e.*, for finite subsets of ω.

Whenever we have a mapping $f\colon G \to H$, there is a standard way to lift or extend it to another mapping $\beta f\colon \beta G \to \beta H$ that is continuous and, if f is a semigroup homomorphism, then so is βf. This extension is given by

$$(\beta f)(p) = \big\{ A \subseteq H \mid f^{-1}[A] \in p \big\} = \big\langle \{ f[A] \mid A \in p \} \big\rangle,$$

where the rightmost expression means that we take the filter on H generated by the family $\{f[A] \mid A \in p\}$, which has the finite intersection property. It is customary to write just $f(p)$ instead of $(\beta f)(p)$, and we will do so throughout this paper. The ultrafilter $f(p)$ is called the *Rudin–Keisler image of p under f*.

The cardinal invariant $\mathrm{cov}(\mathcal{M})$ (read "covering of meagre") is the least cardinal for which Martin's Axiom fails at a countable partial order. That is, $\mathrm{cov}(\mathcal{M})$ is the least κ such that one can find κ-many dense subsets of some countable partial order with no filter meeting them all (this notation is explained by the fact that this cardinal is also the least possible number of meagre sets needed to cover all of the real line). Thus, the equality $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$ means that Martin's Axiom holds for countable partial orders, whilst the failure of this principle is expressed by the inequality $\mathrm{cov}(\mathcal{M}) < \mathfrak{c}$.

One of the most important groups dealt with in this paper is the circle group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. When talking about this group, we will freely identify real numbers with their corresponding cosets modulo $\mathbb{Z}$, and conversely, we will identify elements of $\mathbb{T}$ (cosets modulo $\mathbb{Z}$) with any of the elements of $\mathbb{R}$ representing them. Therefore, when we refer to an element of $\mathbb{T}$ as a real number t, we really mean the coset of that number modulo $\mathbb{Z}$, thus *e.g.,* we may write $t = 0$ and really mean that $t \in \mathbb{Z}$. This should not cause confusion as the context will always clearly indicate whether we are viewing t as a real number or as an element of $\mathbb{T}$. If there is the need to specify a single representative for an element of $\mathbb{T}$, we will pick the unique representative t satisfying $-\frac{1}{2} < t \leq \frac{1}{2}$.

We will now proceed to introduce the main objects of study of this paper.

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Definition 1.1 Let G be an abelian group.

(i) Given a k-sequence $\vec{x} = \langle x_i \mid i < k \rangle$ of elements of G (where $k \leq \omega$), we define the *set of finite sums of the sequence* $\vec{x}$ as:

$$\mathrm{FS}(\vec{x}) = \Big\{ \sum_{n \in a} x_n \mid a \in [k]^{<\omega} \setminus \{\varnothing\} \Big\}.$$

(ii) An *FS-set* is just a set of the form $\mathrm{FS}(\vec{x})$ for some sequence $\vec{x}$ of elements of G with infinite range.

(iii) An ultrafilter $p \in \beta G$ is *strongly summable* if it has a base of FS-sets; *i.e.*, if for every $A \in p$ there exists an ω-sequence with infinite range, $\vec{x} = \langle x_n \mid n < \omega \rangle$, such that $p \ni \mathrm{FS}(\vec{x}) \subseteq A$.

Note that the only principal strongly summable ultrafilter is 0. Strongly summable ultrafilters on $(\mathbb{N}, +)$ were first constructed, under CH, by Hindman in [5, Th. 3.3] (here he claims to construct an idempotent, but a closer look at the proof reveals that the ultrafilter under construction is in fact strongly summable), although at that time this terminology was not in use. The terminology was introduced later on, in [6, Def. 2.1]. Blass and Hindman showed in [2, Th. 3] that the existence of strongly summable ultrafilters is not provable from the axioms of ZFC alone, because it implies the existence of P-points. The sharpest result so far in terms of existence is due to Eisworth, who shows in [3, Th. 9] that $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$ suffices for ensuring the existence of a strongly summable ultrafilter. In a forthcoming paper, this author shows that the existence of strongly summable ultrafilters on any abelian group is consistent with ZFC together with $\mathrm{cov}(\mathcal{M}) < \mathfrak{c}$.

The importance of this type of ultrafilter came at first from the fact that they are examples of idempotents in $\beta\mathbb{N}$, but among idempotents they are special in that the largest subgroup of $\mathbb{N}^*$ containing one of them as the identity is just a copy of $\mathbb{Z}$. More concretely, [10, Th. 12.42] establishes that if $p \in \mathbb{N}^*$ is a strongly summable ultrafilter, and $q, r \in \beta\mathbb{N}$ are such that $q + r = r + q = p$, then $q, r \in \mathbb{Z} + p$. In [8], the authors generalize some results previously only known to hold for ultrafilters on $\beta\mathbb{N}$ or $\beta\mathbb{Z}$. In particular, they proved there ([8, Th. 2.3]) that every strongly summable ultrafilter p on any abelian group G is an idempotent ultrafilter. And [8, Th. 4.6] states that if G can be embedded in $\mathbb{T}$, then whenever $q, r \in G^* = \beta G \setminus G$ are such that $q + r = r + q = p$, it must be the case that $q, r \in G + p$. The following definition captures an even stronger property than the one just mentioned.

Definition 1.2 If $p \in \beta G$ is an idempotent element, we say that p has the *trivial sums property* if whenever $q, r \in \beta G$ are such that $q + r = p$, then it must be the case that $q, r \in G + p$.

Note that 0 always has the trivial sums property, because G^* is an ideal of βG. Idempotents satisfying the trivial sums property would be examples of so-called *maximal idempotents*, *i.e.,* maximal elements with respect to the two partial orders $\leq_R, \leq_L$ defined among idempotents by $q \leq_R r$ if and only if $r + q = q$ and $q \leq_L r$ if and only if $q + r = q$. It is possible to improve the result just mentioned for strongly summable ultrafilters if one strengthens the definition of strongly summable.

Definition 1.3 An ultrafilter $p \in \beta G$ is *sparse* if for every $A \in p$ there exist two sequences $\vec{x} = \langle x_n \mid n < \omega \rangle$, $\vec{y} = \langle y_n \mid n < \omega \rangle$, where $\vec{y}$ is a subsequence of $\vec{x}$ such that $\{x_n \mid n < \omega\} \setminus \{y_n \mid n < \omega\}$ is infinite, $\mathrm{FS}(\vec{x}) \subseteq A$, and $\mathrm{FS}(\vec{y}) \in p$.

Then, obviously, every sparse ultrafilter will be nonprincipal and strongly summable. And by [8, Th. 4.5], if G can be embedded in $\mathbb{T}$ and $p \in G^*$ is sparse, then p has the trivial sums property. In some non-commutative settings (adapting the relevant definitions appropriately), the relationship between sparseness and an analogue of the trivial sums property has been further explored (see [7]).

It follows from results of Krautzberger ([11, Props. 4 and 5, and Th. 4]) that every nonprincipal strongly summable ultrafilter $p \in \mathbb{N}^*$ must actually be sparse. Thus the previous theorem holds for nonprincipal strongly summable ultrafilters on $\mathbb{N}$; *i.e.*, every such ultrafilter, being sparse, has the trivial sums property. In [9], the authors followed this idea and started investigating the different kinds of abelian semigroups on which every nonprincipal strongly summable ultrafilter must be sparse. In particular, [9, Th. 4.2] establishes that if S is a countable subsemigroup of $\mathbb{T}$, then every nonprincipal strongly summable ultrafilter on S is sparse, so this generalizes the previous observation about strongly summable ultrafilters on $\mathbb{N}$. The authors built on this result to get a more general result ([9, Th. 4.5 and Cor. 4.6]) outlining a large class of abelian groups, whose nonprincipal strongly summable ultrafilters must all be sparse. More or less concurrently, this author showed ([4, Th. 2.1]) that every nonprincipal strongly summable ultrafilter on the Boolean group is also sparse. Thus Hindman, Steprāns, and Strauss ([9, Question 4.12]) asked whether every strongly summable ultrafilter on a countable abelian group is sparse.

Although it is not immediately clear that, for groups that are not embeddable in $\mathbb{T}$, sparseness implies the trivial sums property, Hindman, Steprāns, and Strauss were able to get a result, analogous to the ones mentioned in the previous paragraph, concerning the latter property; namely, they proved ([9, Th. 4.8 and Cor. 4.9]) that for the same class of abelian groups, all nonprincipal strongly summable ultrafilters must have the trivial sums property. The analogous result for the Boolean group had already been proved by Protasov ([13, Cor. 4.4]). Thus, Hindman, Steprāns, and Strauss ([9, Question 4.11]) also asked whether every strongly summable ultrafilter on a countable abelian group G has the property that it can only be expressed trivially as a product (*i.e.*, a sum) in G^*.

Section 2 develops some preliminary results that deal with union ultrafilters, additive isomorphisms, and what we call here the 2-uniqueness of finite sums. Section 3 contains the answer to the two questions from [9] mentioned in the previous paragraphs. From the proof of this result, it will turn out that, unless p is a strongly summable ultrafilter on the Boolean group, it will be additively isomorphic to a union ultrafilter. Thus Section 4 deals with the Boolean group, the main result being that, under the assumption that $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$ (this is, under Martin's Axiom for countable forcing notions), there exists a strongly summable ultrafilter on the Boolean group that is not additively isomorphic to any union ultrafilter.

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2 Union Ultrafilters and 2-uniqueness of Finite Sums

Union ultrafilters were first defined by Blass in [1, p. 92], an article that appeared in the same volume as that of Hindman [6], where strongly summable ultrafilters are first defined. So ever since their inception, the notions of union ultrafilter and of strongly summable ultrafilter have always been inextricably related. The results of this paper are no exception, and the notion of union ultrafilter is essential to them. We thus introduce this notion. For a pairwise disjoint family $X \subseteq [\omega]^{<\omega}$, we denote the set of its *finite unions* by

$$\mathrm{FU}(X) = \Big\{ \bigcup_{x \in a} x \mid a \in [X]^{<\omega} \setminus \{\varnothing\} \Big\}.$$

Definition 2.1 A *union ultrafilter* is an ultrafilter p on $[\omega]^{<\omega}$ such that for every $A \in p$ it is possible to find a pairwise disjoint $X \subseteq [\omega]^{<\omega}$ such that $p \ni \mathrm{FU}(X) \subseteq A$.

The reason union ultrafilters are so important when studying strongly summable ultrafilters, is that sometimes strongly summable ultrafilters can be used to construct union ultrafilters, which in turn are often easier to handle. We will state a definition that captures the precise sense in which strongly summable ultrafilters give rise to union ultrafilters. In order to do this, we need to introduce a further notion, which stems from the fact that when dealing with sets of the form $\mathrm{FS}(\vec{x})$, if each finite sum from this set can be expressed uniquely as such, then the situation is much more comfortable. To simplify notation, we make the convention that for any sequence $\vec{x}$ of elements of some abelian group G, the *empty sum* equals zero:

$$\sum_{n \in \varnothing} x_n = 0.$$

Definition 2.2 A sequence $\vec{x}$ on an abelian group G is said to satisfy *uniqueness of finite sums* if whenever $a, b \in [\omega]^{<\omega}$ are such that

$$\sum_{n \in a} x_n = \sum_{n \in b} x_n,$$

it must be the case that $a = b$.

