Bull. Aust. Math. Soc. **80** (2009), 159–