In particular, if $\vec{x}$ satisfies uniqueness of finite sums, then $0 \notin \mathrm{FS}(\vec{x})$. Now we are ready to introduce the notion that will provide the connection between strongly summable ultrafilters and union ultrafilters.

Definition 2.3 Let p be an ultrafilter on an abelian group G and let q be a union ultrafilter. We say that p and q are *additively isomorphic* if there is a sequence $\vec{x}$ of elements of G satisfying uniqueness of finite sums, such that $\mathrm{FS}(\vec{x}) \in p$, and there is a pairwise disjoint family $Y = \{y_n \mid n < \omega\}$ of elements of $[\omega]^{<\omega}$, in such a way that the mapping $\phi\colon \mathrm{FS}(\vec{x}) \mapsto \mathrm{FU}(Y)$ given by $\phi(\sum_{n \in a} x_n) = \bigcup_{n \in a} y_n$ maps p to q.

If we are only interested in determining whether a given strongly summable ultrafilter p is additively isomorphic to *some* union ultrafilter, without worrying about which ultrafilter, then we can assume without loss of generality that the isomorphism is fairly simple. This is established formally and precisely in the following proposition.

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Proposition 2.4 *If p is additively isomorphic to a union ultrafilter, and this is witnessed by the mapping $\sum_{n\in a} x_n \mapsto \bigcup_{n\in a} y_n$ from $\mathrm{FS}(\vec{x})$ to $\mathrm{FU}(Y)$, then the mapping $\psi\colon \mathrm{FS}(\vec{x}) \to [\omega]^{<\omega}$ given by $\psi(\sum_{n\in a} x_n) = a$ also maps p to a union ultrafilter.*

Proof We only need to show that for any union ultrafilter q and any pairwise disjoint $Y = \{y_n \mid n < \omega\}$ such that $\mathrm{FU}(Y) \in q$, the mapping ϕ given by $\bigcup_{n\in a} y_n \mapsto a$ maps q to another union ultrafilter. Once we prove this, then given the hypothesis of the theorem we can simply compose the mapping ϕ with the original isomorphism to get the ψ that we need. So let r be the image of q under such mapping, and let $A \in r$. Then since $B = \phi^{-1}[A] \in q$, there is a pairwise disjoint X such that $q \ni \mathrm{FU}(X) \subseteq B \cap \mathrm{FU}(Y)$. Since X is pairwise disjoint and contained in $\mathrm{FU}(Y)$, it is readily checked that for distinct $x, w \in X$, if $x = \bigcup_{n\in a} y_n$ and $w = \bigcup_{n\in b} y_n$, then $a \cap b = \varnothing$. Hence the family $Z = \{a \in [\omega]^{<\omega} \mid \bigcup_{n\in a} y_n \in X\}$ is pairwise disjoint. Note, moreover, that all finite unions are preserved in the sense that, for $x_0, \dots, x_n \in X$ such that $x_i = \bigcup_{k\in a_i} y_k$, we have that $\bigcup_{i=0}^n x_i = \bigcup_{k\in a} y_k$, where

$$a = \bigcup_{i=0}^n a_i, \textit{i.e.}, \phi\Big(\bigcup_{i=0}^n x_i\Big) = \bigcup_{i=0}^n \phi(x_i).$$

This means that $\phi[\mathrm{FU}(X)] = \mathrm{FU}(Z)$, thus $r \ni \mathrm{FU}(Z) \subseteq A$, and we are done. ∎

We will develop a useful criterion for knowing when a strongly summable ultrafilter is additively isomorphic to some union ultrafilter. For that, it will be helpful to think of the uniqueness of finite sums as a 1-*uniqueness of finite sums*, in the sense that the expressions under consideration only have coefficients equal to 1. With this in mind, it is natural to try and define a corresponding 2-uniqueness where we allow coefficients 1 and 2. More formally, we have the following definition.

Definition 2.5 A sequence $\vec{x}$ on an abelian group G is said to satisfy the 2-*uniqueness of finite sums* if whenever $a, b \in [\omega]^{<\omega}$ and $\varepsilon\colon a \to \{1,2\}$, $\delta\colon b \to \{1,2\}$ are such that

$$\sum_{n\in a} \varepsilon(n) x_n = \sum_{n\in b} \delta(n) x_n,$$

it must be the case that $a = b$ and $\varepsilon = \delta$.

In particular, if $\vec{x}$ satisfies 2-uniqueness of finite sums, then no element of $\mathrm{FS}(\vec{x})$ can have order 2. Thus, Boolean groups do not contain sequences satisfying 2-uniqueness of finite sums. It is of course possible to analogously define n-uniqueness of finite sums, for every n, but for the results of this paper we only need to consider the case $n = 2$.

Proposition 2.6 *For a sequence $\vec{x}$ on an abelian group G, the following are equivalent.*

(i) *$\vec{x}$ satisfies the 2-uniqueness of finite sums.*
(ii) *Whenever $a, b, c, d \in [\omega]^{<\omega}$ are such that $a \cap b = \varnothing = c \cap d$, if*

$$2\sum_{n\in a} x_n + \sum_{n\in b} x_n = 2\sum_{n\in c} x_n + \sum_{n\in d} x_n,$$

then $a = c$ and $b = d$.

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(iii) *Whenever $a, b, c, d \in [\omega]^{<\omega}$ are such that*

$$\sum_{n\in a} x_n + \sum_{n\in b} x_n = \sum_{n\in c} x_n + \sum_{n\in d} x_n,$$

it must be the case that $a \triangle b = c \triangle d$ and $a \cap b = c \cap d$.

Proof The proof is straightforward. ∎

The following two theorems do not contain any new ideas, but rather they are just a useful reformulation of [9, Th. 3.2] (although that theorem uses a condition that is slightly weaker than the 2-uniqueness of finite sums, namely what the authors call the "strong uniqueness of finite sums"; however, the version that we present here will be enough for our purposes) that divides it into two parts, each of which will be of some use in the future. We think that the distinction made here is illuminating.

Theorem 2.7 *Let p be a strongly summable ultrafilter such that for some $\vec{x}$ satisfying 2-uniqueness of finite sums, $\mathrm{FS}(\vec{x}) \in p$. Then p is additively isomorphic to a union ultrafilter.*

Proof We just need to check that the mapping ϕ given by $\phi(\sum_{n\in a} x_n) = a$ sends p to a union ultrafilter. So let $A \in q = \phi(p)$. Pick a sequence $\vec{y}$ such that $p \ni \mathrm{FS}(\vec{y}) \subseteq \phi^{-1}[A]$. Then $\phi[\mathrm{FS}(\vec{y})] \subseteq A$. Now $\phi^{-1}[A] \subseteq \mathrm{FS}(\vec{x})$; thus, for each $n < \omega$ we can define $c_n \in [\omega]^{<\omega}$ by $c_n = \phi(y_n)$ or, equivalently, by $y_n = \sum_{i\in c_n} x_i$. We claim that the family $C = \{c_n \mid n < \omega\}$ is pairwise disjoint. This is because if $n \neq m$, since $y_n + y_m \in \mathrm{FS}(\vec{y}) \subseteq \mathrm{FS}(\vec{x})$, then there must be a $c \in [\omega]^{<\omega}$ such that

$$\sum_{i\in c} x_i = y_n + y_m = \sum_{i\in c_n} x_i + \sum_{i\in c_m} x_i.$$

Since $\vec{x}$ satisfies 2-uniqueness of finite sums, by Proposition 2.6 we can conclude that $c = c_n \cup c_m$ and $c_n \cap c_m = \varnothing$. This argument shows at once that C is a pairwise disjoint family, and that $\phi(y_n + y_m) = c_n \cup c_m = \phi(y_n) \cup \phi(y_m)$. From this, it is easy to prove by induction that $\phi(\sum_{n\in a} y_n) = \bigcup_{n\in a} \phi(y_n)$, for all $a \in [\omega]^{<\omega}$, hence $\phi[\mathrm{FS}(\vec{y})] = \mathrm{FU}(C)$, therefore $q \ni \mathrm{FU}(C) \subseteq A$, and we are done. ∎

Theorem 2.8 *Let p be an ultrafilter that is additively isomorphic to a union ultrafilter. Then p is sparse.*

Proof If p is additively isomorphic to some union ultrafilter, by Proposition 2.4 we can pick a sequence $\vec{x}$ satisfying uniqueness of finite sums such that $\mathrm{FS}(\vec{x}) \in p$, and such that the mapping ϕ given by $\phi(\sum_{n\in a} x_n) = a$ maps p to a union ultrafilter q. Let $A \in p$, and let X be pairwise disjoint such that $q \ni \mathrm{FU}(X) \subseteq \phi[A \cap \mathrm{FS}(\vec{x})]$. Now let $M = \bigcup X$. Since q is a union ultrafilter, [11, Th. 4] (*cf.* also [9, Th. 2.6]) ensures that there is $B \in q$ such that $M \setminus \bigcup B$ is infinite. Without loss of generality we can assume $B \subseteq \mathrm{FU}(X)$, so that $\bigcup B$ is a coinfinite subset of M. Grab a pairwise disjoint family Y such that $q \ni \mathrm{FU}(Y) \subseteq B$; then $\bigcup Y$ is a coinfinite subset of $M = \bigcup X$, and thus there are infinitely many $x \in X$ that do not intersect $\bigcup Y$ (because $Y \subseteq \mathrm{FU}(X)$ and X is a pairwise disjoint family, so if $x \in X$ intersects $\bigcup Y$ then $x \subseteq \bigcup Y$). Thus, if we let $Z = \{x \in X \mid x \cap \bigcup Y = \varnothing\} \cup Y$, then Z is a pairwise disjoint family and

$\mathrm{FU}(Z) \subseteq \mathrm{FU}(X) \subseteq \phi[A \cap \mathrm{FS}(\vec{x})]$. Enumerate $Z = \{z_n \mid n < \omega\}$ in such a way that $Y = \{z_{2n} \mid n < \omega\}$ and $\{x \in X \mid x \cap \bigcup Y = \varnothing\} = \{z_{2n+1} \mid n < \omega\}$. Then let $\vec{w}$ be given by $w_n = \sum_{i \in z_n} x_i$. We get that $\mathrm{FS}(\vec{w}) = \phi^{-1}[\mathrm{FU}(Z)] \subseteq A$, and if $\vec{y}$ is the subsequence of even elements of $\vec{w}$, then we will have that $|\{w_n \mid n < \omega\} \setminus \{y_n \mid n < \omega\}|$ is infinite and $\mathrm{FS}(\vec{y}) = \phi^{-1}[\mathrm{FU}(Y)] \in p$. ∎

Corollary 2.9 ([9, Th. 3.2]) *Let p be a strongly summable ultrafilter on some abelian group G such that there exists a sequence $\vec{x}$ satisfying the 2-uniqueness of finite sums with $\mathrm{FS}(\vec{x}) \in p$. Then p is sparse.* ∎

To finish this section, we quote another result from [9] that will be relevant in the subsequent section, and that illustrates another application of the concept of 2-uniqueness of finite sums.

Theorem 2.10 ([9, Th. 4.8]) *Let G be an abelian group, and $p \in G^*$ be a strongly summable ultrafilter such that there exists a sequence $\vec{x}$ satisfying the 2-uniqueness of finite sums, with $\mathrm{FS}(\vec{x}) \in p$. Then p has the trivial sums property.*

3 Strongly Summable Ultrafilters are Sparse and have the Trivial Sums Property

The main result of this section tells us that almost all strongly summable ultrafilters on abelian groups have FS-sets generated from sequences that satisfy 2-uniqueness of finite sums. As a consequence, almost all strongly summable ultrafilters on abelian groups are essentially union ultrafilters (because of Theorem 2.7), and this helps solve [9, Questions 4.11 and 4.12]. More precisely, we have the following theorem and corollary.

Theorem 3.1 *Let G be an abelian group, and let $p \in G^*$ be a strongly summable ultrafilter such that*

$$\{x \in G \mid o(x) = 2\} \notin p.$$

Then there exists a sequence $\vec{x}$ of elements of G satisfying the 2-uniqueness of finite sums such that $\mathrm{FS}(\vec{x}) \in p$.

Corollary 3.2 *Let G be an abelian group and let $p \in G^*$ be a strongly summable ultrafilter such that*

$$\{x \in G \mid o(x) = 2\} \notin p.$$

Then p is additively isomorphic to some union ultrafilter.

In order to prove this result, we will need to break the proof down into several subcases.

Lemma 3.3 *Let G be an abelian group, and let $X = \{x \in G \mid o(x) = 4\}$. If $\vec{x}$ is a sequence of elements of G such that $\mathrm{FS}(\vec{x}) \subseteq X$, then $\vec{x}$ must satisfy 2-uniqueness of finite sums.*

Proof Assume that $\vec{x}$ is such that $\mathrm{FS}(\vec{x}) \subseteq X$. By Proposition 2.6, in order to prove that $\vec{x}$ satisfies 2-uniqueness of finite sums, it suffices to show that whenever a, b, c, d are such that $a \cap b = \varnothing = c \cap d$ and

$$2\sum_{n \in a} x_n + \sum_{n \in b} x_n = 2\sum_{n \in c} x_n + \sum_{n \in d} x_n,$$

then $a = c$ and $b = d$. Now, for each $n \in b \cap d$ we can cancel the term x_n from both sides of the previous equation, and similarly for each $n \in a \cap c$, we can cancel the term $2x_n$ from both sides of the equation, which thus becomes

$$2\sum_{n \in a'} x_n + \sum_{n \in b'} x_n = 2\sum_{n \in c'} x_n + \sum_{n \in d'} x_n, \tag{3.1}$$

where $a' = a \smallsetminus (a \cap c)$, $b' = b \smallsetminus (b \cap d)$, $c' = c \smallsetminus (a \cap c)$, and $d' = d \smallsetminus (b \cap d)$. Since b' is disjoint from d', equation (3.1) yields

$$\sum_{n \in b' \cup d'} x_n = \sum_{n \in b'} x_n + \sum_{n \in d'} x_n = -2\sum_{n \in a'} x_n + 2\sum_{n \in c'} x_n + 2\sum_{n \in d'} x_n,$$

where the right-hand side is either the identity or has order 2, while the left-hand side is either the identity or has order 4. Hence both sides of this equation must be the identity, and so $b' \cup d' = \varnothing$; that is, $b' = d' = \varnothing$ and hence $b = b \cap d = d$. Therefore, (3.1) becomes

$$2\sum_{n \in a'} x_n = 2\sum_{n \in c'} x_n,$$

which in turn implies that

$$2\sum_{n \in a' \cup c'} x_n = 4\sum_{n \in c'} x_n = 0,$$

and this can only happen if $a' \cup c' = \varnothing$, which means that $a' = c' = \varnothing$, and hence $a = a \cap c = d$. So we have that $\vec{x}$ satisfies 2-uniqueness of finite sums. ∎

If G is any abelian group, and $p \in G^*$ is strongly summable, then there must be a countable subgroup H such that $H \in p$ (*e.g.*, take any FS set in p because of strong summability, and then let H be the subgroup generated by such FS set), and certainly the restricted ultrafilter $p \upharpoonright H = p \cap \mathfrak{P}(H)$ will also be strongly summable. If we prove that $p \upharpoonright H$ contains a set of the form $\mathrm{FS}(\vec{x})$ for a sequence $\vec{x}$ satisfying 2-uniqueness of finite sums, then certainly so does p itself, because p is just the ultrafilter generated in G by $p \upharpoonright H$ and in particular $p \upharpoonright H \subseteq p$. Hence, in order to prove Theorem 3.1, it suffices to consider only countable abelian groups G, and we will do so in the remainder of this section.

Now, it is a well-known result (this is mentioned in [8, p. 123, Sect. 1], and thoroughly discussed at the beginning of [4, Section 3]) that every countable abelian group G can be embedded in a countable direct sum of circle groups $\bigoplus_{n<\omega} \mathbb{T}$. Thus, from now on we will use this fact liberally; in particular, all elements x of the abelian group under consideration will be thought of as ω-sequences, each of whose terms is an element of $\mathbb{T}$. We will denote by π_n the projection map onto the n-th coordinate; *i.e.,* $\pi_n(x)$ is the n-th term of the sequence that x represents.

Definition 3.4 When dealing with an arbitrary (countable) abelian group G, we will denote by $Q(G) = \{x \in G \mid o(x) > 4\}$. Since elements of G are elements of

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$\bigoplus_{n<\omega}\mathbb{T}$, if $x \in Q(G)$, then there is an $n < \omega$ such that $\pi_n(x) \notin \left\{0, \frac{1}{4}, -\frac{1}{4}, \frac{1}{2}\right\}$. We will denote the least such n by $\rho(x)$.

At this point, it is worth recalling the following theorem of Hindman, Steprāns, and Strauss.

Theorem 3.5 ([9, Th. 4.5]) *Let S be a countable subsemigroup of $\bigoplus_{n<\omega}\mathbb{T}$, and let p be a nonprincipal strongly summable ultrafilter on S. If*

$$\left\{x \in S \mid \pi_{\min(x)}(x) \neq \tfrac{1}{2}\right\} \in p,$$

where $\min(x)$ denotes the least n such that $\pi_n(x) \neq 0$, then there exists a set $X \in p$ such that for every sequence $\vec{x}$ of elements of $\bigoplus_{n<\omega}\mathbb{T}$, if $\mathrm{FS}(\vec{x}) \subseteq X$, then $\vec{x}$ must satisfy 2-uniqueness of finite sums.

This theorem is the tool that will allow us to prove the following lemma.

Lemma 3.6 *Let G be an abelian group, and let $p \in G^*$ be a strongly summable ultrafilter. If*

$$\left\{x \in Q(G) \mid \pi_{\rho(x)}(x) \notin \left\{\tfrac{1}{8}, -\tfrac{1}{8}, \tfrac{3}{8}, -\tfrac{3}{8}\right\}\right\} \in p,$$

then there exists a set $X \in p$ such that for every sequence $\vec{x}$ of elements of $\bigoplus_{n<\omega}\mathbb{T}$, if $\mathrm{FS}(\vec{x}) \subseteq X$ then $\vec{x}$ must satisfy 2-uniqueness of finite sums.

Proof Consider the morphism $\phi\colon G \to G \subseteq \bigoplus_{n<\omega}\mathbb{T}$ given by $\phi(x) = 4x$, whose kernel is exactly $G \setminus Q(G)$. Since the latter is not an element of p, then $\phi(p)$ is a nonprincipal ultrafilter. Moreover, since p is strongly summable, so is $\phi(p)$ by [9, Lemma 4.4]. Now notice that for $x \in G\setminus\ker(\phi) = Q(G)$, we have $\rho(x) = \min(\phi(x))$. Thus $\phi(p)$ contains the set $\{x \in G \setminus \{0\} \mid \pi_{\min(x)}(x) \neq 1/2\}$, since its preimage under ϕ is exactly

$$\left\{x \in Q(G) \mid \pi_{\rho(x)}(x) \notin \left\{\tfrac{1}{8}, -\tfrac{1}{8}, \tfrac{3}{8}, -\tfrac{3}{8}\right\}\right\}.$$

Therefore, by Theorem 3.5, there is a set $Y \in \phi(p)$ such that whenever $\mathrm{FS}(\vec{y}) \subseteq Y$, $\vec{y}$ must satisfy 2-uniqueness of finite sums. If we let $X = \phi^{-1}[Y]$, we claim that $X \in p$ is the set that we need. So let $\vec{x}$ be a sequence such that $\mathrm{FS}(\vec{x}) \subseteq X$. Then letting $\vec{y}$ be the sequence given by $y_n = \phi(x_n)$, since ϕ is a group homomorphism, we get that $\mathrm{FS}(\vec{y}) = \phi[\mathrm{FS}(\vec{x})] \subseteq \phi[X] \subseteq Y$; thus, $\vec{y}$ must satisfy 2-uniqueness of finite sums. Again, since ϕ is a group homomorphism, it is not hard to see that this implies that $\vec{x}$ satisfies 2-uniqueness of finite sums as well, and we are done. ■

The following theorem is the last piece needed for proving Theorem 3.1.

Theorem 3.7 *Let G be an abelian group, and let $p \in G^*$ be a strongly summable ultrafilter. If*

$$\left\{x \in Q(G) \mid \pi_{\rho(x)}(x) \in \left\{\tfrac{1}{8}, -\tfrac{1}{8}, \tfrac{3}{8}, -\tfrac{3}{8}\right\}\right\} \in p,$$

then there exists a set $X \in p$ such that for every sequence $\vec{x}$ of elements of $\bigoplus_{n<\omega}\mathbb{T}$, if $\mathrm{FS}(\vec{x}) \subseteq X$, then $\vec{x}$ must satisfy 2-uniqueness of finite sums.

Proof If $p \in G^*$ is as described in the hypothesis, then there is an $i \in \{1, -1, 3, -3\}$ such that

$$Q_i = \left\{x \in Q(G) \mid \pi_{\rho(x)}(x) = \tfrac{i}{8}\right\} \in p.$$

Let $\vec{x}$ be such that $p \ni \mathrm{FS}(\vec{x}) \subseteq Q_i$. For $j < \omega$ let $M_j = \{n < \omega \mid \rho(x_n) = j\}$.

Claim 3.8 For each $j < \omega$, $|M_j| \le 2$.

Proof of Claim Assume, by way of contradiction, that there are three distinct $n, m, k \in M_j$, and let $x = x_n + x_m + x_k$. For $l < j$, $\pi_l(x)$ must be an element of $\left\{0, \frac{1}{4}, -\frac{1}{4}, \frac{1}{2}\right\}$, because so are $\pi_l(x_n), \pi_l(x_m)$ and $\pi_l(x_k)$. On the other hand, $\pi_j(x_n) = \pi_j(x_m) = \pi_j(x_k) = \frac{i}{8}$, so $\rho(x) = j$, but $\pi_j(x) = \frac{3i}{8} \ne \frac{i}{8}$. ∎

Thus, we can rearrange the sequence $\vec{x}$ in such a way that $n < m$ implies $\rho(x_n) \le \rho(x_m)$, where the inequality is strict if $m > n + 1$. Let $M = \{\rho(x_n) \mid n < \omega\}$.

Claim 3.9 Let $n < m < \omega$ and assume that $j = \rho(x_n) < \rho(x_m)$ (which may or may not hold if $m = n + 1$, but must hold if $m > n + 1$). Then $\pi_j(x_m) = 0$.

Proof of Claim Let $x = x_n + x_m$. Arguing as in the proof of Claim 3.8, we get that $\rho(x) = j$, and thus since $x \in Q_i$, $\pi_j(x_n) + \pi_j(x_m) = \pi_j(x) = \frac{i}{8}$. Now on the one hand we know that $\pi_j(x_m) \in \left\{0, \frac{1}{4}, -\frac{1}{4}, \frac{1}{2}\right\}$, while on the other hand $\pi_j(x_n) = \frac{i}{8}$. Hence, the only possibility that does not lead to contradiction is that $\pi_j(x_m) = 0$. ∎

Claim 3.10 For every $x \in \mathrm{FS}(\vec{x})$ there is a $j \in M$ such that $\pi_j(x) \ne 0$. Moreover, for the least such j we actually have that $\pi_j(x) \in \left\{\frac{i}{8}, \frac{2i}{8}\right\}$.

Proof of Claim For if $x = \sum_{n \in a} x_n$ and if $m = \min(a)$, then we can let $j = \rho(x_m) \in M$ so that for every $n \in a$, we have $\rho(x_n) \ge j$, with a strict inequality if $n > m + 1$. Now, we have that

$$\pi_j(x) = \sum_{n \in a} \pi_j(x),$$

where, by Claim 3.9, each of the terms on the right-hand side of this expression are zero, except for $\pi_j(x_m) = \frac{1}{8}$ and possibly $\pi_j(x_{m+1})$ (which will appear on the summation only if $m+1 \in a$, and if so it will equal $\frac{1}{8}$ if $\rho(x_{m+1}) = \rho(x_m)$, and zero otherwise). Thus, $\pi_j(x) \in \left\{\frac{i}{8}, \frac{2i}{8}\right\}$. In particular, $\pi_j(x) \ne 0$. Now in order to prove the "moreover" part, we will argue that for all $l < j$ such that $l \in M$, $\pi_l(x) = 0$. This is because if $l \in M$, then there is $k < \omega$ such that $\rho(x_k) = l$, and if $l < j$, then we must necessarily have $k < m$ because of the way we arranged our sequence $\vec{x}$. Hence, again by Claim 3.9 and since $m = \min(a)$, it will be the case that $\pi_l(x_n) = 0$ for all $n \in a$, and hence

$$\pi_l(x) = \sum_{n \in a} \pi_l(x_n) = 0.$$

Therefore, j is actually the least $l \in M$ such that $\pi_l(x) \ne 0$, and we are done. ∎

The previous claim allows us to define $\tau \colon \mathrm{FS}(\vec{x}) \mapsto M$ by

$$\tau(x) = \min\{j \in M \mid \pi_j(x) \ne 0\},$$

and ensures that $\pi_{\tau(x)}(x) \in \left\{\frac{i}{8}, \frac{2i}{8}\right\}$. We can thus let

$$C_k = \left\{x \in \mathrm{FS}(\vec{x}) \mid \pi_{\tau(x)}(x) = \frac{ki}{8}\right\}$$

for $k \in \{1,2\}$, and choose from among those k such that $C_k \in p$. We let $X = C_k$ and claim that X is as in the conclusion of the theorem. In order to see this, let $\vec{y}$ be such that $\mathrm{FS}(\vec{y}) \subseteq C_k$.

Notice first that for distinct $n, m < \omega$ we must have $\tau(y_n) \neq \tau(y_m)$, for otherwise we would get, arguing in a similar way as in the proofs of Claims 3.8 and 3.9, that

$$\tau(y_n + y_m) = \tau(y_n) = \tau(y_m) \quad \text{and} \quad \pi_{\tau(y_n+y_m)}(y_n + y_m) = \frac{2ki}{8} \neq \frac{ki}{8},$$

a contradiction. Thus, by rearranging $\vec{y}$ if necessary, we can assume that $n < m$ implies $\tau(y_n) < \tau(y_m)$.

Now an observation is in order. Consider $a \in [\omega]^{<\omega} \setminus \varnothing$ and $\varepsilon\colon a \to \{1,2\}$. Let $m = \min(a)$ and $j = \tau(y_m)$. Since τ is increasing on $\vec{y}$, $\pi_j(y_n) = 0$ for all $n \in a \setminus \{m\}$, while $\pi_j(y_m) = \frac{ki}{8}$. Thus

$$\pi_j\Big(\sum_{n\in a} \varepsilon(n) y_n\Big) = \varepsilon(m)\frac{ki}{8} \neq 0.$$

From this we can conclude that $\vec{y}$ satisfies 2-uniqueness of finite sums. Assume that $a, b \in [\omega]^{<\omega}$ and $\varepsilon\colon a \to \{1,2\}$, $\delta\colon b \to \{1,2\}$ are such that

$$\sum_{n\in a} \varepsilon(n) x_n = \sum_{n\in b} \delta(n) x_n. \tag{3.2}$$

We will proceed by induction on $\min\{|a|, |b|\}$. If $a = b = \varnothing$, we are done. Otherwise, let $m = \min(a \cup b)$. Assume without loss of generality that $m \in a$, so that $m = \min(a)$. Let $j = \tau(y_m)$. Then by the previous observation, the value of each side of (3.2) under π_j is nonzero, while $\pi_j(y_n) = 0$ for all $n > m$. Thus, by looking at the right-hand side of (3.2) we conclude that we must have $m \in b$ as well. Then it is also the case that $\min(b) = m$. Now again, by the observation from last paragraph, we get that the value of each side of (3.2) under the function π_j must equal, at the same time, $\varepsilon(m)\frac{ki}{8}$ and $\delta(m)\frac{ki}{8}$. This can only happen if $\varepsilon(m) = \delta(m)$; therefore, we can cancel the term $\varepsilon(m) y_m$ from both sides of (3.2) and get

$$\sum_{n\in a\setminus\{m\}} \varepsilon(n) x_n = \sum_{n\in b\setminus\{m\}} \delta(n) x_n.$$

Now we can apply the inductive hypothesis and conclude that $a \setminus \{m\} = b \setminus \{m\}$ and $\varepsilon \upharpoonright (a \setminus \{m\}) = \delta \upharpoonright (b \setminus \{m\})$. Since m is an element of both a and b, with $\varepsilon(m) = \delta(m)$, we have proved that $a = b$ and $\varepsilon = \delta$, and we are done. ∎

Proof of Theorem 3.1 Let G be an abelian group and let $p \in G^*$ be a strongly summable ultrafilter such that $\{x \in G \mid o(x) = 2\} \notin p$. Since p is nonprincipal and the only $x \in G$ with $o(x) = 1$ is 0, we have that $B = \{x \in G \mid o(x) > 2\} \in p$. If $C = \{x \in G \mid o(x) = 3\} \in p$, then notice that, since $C \subseteq \{x \in G \mid \pi_{\min(x)}(x) \neq \frac{1}{2}\}$ (because $C = \{x \in G \mid (\forall n < \omega)(\pi_n(x) \in \{0, \frac{1}{3}, -\frac{1}{3}\})\}$), we can apply Theorem 3.5 and get an $X \in p$ such that, if $\vec{x}$ is such that $\mathrm{FS}(\vec{x}) \subseteq X$ (and there is such an $\vec{x}$ with $\mathrm{FS}(\vec{x}) \in p$ because of strong summability), then $\vec{x}$ must satisfy 2-uniqueness of finite

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sums. If $D = \{x \in G \mid o(x) = 4\} \in p$, then we can pick a sequence $\vec{x}$ such that $p \ni \mathrm{FS}(\vec{x}) \subseteq D$, so by Lemma 3.3 this sequence must satisfy 2-uniqueness of finite sums, and we are done. Otherwise, if $C \notin p$ and $D \notin p$, then

$$Q(G) = \{x \in G \mid o(x) > 4\} = (G \setminus D) \cap (G \setminus C) \cap B \in p.$$

Now $Q(G) = Q_0 \cup Q_1$, where

$$Q_0 = \left\{ x \in Q(G) \mid \pi_{\rho(x)}(x) \notin \left\{ \frac{1}{8}, -\frac{1}{8}, \frac{3}{8}, -\frac{3}{8} \right\} \right\},$$

and

$$Q_1 = \left\{ x \in Q(G) \mid \pi_{\rho(x)}(x) \in \left\{ \frac{1}{8}, -\frac{1}{8}, \frac{3}{8}, -\frac{3}{8} \right\} \right\},$$

so pick $i \in 2$ such that $Q_i \in p$. If $i = 0$ apply Lemma 3.6 and if $i = 1$ apply Theorem 3.7, in either case, there is an $X \in p$ such that whenever $\vec{x}$ is such that $\mathrm{FS}(\vec{x}) \subseteq X$, then $\vec{x}$ must satisfy 2-uniqueness of finite sums. By strong summability of p there is such a sequence $\vec{x}$ that additionally satisfies $\mathrm{FS}(\vec{x}) \in p$, and we are done. ■

Corollary 3.11 ([9, Question 4.12]) *Let p be a nonprincipal strongly summable ultrafilter on an abelian group G. Then p is sparse.*

Proof Let G be any abelian group and let $p \in G^*$ be a strongly summable ultrafilter. Let

$$B = \{x \in G \mid o(x) \leq 2\}.$$

Then B is a subgroup of G. If $B \in p$, then since p is nonprincipal, B must be infinite; and since G is countable, B must be isomorphic to the (unique up to isomorphism) countably infinite Boolean group. Consider the restricted ultrafilter $q = p \upharpoonright B = p \cap \mathfrak{P}(B)$. Then q is also strongly summable, so q is a nonprincipal strongly summable ultrafilter on the Boolean group, and therefore by [4, Th. 2.1], it is sparse. It is easy to see that this implies that p is sparse as well. Thus, the only case that remains to be proved is when $B \notin p$, but this is handled by Theorem 3.1 together with Corollary 2.9, and we are done. ■

Corollary 3.12 ([9], Question 4.11) *Let p be a nonprincipal strongly summable ultrafilter on an abelian group G. Then p has the trivial sums property.*

Proof Let G be any abelian group ,and let $p \in G^*$ be a strongly summable ultrafilter. If p does not contain the subgroup $B = \{x \in G \mid o(x) \leq 2\}$, then we just need to apply Theorems 3.1 and 2.10. So assume that $B \in p$ and let $q, r \in \beta G$ be such that $q + r = p$. Then we have that

$$\{x \in G \mid B - x \in r\} \in q.$$

In particular, this set is nonempty, and so we can pick an $x \in G$ such that $B - x \in r$, or equivalently $B \in r + x$. Since $x \in G$ (hence it commutes with all ultrafilters), the equation $(q - x) + (r + x) = p$ holds; thus,

$$A = \{y \in G \mid B - y \in r + x\} \in q - x.$$

Notice that $A \subseteq B$, because if $y \in G$ is such that $B - y \in r + x$, then $B \cap (B - y) \in r + x$. In particular the latter set is nonempty, and so there are $z, w \in B$ such that $z = w - y$

which means that $y = w - z \in B$. Therefore, $B \in q - x$, so we can define $u = (q - x) \upharpoonright B$ and $v = (r + x) \upharpoonright B$. We then get that $u, v \in \beta B$ and $p \upharpoonright B \in B^*$ is a strongly summable ultrafilter such that $u + v = p \upharpoonright B$. Notice that in B, FS-sets are just subgroups from which the element 0 might have been removed; thus, the filter $\{A \cup \{0\} \mid A \in p \upharpoonright B\}$ has a base of subgroups, and hence it is the neighbourhood filter of 0 for some group topology. This means that $p \upharpoonright B$ satisfies the hypothesis of [13, Cor. 4.4], so it must be the case that $u, v \in B + p \upharpoonright B$. This is easily seen to imply that $q - x, r + x \in B + p$, and therefore, since $x \in G$, we conclude that $q, r \in G + p$, and we are done. ∎

4 The Boolean Group

Theorem 3.1 depends heavily on the hypothesis that the ultrafilter p at hand does not contain the subgroup $B(G) = \{x \in G \mid o(x) = 2\}$, since there are no sequences $\vec{x}$ satisfying the 2-uniqueness of finite sums in $B(G)$. Corollary 3.2 also has that $B(G) \notin p$ as a hypothesis, but it is not entirely clear *a priori* that this hypothesis is necessary for the result. The main objective of this section is to prove that we do in fact need such a hypothesis. That is, if $p \in G^*$ is strongly summable and $B(G) \in p$, then there is no guarantee that p is additively isomorphic to a union ultrafilter. For this, of course, we only need to consider the case where $B(G)$ is infinite (otherwise, the only ultrafilters that can contain it are the principal ones). And, as noted in the previous section, when dealing with strongly summable ultrafilters, we can assume without loss of generality that G (and hence $B(G)$) is countable. Since there is (up to isomorphism) only one countably infinite group all of whose nonidentity elements have order 2, it will be enough for our purposes to look at strongly summable ultrafilters in this group (which we will from now on simply call "the Boolean group"), by focusing our attention on the restricted ultrafilter $p \upharpoonright B(G)$.

We will choose a particularly nice "realization" of the Boolean group to work with. We think of the Boolean group as the set $\mathbb{B} = [\omega]^{<\omega}$ equipped with the symmetric difference $\triangle$ as group operation. Since every element of $\mathbb{B}$ has order 2, we have that for any sequence $\vec{x}$ of elements of $\mathbb{B}$, we can ignore the repeated elements from the sequence and still get the same set $\mathrm{FS}(\vec{x})$. Thus, we will talk about $\mathrm{FS}(X)$ for $X \subseteq \mathbb{B}$, and it is easy to see that for $p \in \mathbb{B}^*$, p is strongly summable if and only if for every $A \in p$ there is an infinite set $X \subseteq \mathbb{B}$ such that $p \ni \mathrm{FS}(X) \subseteq A$.

We will use the fact that $\mathbb{B}$ is a vector space over the field with two elements $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$ (scalar multiplication being the obvious one). Note that for $X \subseteq \mathbb{B}$, the subspace spanned (which in $\mathbb{B}$ coincides with the subgroup generated) by X is exactly $\mathrm{FS}(X) \cup \{\varnothing\}$, because nontrivial linear combinations (*i.e.,* linear combinations in which not all scalars equal zero) of elements of X are exactly finite sums (or symmetric differences) of elements of X. The following proposition, whose proof is obvious, tells us how do subsets $X \subseteq G$ satisfying uniqueness of finite sums look like.

Proposition 4.1 *For $X \subseteq G$, the following are equivalent:*

(i) *X satisfies uniqueness of finite sums.*
(ii) $\varnothing \notin \mathrm{FS}(X)$.
(iii) *X is linearly independent.*

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Thus, when we have a set $\mathrm{FS}(Y)$ such that Y is not linearly independent, we can always choose a basis X for the subspace $\mathrm{FS}(Y)$ spanned by Y, and we will have that $\mathrm{FS}(X) = \mathrm{FS}(Y) \smallsetminus \{\varnothing\}$. This means that, when considering sets of the form $\mathrm{FS}(X)$, we can assume without loss of generality that X is linearly independent. Another way to see this as follows: let $p \in B^*$ be a strongly summable ultrafilter, and let $A \in p$. Since p is nonprincipal, $\{\varnothing\} \notin p$ and hence $A \smallsetminus \{\varnothing\} \in p$. Therefore, we can choose an X such that $p \ni \mathrm{FS}(X) \subseteq A \smallsetminus \{0\}$, so $\mathrm{FS}(X) \subseteq A$ and X must be linearly independent.

Definition 4.2 For a linearly independent set $X \subseteq \mathbb{B}$, we define for an element $y \in \mathrm{FS}(X)$ the X-*support* of y, denoted by $X - \mathrm{supp}(y)$, as the (unique, by linear independence of X) finite set of elements of X whose sum equals y. That is,

$$y = \sum_{x \in X - \mathrm{supp}(y)} x.$$

If $Y \subseteq \mathrm{FS}(X)$, then we also define the X-support of Y as

$$X - \mathrm{supp}(Y) = \bigcup_{y \in Y} X - \mathrm{supp}(y).$$

Similarly, we define the X-support of a sequence of elements of $\mathrm{FS}(X)$ as the X-support of its range.

It will be convenient to stipulate the convention that $X - \mathrm{supp}(\varnothing) = \varnothing$. Then it is readily checked that the function $X - \mathrm{supp}\colon \mathrm{FS}(X) \cup \{\varnothing\} \to ([X]^{<\omega}, \triangle)$ is a group isomorphism (in fact, a linear transformation between the two vector spaces), in other words, $X - \mathrm{supp}(x \triangle y) = X - \mathrm{supp}(x) \triangle X - \mathrm{supp}(y)$ for all $x, y \in \mathrm{FS}(X)$, and more generally, $X - \mathrm{supp}\left(\sum_{x \in A} x\right) = \sum_{x \in A} X - \mathrm{supp}(x)$ for all $A \in [\mathrm{FS}(X)]^{<\omega}$. This is the really crucial feature of the X-support, and it will be used ubiquitously in what follows.

As an application of the previous definitions and properties, we will provide another proof of the fact that every strongly summable ultrafilter on $\mathbb{B}$ is sparse, much simpler than the original one from [4, Th. 2.1]. So let $p \in \mathbb{B}^*$ be a strongly summable ultrafilter, and let $A \in p$. Because of strong summability, there is an infinite linearly independent Z such that $p \ni \mathrm{FS}(Z) \subseteq A$.

Claim 4.3 There is a $B \in p$ such that for some infinite $W \subseteq Z$, $\mathrm{FS}(W) \cap B = \varnothing$.

The result follows easily from the claim. Just pick a linearly independent Y such that $p \ni \mathrm{FS}(Y) \subseteq B \cap \mathrm{FS}(Z)$, and let $X = Y \cup W$. Then it is straightforward to prove that X is linearly independent, since Y and W are linearly independent, and $\mathrm{FS}(W)$ is disjoint from $\mathrm{FS}(Y)$. Since $X \smallsetminus Y = W$ we also have that $|X \smallsetminus Y| = \omega$, and since $Y, W \subseteq \mathrm{FS}(Z)$, we will have that $\mathrm{FS}(X) \subseteq \mathrm{FS}(Z) \subseteq A$, and we are done.

Proof of Claim 4.3 Let Z' be an infinite, coinfinite subset of Z. Let

$$B_0 = \left\{ w \in \mathrm{FS}(Z) \mid Z - \mathrm{supp}(w) \cap Z' \neq \varnothing \right\},$$

$$B_1 = \mathrm{FS}(Z) \smallsetminus B_0 = \left\{ w \in \mathrm{FS}(Z) \mid Z - \mathrm{supp}(w) \cap Z' = \varnothing \right\}.$$

There is $i \in 2$ such that $B_i \in p$. If $B_0 \in p$, then we let $W = Z \smallsetminus Z'$; otherwise, if $B_1 \in p$ we let $W = Z'$. In any case it is easy to see that $\mathrm{FS}(W) \cap B_i = \varnothing$. ■

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The remainder of this section is devoted to showing that the hypothesis that $\{x \in G \mid o(x) = 2\} \notin p$ in Corollary 3.2 is necessary, by constructing a nonprincipal strongly summable ultrafilter on $\mathbb{B}$ that is not additively isomorphic to a union ultrafilter. This construction borrows a lot of ideas from the constructions of unordered union ultrafilters that can be found in [2, Th. 4] and [12, Cor. 5.2]. We first show an effective way to look at additive isomorphisms to union ultrafilters.

Lemma 4.4 *Let $p \in \mathbb{B}^*$ be a strongly summable ultrafilter that is additively isomorphic to some union ultrafilter. Then there exists a linearly independent X such that $\mathrm{FS}(X) \in p$ and satisfying that whenever $A \subseteq \mathrm{FS}(X)$ is such that $A \in p$, there exists a set Z, whose elements have pairwise disjoint X-supports, with $p \ni \mathrm{FS}(Z) \subseteq A$.*

Proof If the strongly summable ultrafilter $p \in \mathbb{B}^*$ is additively isomorphic to a union ultrafilter, by Propositions 2.4 and 4.1, we have that for some linearly independent X such that $\mathrm{FS}(X) \in p$ and for some enumeration of X as $X = \{x_n \mid n < \omega\}$, the mapping $\phi\colon \mathrm{FS}(X) \to [\omega]^{<\omega}$ given by $\sum_{n\in a} x_n \mapsto a$ sends p to a union ultrafilter. Note that the mapping ϕ is a vector space isomorphism from the subspace spanned by X, to all of $\mathbb{B}$ (in fact it is the unique linear extension of the mapping $x_n \mapsto \{n\}$). The fact that $\phi(p)$ is a union ultrafilter means that, for every $A \subseteq \mathrm{FS}(X)$ such that $A \in p$, there is a pairwise disjoint family Y such that $\phi(p) \ni \mathrm{FU}(Y) \subseteq \phi[A]$. Since Y is pairwise disjoint, we get that $\mathrm{FU}(Y) = \mathrm{FS}(Y)$, and since ϕ is an isomorphism, $\phi^{-1}[\mathrm{FS}(Y)] = \mathrm{FS}(Z)$ where $Z = \phi^{-1}[Y]$. Now the fact that Y is pairwise disjoint means that the X-supports of the elements of Z are pairwise disjoint, and we have that $p \ni \mathrm{FS}(Z) \subseteq A$. ■

Thus, our goal is to construct, by a transfinite recursion, a strongly summable ultrafilter and somehow, at the same time, for each linearly independent X such that $\mathrm{FS}(X)$ will end up in the ultrafilter, at some stage we need to start making sure that, for every new set of the form $\mathrm{FS}(Z)$ that we are adding to the ultrafilter, the generators Z do not have pairwise disjoint X-support. The notions of suitable and adequate families for X will precisely code the way in which we are going to ensure that.

Definition 4.5 For a linearly independent subset $X \subseteq G$, we will say that a subset $Y \subseteq \mathrm{FS}(X)$ is *suitable* for X if the following hold.

(i) For each $m < \omega$ there exists an m-sequence $\langle y_i \mid i < m\rangle$ of elements of Y such that whenever $i < j < m$, the set $X - \mathrm{supp}(y_i) \cap X - \mathrm{supp}(y_j)$ is nonempty. This sequence will be called an *m-witness for suitability*.

(ii) Whenever $y, y' \in Y$ are such that $X - \mathrm{supp}(y) \cap X - \mathrm{supp}(y')$ is nonempty, the set $[X - \mathrm{supp}(y) \cap X - \mathrm{supp}(y')] \setminus X - \mathrm{supp}(Y \setminus \{y, y'\})$ is also nonempty. (We do not require here that $y \neq y'$; in particular, for each $y \in Y$, $X - \mathrm{supp}(y) \setminus X - \mathrm{supp}(Y \setminus \{y\})$ is nonempty, and this is easily seen to imply that Y must be linearly independent).

Thus, a suitable set Y for X contains, in a carefully controlled way, arbitrarily large bunches of elements whose X-supports always pairwise intersect. Given a linearly independent set X, it is easy to inductively build a set Y that is suitable for X. And once we have such a suitable set, we can look at subsets of $\mathrm{FS}(Y)$ that, in a sense, borrow from Y the non-disjointness of their X-supports. This is captured in a precise sense by the following definition, which also captures the fact that we will want to handle the non-disjointness of the X-supports for several distinct linearly independent sets X simultaneously.

Definition 4.6 Let $A \subseteq \mathbb{B}$ and let $\mathscr{Y} = \{(X_i, Y_i) \mid i < n\}$ be a finite family such that for each $i < n$, X_i is a linearly independent subset of G and Y_i is suitable for X_i. Also, let $m < \omega$. Then we will say that A is $(\mathscr{Y}, m)$-*adequate* if there exists an m-sequence $\langle a_j \mid j < m\rangle$, called a $(\mathscr{Y}, m)$-*witness for adequacy*, such that for each $i < n$,

(i) $\mathrm{FS}(\vec{a}) \subseteq A \cap \mathrm{FS}(Y_i)$ (which is in turn a subset of $\mathrm{FS}(X_i)$);
(ii) There exists an m-witness for the suitability of Y_i, $\langle y_j \mid j < m\rangle$ such that for each two distinct $j, k < m$, $y_j \in Y_i - \mathrm{supp}(a_j)$ and $y_j \notin Y_i - \mathrm{supp}(a_k)$.

If we are given a family of ordered pairs $\mathscr{X}$ all of whose first entries are linearly independent subsets of $\mathbb{B}$, while every second entry is suitable for the corresponding first entry, then we will say that A is $\mathscr{X}$-adequate if it is $(\mathscr{Y}, m)$-adequate for all finite $\mathscr{Y} \subseteq \mathscr{X}$ and for all $m < \omega$. When $\mathscr{Y}$ is a singleton $\{(X, Y)\}$, we will just say that A is (X, Y)-adequate.

Definition 4.6(ii) in particular implies that, for $j < k < m$, the set $X_i - \mathrm{supp}(a_j) \cap X_i - \mathrm{supp}(a_k)$ is nonempty. Thus, the X_i-supports of the terms of a witness for adequacy are not pairwise disjoint, and, moreover, their non-disjointness does not happen randomly, but is rather induced by some non-disjointness going on at the level of Y_i. Also, note that if Y is suitable for X, then $\mathrm{FS}(Y)$ is (X, Y)-adequate, with the witnesses for suitability witnessing adequacy at the same time. The following lemma, along with the observation that an $\mathscr{X}$-adequate set is also (X, Y)-adequate for each $(X, Y) \in \mathscr{X}$, tells us that this notion of adequacy is adequate (pun intended) for our purpose of banishing sets of the form $\mathrm{FS}(Z)$ for which the elements of Z have pairwise disjoint X-supports.

Lemma 4.7 *Let X and Z be both linearly independent and let Y be suitable for X. Assume that $Z \subseteq \mathrm{FS}(Y)$. If the elements of Z have pairwise disjoint X-supports, then $\mathrm{FS}(Z)$ is not (X, Y)-adequate.*

Proof Definition 4.5(ii) implies that, for two distinct $z, z' \in Z$, if $y \in Y - \mathrm{supp}(z)$ and $y' \in Y - \mathrm{supp}(z')$, then $X - \mathrm{supp}(y) \cap X - \mathrm{supp}(y') = \varnothing$, for otherwise $X - \mathrm{supp}(z)$ would not be disjoint from $X - \mathrm{supp}(z')$. Thus, $\langle z, z'\rangle$ cannot be an $((X, Y), 2)$-witness. More generally, for any two $w, w' \in \mathrm{FS}(Z)$, the only way that there could exist two distinct $y \in Y - \mathrm{supp}(w)$ and $y' \in Y - \mathrm{supp}(w')$ such that $X - \mathrm{supp}(y) \cap X - \mathrm{supp}(y') \neq \varnothing$ would be if $y, y' \in Y - \mathrm{supp}(z)$ for some $z \in Z$ such that $z \in Z - \mathrm{supp}(w) \cap Z - \mathrm{supp}(w')$. But then $y \in Y - \mathrm{supp}(w')$ and $y' \in Y - \mathrm{supp}(w)$. Hence, $\langle w, w'\rangle$ cannot be an $((X, Y), 2)$-witness and we are done. ∎

Given this, the idea for the recursive construction of an ultrafilter would be as follows. At each stage we choose some set $\mathrm{FS}(X)$ that has already been added to the ultrafilter, and then we choose a suitable (for X) set Y. At every stage we make sure that the subsets of $\mathbb{B}$ that we are adding to the ultrafilter are $\mathscr{X}$-adequate, where $\mathscr{X}$ is the collection of all pairs (X, Y) that have been thus chosen so far. If we want to have a hope of succeeding in such a construction, we need to make sure that the notion of being $\mathscr{X}$-adequate behaves well with respect to partitions. For this we will need the following lemma.

Lemma 4.8 *Let $\mathscr{Y} = \{(X_i, Y_i) \mid i < n\}$, where each X_i is linearly independent and each Y_i is suitable for X_i. Let $\vec{a} = \langle a_j \mid j < M\rangle$ be a $(\mathscr{Y}, M)$-witness for adequacy, and let $\langle b_i \mid i < m\rangle$ be an m-sequence of pairwise disjoint subsets of M. If we define $\vec{c} = \langle c_j \mid j < m\rangle$ by $c_j = \sum_{k\in b_j} a_k$, then $\vec{c}$ will be a $(\mathscr{Y}, m)$-witness for adequacy.*

Proof Let us check that $\vec{c}$ satisfies both requirements of Definition 4.6 for a $(\mathscr{Y}, m)$-witness. Fix $i < n$. Since the b_j are pairwise disjoint, we have that $\mathrm{FS}(\vec{c}) \subseteq \mathrm{FS}(\vec{a}) \subseteq A \cap \mathrm{FS}(Y_i)$, thus requirement (i) is satisfied. In order to see that requirement (ii) holds, grab the corresponding m-witness for suitability, $\langle y_j \mid j < M\rangle$, as of Definition 4.6(ii) for $\vec{a}$. Now for $j < m$, pick a $k_j \in b_j$ and let $w_j = y_{k_j}$. Since the w_j were chosen from among the y_k, the sequence $\vec{w} = \langle w_j \mid j < m\rangle$ is an m-witness for suitability. Now for $j < m$, since $w_j \in Y_i - \mathrm{supp}(a_{k_j})$ and $w_j \notin Y_i - \mathrm{supp}(a_l)$ for $l \neq k_j$, it follows that $w_j \in Y_i - \mathrm{supp}(c_j)$ and $w_j \notin Y_i - \mathrm{supp}(c_{j'})$ for $j \neq j'$, and we are done. ∎

An easy consequence of the previous lemma is the observation that any $(\mathscr{Y}, M)$-adequate set is also $(\mathscr{Y}, m)$-adequate for any $m \leq M$. Lemma 4.8 will allow us to prove the following lemma, which is crucial.

Lemma 4.9 *For each $m < \omega$ there is an $M < \omega$ such that whenever $\mathscr{Y}$ is a finite family of ordered pairs of the form (X, Y), with X a linearly independent set and Y suitable for X, and whenever a $(\mathscr{Y}, M)$-adequate set is partitioned into two cells, one of the cells must be $(\mathscr{Y}, m)$-adequate.*

Proof For this, we will use a theorem of Graham and Rothschild that is a finitary version of Hindman's theorem; namely, for every $m < \omega$ there is an $M < \omega$ such that whenever we partition $\mathfrak{P}(M) \setminus \{\varnothing\}$ into two cells, then one of the cells contains $\mathrm{FU}(\vec{b})$ for some pairwise disjoint m-sequence $\vec{b} = \langle b_i \mid i < m\rangle$ of nonempty subsets of M (this result is sometimes referred to as the *Folkman–Rado–Saunders* theorem). An elegant proof of this theorem from the infinitary version, using a so-called compactness argument, can be obtained by following the proof of [10, Th. 5.29] as a template, applied to the semigroup whose underlying set is $[\omega]^{<\omega}$ and whose semigroup operation is the union $\cup$.

Thus, for $m < \omega$, let M be given by this finitary theorem, and let A be a $(\mathscr{Y}, M)$-adequate set. Let $\vec{a} = \langle a_j \mid j < M\rangle$ be a $(\mathscr{Y}, M)$-witness for the adequacy of A. If A is partitioned into the two cells A_0, A_1, then since $\mathrm{FS}(a) \subseteq A$, we can induce a partition of $\mathfrak{P}(M) \setminus \{\varnothing\}$ into the two cells B_0, B_1 by declaring a subset $s \subseteq M$ to be an element of B_l if and only if $\sum_{j\in s} a_j \in A_l$ for $l \in 2$. Then the theorem of Graham and Rothschild gives us a pairwise disjoint family $\vec{b} = \langle b_j \mid j < m\rangle$ and an $l \in 2$ such that $\mathrm{FU}(\vec{b}) \subseteq B_l$. Letting $\vec{c} = \langle c_j \mid j < m\rangle$ be given by $c_j = \sum_{k\in b_j} a_k$, we get that $\mathrm{FS}(\vec{c}) \subseteq A_l$ and Lemma 4.8 ensures that $\vec{c}$ is a $(\mathscr{Y}, m)$-witness for adequacy. Therefore, A_l is $(\mathscr{Y}, m)$-adequate, and we are done. ∎

Corollary 4.10 *For any family $\mathscr{X}$ consisting of ordered pairs of the form (X, Y), with X a linearly independent set and Y suitable for X, if we partition an $\mathscr{X}$-adequate set into two cells, then one of them must be $\mathscr{X}$-adequate.*

Proof If $A = A_0 \cup A_1$ is a partition of the $\mathscr{X}$-adequate set A, and neither A_0 nor A_1 are $\mathscr{X}$-adequate, then the reason for this is the existence of finite $\mathscr{Y}_0, \mathscr{Y}_1 \subseteq \mathscr{X}$ and $m_0, m_1 < \omega$ such that A_0 is not $(\mathscr{Y}_0, m_0)$-adequate and A_1 is not $(\mathscr{Y}_1, m_1)$-adequate. Pick the M that works for $\max\{m_0, m_1\}$ in Lemma 4.9. Then for some $i \in 2$, A_i is $(\mathscr{Y}_0 \cup \mathscr{Y}_1, \max\{m_0, m_1\})$-adequate (because A is $(\mathscr{Y}_0 \cup \mathscr{Y}_1, M)$-adequate); in particular, A_i is $(\mathscr{Y}_i, m_i)$-adequate, a contradiction. ∎

Recall that, in an abstract setting, if we have a set X and a family $\mathscr{A} \subseteq \mathfrak{P}(X)$ then we say that $\mathscr{A}$ is *partition regular*, or a *coideal*, if $\mathscr{A}$ is closed under supersets and, whenever an element of $\mathscr{A}$ is partitioned into two cells, the family $\mathscr{A}$ necessarily contains at least one of the cells. Thus, the previous corollary establishes that, for any family $\mathscr{X}$, the collection of $\mathscr{X}$-adequate subsets of $\mathbb{B}$ is partition regular. This is important because of the well-known fact that if $\mathscr{A}$ is partition regular and $\mathcal{F} \subseteq \mathscr{A}$ is a filter on X, then it is possible to extend $\mathcal{F}$ to an ultrafilter $p \subseteq \mathscr{A}$.

With these preliminary results under our belt, we are finally ready to prove the main theorem of this section.

Theorem 4.11 *If $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$, then there exists a strongly summable ultrafilter on $\mathbb{B}$ that is not additively isomorphic to any union ultrafilter.*

Proof Let $\{A_\alpha \mid \alpha < \mathfrak{c}\}$ be an enumeration of all subsets of $\mathbb{B}$ and let $\langle X_\alpha \mid \alpha < \mathfrak{c}\rangle$ be an enumeration of all infinite linearly independent subsets of $\mathbb{B}$ in such a way that each such set appears cofinally often in the enumeration. Now recursively define linearly independent sets $\langle Y_\alpha \mid \alpha < \mathfrak{c}\rangle$ and a strictly increasing sequence of ordinals $\langle \gamma_\alpha \mid \alpha < \mathfrak{c}\rangle$ satisfying the following conditions for each $\alpha < \mathfrak{c}$:

(i) γ_α is the least $\eta \geq \sup_{\xi<\alpha}(\gamma_\xi + 1)$ such that $\mathrm{FS}(Y_\xi) \subseteq \mathrm{FS}(X_\eta)$ for some $\xi < \alpha$;
(ii) Y_α is suitable for X_{γ_α};
(iii) $\mathrm{FS}(Y_\alpha)$ is either contained in or disjoint from A_α;
(iv) the family $\mathcal{F}_\alpha = \{\mathrm{FS}(Y_\xi) \mid \xi \leq \alpha\}$ is centred;
(v) letting $\mathscr{X}_\alpha = \{(X_{\gamma_\xi}, Y_\xi) \mid \xi \leq \alpha\}$, the filter generated by $\mathcal{F}_\alpha$ consists of $\mathscr{X}_\alpha$-adequate sets.

Thus, at each stage α, we first use clause (i) to determine what γ_α will be, and then we work to find a Y_α satisfying (ii)–(v).

Let us first look at what we have at the end of this construction. Clause (iv) tells us that the family $\{\mathrm{FS}(Y_\alpha) \mid \alpha < \mathfrak{c}\}$ generates a filter p, which will be an ultrafilter because of (iii), and it will obviously be nonprincipal and strongly summable. Now notice that (v) implies that if $\mathscr{X}_\mathfrak{c} = \{(X_{\gamma_\alpha}, Y_\alpha) \mid \alpha < \mathfrak{c}\}$, then each $A \in p$ will be $\mathscr{X}_\mathfrak{c}$-adequate, because if $\mathscr{Y} = \{(X_{\gamma_{\alpha_i}}, Y_i) \mid i < n\}$ is a finite subfamily of $\mathscr{X}_\mathfrak{c}$, $m < \omega$, and $A \in p$, then we can grab an $\alpha < \mathfrak{c}$ larger than all γ_{α_i} and also larger than the β witnessing $\mathrm{FS}(Y_\beta) \subseteq A$. By (v), $\mathrm{FS}(Y_\alpha) \cap \mathrm{FS}(Y_\beta)$ is $\mathscr{X}_\alpha$-adequate; in particular, it is $(\mathscr{Y}, m)$-adequate, and thus so is A.

The last observation is crucial for the argument that p cannot be additively isomorphic to any union ultrafilter. If it was, by Lemma 4.4, there would be a linearly independent X such that $\mathrm{FS}(X) \in p$ and such that for each $A \in p$ satisfying $A \subseteq \mathrm{FS}(X)$, we would be able to find a family Z whose elements have pairwise disjoint X-supports and such that $p \ni \mathrm{FS}(Z) \subseteq A$. Now since $\mathrm{FS}(X) \in p$, there is an $\alpha < \mathfrak{c}$ such that $\mathrm{FS}(Y_\alpha) \subseteq \mathrm{FS}(X)$, let η be the least ordinal $\geq \sup_{\xi\leq\alpha}(\gamma_\xi + 1)$ such that $X = X_\eta$. By (i) we will have that $\gamma_{\alpha+1} \leq \eta$ and, in fact, whenever $\xi > \alpha$ is such that no γ_β equals η for any $\alpha < \beta < \xi$; then $\gamma_\xi \leq \eta$. Thus, there will eventually be some $\zeta > \alpha$ such that $\gamma_\zeta = \eta$, and by (ii) this means that Y_ζ is suitable for X. Since every element of p is $\mathscr{X}_\mathfrak{c}$-adequate, in particular (X, Y_ζ)-adequate, then by Lemma 4.7 we get that for no set Z with pairwise disjoint X-supports can we have that $p \ni \mathrm{FS}(Z) \subseteq \mathrm{FS}(Y_\zeta)$. This shows that p cannot be additively isomorphic to any union ultrafilter, and we are done.

We now proceed to show how is it possible to carry out such a construction. So let $\alpha < \mathfrak{c}$ and assume that for all $\xi < \alpha$, conditions (i)–(v) are satisfied. As mentioned before, condition (i) uniquely determines γ_α, so we only need to focus on constructing Y_α satisfying conditions (ii)–(v). Let

$$\mathcal{F} = \left\{ \mathrm{FS}(Y_\xi) \mid \xi < \alpha \right\} \quad \text{and} \quad \mathscr{X} = \left\{ (X_{\gamma_\xi}, Y_\xi) \mid \xi < \alpha \right\}.$$

Condition (v) implies that the filter generated by $\mathcal{F}$ consists of $\mathscr{X}$-adequate sets, if α is limit, by the same argument as in the proof that p consists of $\mathscr{X}_\mathfrak{c}$-adequate sets, and if $\alpha = \xi + 1$ just because $\mathcal{F} = \mathcal{F}_\xi$ and $\mathscr{X} = \mathscr{X}_\xi$. Thus if we define

$$H = \left\{ q \in \beta\mathbb{B} \mid (q \supseteq \mathcal{F}) \wedge (\forall A \in q)(A \text{ is } \mathscr{X}\text{–adequate}) \right\},$$

then H will be a nonempty subset of $\beta\mathbb{B}$ by Corollary 4.10 (*cf.* the discussion following that corollary). Since finite sets cannot be $\mathscr{X}$-adequate, we have that, in fact, $H \subseteq B^*$. In what follows, in order to avoid confusion, we will use the symbol $\blacktriangle$ to denote the extension of the group operation $\triangle$ on $\mathbb{B}$ to all of $\beta\mathbb{B}$. We will also use that symbol to denote translates of sets, $x \blacktriangle A = \{x \triangle y \mid y \in A\}$. Thus, with this notation,

$$p \blacktriangle q = \left\{ A \subseteq \mathbb{B} \mid \{x \in \mathbb{B} \mid x \blacktriangle A \in q\} \in p \right\}.$$

Claim 4.12 H is a closed subsemigroup of $\mathbb{B}$.

Proof of Claim The fact that H is closed is fairly straightforward and is left to the reader. To prove that H is a subsemigroup, let $p, q \in H$. We first show that $\mathcal{F} \subseteq p \blacktriangle q$. Fix a $\xi < \alpha$, and note that we have, for each $w \in \mathrm{FS}(Y_\xi)$, that $w \blacktriangle \mathrm{FS}(Y_\xi) = \mathrm{FS}(Y_\xi) \cup \{\varnothing\} \in q$. Hence $p \ni \mathrm{FS}(Y_\xi) \subseteq \{x \in \mathbb{B} \mid x \blacktriangle \mathrm{FS}(Y_\xi) \in q\}$, which means that $\mathrm{FS}(Y_\xi) \in p \blacktriangle q$.

Now we only need to show that if $A \in p \blacktriangle q$, then A is $\mathscr{X}$-adequate. So fix a finite $\mathscr{Y} = \{(X_i, Y_i) \mid i < n\} \subseteq \mathscr{X}$ and an $m < \omega$. We will see that there is a $(\mathscr{Y}, m)$-witness for the adequacy of A. Let $B = \{x \in \mathbb{B} \mid x \blacktriangle A \in q\}$. We have that $B \in p$, because $A \in p \blacktriangle q$, so B is $\mathscr{X}$-adequate, and thus we can grab a $(\mathscr{Y}, m)$-witness $\langle a_j \mid j < m\rangle$ for the adequacy of B. For each $i < n$, $\mathrm{FS}(\vec{a}) \subseteq \mathrm{FS}(Y_i)$, so we can define $Z_i \in [Y_i]^{<\omega}$ by $Z_i = Y_i - \mathrm{supp}(\vec{a})$. Consider the set

$$C = \bigcap_{a\in\mathrm{FS}(\vec{a})} a \blacktriangle A,$$

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which is an element of q, because $\mathrm{FS}(\vec{a}) \subseteq B$, and hence it is $\mathscr{X}$-adequate. Therefore, we can grab a $(\mathscr{Y}, 2^{\sum_{i<n}|Z_i|} + 2m - 1)$-witness for the adequacy of C, $\langle b_j \mid j < 2^{\sum_{i<n}|Z_i|} + 2m - 1\rangle$. Associate with any element $x \in \bigcap_{i<n} \mathrm{FS}(Y_i)$ the vector $\langle Z_i \cap Y_i - \mathrm{supp}(x) \mid i < n\rangle$, and notice that there are exactly $2^{\sum_{i<n}|Z_i|}$ many possible distinct such vectors. Thus, there exist $2m$ distinct numbers $k_0, \dots, k_{2m-1} < 2^{\sum_{i<n}|Z_i|} + 2m - 1$ such that for each $j < m$, the vector associated with $b_{k_{2j}}$ is exactly the same as the one associated with $b_{k_{2j+1}}$, and so if we let $c_j = b_{k_{2j}} \triangle b_{k_{2j+1}}$, then for each $i < n$, $c_j \in \mathrm{FS}(Y_i \smallsetminus Z_i)$. By Lemma 4.8, the m-sequence $\vec{c} = \langle c_j \mid j < m\rangle$ will be an m-witness for the adequacy of C. Now let $\vec{d} = \langle d_j \mid j < m\rangle$ be given by $d_j = a_j \triangle c_j$. We claim that $\vec{d}$ is a $(\mathscr{Y}, m)$-witness for the adequacy of A, so let us fix $i < n$ and verify that $\vec{d}$ satisfies Definition 4.6(i) and (ii). It is certainly the case that $\mathrm{FS}(\vec{c}) \subseteq A \cap \mathrm{FS}(Y_i)$, because if $d \in \mathrm{FS}(\vec{d})$, then there are $a \in \mathrm{FS}(\vec{a})$ and $c \in \mathrm{FS}(\vec{c})$ such that $d = a \triangle c$, and since $c \in C \subseteq a \blacktriangle A$, we get that $d \in A$. Thus, requirement (i) is satisfied. Now for requirement (ii), just grab the m-witness for the suitability of Y_i that works for $\vec{a}$, $\langle y_j \mid j < m\rangle$. We constructed the c_j in such a way that $Y_i - \mathrm{supp}(c_j) \cap Z_i = \varnothing$, while $Y_i - \mathrm{supp}(a_j) \subseteq Z_i$. Hence, for each $j < m$, $Y_i - \mathrm{supp}(d_j) \cap Z_i = Y_i - \mathrm{supp}(a_j)$, and so whenever $j < m$, $y_j \in Y_i - \mathrm{supp}(d_j)$, and $y_j \notin Y_i - \mathrm{supp}(d_k)$ for $k \neq j$. ■

Since H is a closed subset of the compact space $\beta\mathbb{B}$, then H is compact as well, and since it is a semigroup in its own right, we can apply the so-called Ellis–Numakura lemma [10, Th. 2.5], which asserts that every (nonempty) compact right-topological semigroup contains idempotent elements. Hence we can pick an idempotent $q \blacktriangle q = q \in H$. Let $A \in \{A_\alpha, \mathbb{B} \smallsetminus A_\alpha\}$ be such that $A \in q$. We will use q to carefully construct Y_α. Let $X = X_{\gamma_\alpha}$.

Claim 4.13 There is a Y, suitable for X, such that the following hold:

(i) $\mathrm{FS}(Y) \subseteq A$.

(ii) For any finite subfamily $\mathscr{Y} = \{(X_i, Y_i) \mid i < n\} \subseteq \mathscr{X}$, for any $m < \omega$ and for any finitely many $\xi_0, \dots, \xi_k < \alpha$, there is a sequence $\langle a_j \mid j < m\rangle$ of elements of Y that is simultaneously an m-witness for the suitability (for X) of Y and a $(\mathscr{Y}, m)$-witness for the adequacy of $\bigcap_{l\leq k} \mathrm{FS}(Y_{\xi_l})$. In particular, $\vec{a}$ witnesses the $(\mathscr{Y} \cup \{(X, Y)\}, m)$-adequacy of $(\bigcap_{l\leq k} \mathrm{FS}(Y_{\xi_l})) \cap \mathrm{FS}(Y)$.

Proof This is the only place where we will actually use the hypothesis that $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$. Since q is an idempotent and $A \in q$, the set $A^\star = \{x \in A \mid x \blacktriangle A \in q\} \in q$ and by [10, Lemma 4.14], for every $x \in A^\star$, $x \blacktriangle A^\star \in q$. Let $\mathbb{P}$ be the partial order consisting of those finite subsets $W \subseteq \mathrm{FS}(X)$ such that $\mathrm{FS}(W) \subseteq A^\star$ and satisfying Definition 4.5(ii) of suitability for X, ordered by reverse inclusion (thus $Z \leq W$ means that $Z \supseteq W$). This is a countable forcing notion, hence forcing equivalent to Cohen's forcing. For any finite $\mathscr{Y} \subseteq \mathscr{X}$, every $m < \omega$, and all $\xi_0, \dots, \xi_k < \alpha$ as in part (ii) of the conclusions of this claim, we let $D(\mathscr{Y}, m, \xi_0, \dots, \xi_k)$ be the set consisting of all conditions $Z \in \mathbb{P}$ such that there is an m-sequence $\vec{a}$ of elements of Z that simultaneously witnesses the suitability of Z for X and the $(\mathscr{Y}, m)$-adequacy of $\bigcap_{l\leq k} \mathrm{FS}(Y_{\xi_l})$. The heart of this proof will be the argument that all of these sets $D(\mathscr{Y}, m, \xi_0, \dots, \xi_k)$ are dense in $\mathbb{P}$. Once we have that, we just need to notice that there are $|\alpha| < \mathfrak{c} = \mathrm{cov}(\mathcal{M})$

many such dense sets, so we can pick a filter G intersecting them all, and we will clearly be done by defining $Y = \bigcup G$.

So let us prove that $D(\mathscr{Y}, m, \xi_0, \ldots, \xi_k)$ is dense in $\mathbb{P}$. The idea is that we are given a condition $Z \in \mathbb{P}$, and we would like to pick a $(\mathscr{Y}, m)$-witness $\vec{a}$ for the adequacy of $\bigcap_{l \le k} \mathrm{FS}(Y_{\xi_l})$, and extend Z to a stronger condition W by adding the range of $\vec{a}$ to it. The main difficulty is that we want $\vec{a}$ to be at the same time an m-witness for suitability (for X) such that the resulting condition $W = Z \cup \{a_j \mid j < m\}$ still satisfies Definition 4.5(ii).

Let us start with a condition $Z \in \mathbb{P}$, and let $X' = X \smallsetminus X-\mathrm{supp}(Z)$. Notice first that we must have $\mathrm{FS}(X') \in q$, for otherwise we would have

$$\{w \in \mathrm{FS}(X) \mid X-\mathrm{supp}(w) \cap X-\mathrm{supp}(Z) \neq \varnothing\} \in q,$$

but it is easy to see (arguing as in [4, Lemma 2.2 and Cor. 2.3]) that this set cannot contain any FS-set, which it should if it was to belong to any idempotent (because of [10, Th. 5.8]). Let

$$B = \Big(\bigcap_{l \le k} \mathrm{FS}(Y_{\xi_l})\Big) \cap \mathrm{FS}(X') \cap \Big(\bigcap_{z \in \mathrm{FS}(Z)} z \blacktriangle A^\star\Big).$$

Then $B^\star = \{x \in B \mid x \blacktriangle B \in q\} \in q$; thus, $B^\star$ is $\mathscr{X}$-adequate, so there is a $(\mathscr{Y}, m)$-witness $\vec{a} = \langle a_j \mid j < m\rangle$ for the adequacy of $B^\star$. We will now recursively construct an $m + \binom{m}{2}$-sequence of elements $\vec{x} = \langle x_k \mid k < m + \binom{m}{2}\rangle$ such that $\mathrm{FS}(\vec{x}) \subseteq \bigcap_{a \in \mathrm{FS}(\vec{a})} a \blacktriangle B^\star$ and such that the X-supports of its elements are pairwise disjoint and also disjoint from $X-\mathrm{supp}(\vec{a})$, and whose Y_i-supports are disjoint from $Y_i - \mathrm{supp}(\vec{a})$ for each $i < n$. If we succeed in this construction, picking a bijection $f\colon [m]^2 \to (m + \binom{m}{2}) \smallsetminus m$ will enable us to define the sequence $\vec{b} = \langle b_j \mid j < m\rangle$ by

$$b_j = a_j \vartriangle x_j \vartriangle \Big(\sum_{\substack{k<m \\ k \neq j}} x_{f(\{j,k\})}\Big).$$

Since the Y_i-supports of all the x_k are disjoint from $Y_i - \mathrm{supp}(\vec{a})$, arguing as in the proof of Claim 4.12 we conclude that $\vec{b}$ is a $(\mathscr{Y}, m)$-witness for the adequacy of $B^\star$, hence also for the adequacy of $\bigcap_{l \le k} \mathrm{FS}(Y_{\xi_l})$. And the careful choice of the X-supports of the x_k ensures that $\vec{b}$ is at the same time an m-witness for suitability for X, hence letting $W = Z \cup \{b_j \mid j < m\}$ yields a condition in $\mathbb{P}$ (*i.e.* , W satisfies Definition 4.5(ii)).

Thus, the only remaining issue is that of picking the x_k. Assume that we have picked x_l for $l < k$, and we will show how to pick x_k. Since q is an idempotent and

$$C = \bigcap_{a \in \mathrm{FS}(\vec{a} \frown \langle x_l \mid l < k\rangle)} a \blacktriangle B^\star \in q,$$

then there is a set of the form $\mathrm{FS}(V) \subseteq C$ (as before, this follows from [10, Th. 5.8]). As in the argument for the proof of Claim 4.12, with each element $x \in C$ we associate the vector

$$\Big\langle Y_i - \mathrm{supp}(\vec{a}) \cap Y_i - \mathrm{supp}(x) \mid i < n\Big\rangle \frown$$
$$\Big\langle X - \mathrm{supp}\Big(\{a_j \mid j < m\} \cup \{x_l \mid l < k\}\Big) \cap X - \mathrm{supp}(x)\Big\rangle,$$

and notice that, since there are only finitely many possible distinct such vectors, the infinite set V must contain at least one pair of distinct elements v, w that have the same associated vector. Hence, by letting $x_k = v \vartriangle w \in \mathrm{FS}(V) \subseteq C$, we get that $Y_i - \mathrm{supp}(x_k) \cap Y_i - \mathrm{supp}(\vec{a}) = \varnothing$ for all $i < n$, and

$$X - \mathrm{supp}(x_k) \cap X - \mathrm{supp}\big(\{a_j \mid j < m\} \cup \{x_l \mid l < k\}\big) = \varnothing,$$

so the construction can go on, and we are done. ■

Let $Y_\alpha = Y$. Obviously requirement (ii) is satisfied, and since $\mathrm{FS}(Y_\alpha) \subseteq A \in \{A_\alpha, \mathbb{B} \smallsetminus A_\alpha\}$, requirement (iii) is satisfied as well. It is easy to see that condition (ii) from the conclusion of the claim ensures at once that requirements (iv) and (v) are fulfilled, and we are done. ■

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References

[1] A. Blass, *Ultrafilters related to Hindman's finite-unions theorem and its extensions.* In: Logic and Combinatorics, Contemp. Math., 65, American Mathematical Society, Providence, RI, 1987, pp. 89–124. http://dx.doi.org/10.1090/conm/065/891244

[2] A. Blass and N. Hindman, *On strongly summable ultrafilters and union ultrafilters.* Trans. Amer. Math. Soc. **304**(1987), no. 1, 83–99. http://dx.doi.org/10.1090/S0002-9947-1987-0906807-4

[3] T. Eisworth, *Forcing and stable-ordered union ultrafilters.* J. Symbolic Logic **67**(2002), no. 1, 449–464. http://dx.doi.org/10.2178/jsl/1190150054

[4] D. J. Fernández Bretón, *Every strongly summable ultrafilter on $\bigoplus \mathbb{Z}_2$ is sparse.* New York J. Math. **19**(2013), 117–129.

[5] N. Hindman, *The existence of certain ultrafilters on $\mathbb{N}$ and a conjecture of Graham and Rothschild.* Proc. Amer. Math. Soc. **36**(1972), no. 2, 341–346.

[6] ———, *Summable ultrafilters and finite sums.* In: Logic and Combinatorics, Contemp. Math., 65, American Mathematical Society, Providence, RI, 1987, 263–274. http://dx.doi.org/10.1090/conm/065/891252

[7] N. Hindman and L. Legette Jones, *Idempotents in βS that are only products trivially.* New York J. Math. **20**(2014), 57–80.

[8] N. Hindman, I. Protasov, and D. Strauss, *Strongly summable ultrafilters on Abelian groups.* Mat. Stud. **10**(1998), no. 2, 121–132.

[9] N. Hindman, J. Steprāns, and D. Strauss, *Semigroups in which all strongly summable ultrafilters are sparse.* New York J. Math. **18**(2012), 835–848.

[10] N. Hindman and D. Strauss, *Algebra in the Stone-Čech compactification.* Second ed., de Gruyter Textbook, Walter de Gruyter, Berlin, 2012.

[11] P. Krautzberger, *On strongly summable ultrafilters.* New York J. Math. **16**(2010), 629–649.

[12] ———, *On union ultrafilters.* Order **29**(2012), 317–343. http://dx.doi.org/10.1007/s11083-011-9223-3

[13] I. V. Protasov, *Finite groups in βG.* Mat. Stud. **10**(1998), no. 1, 17–22.

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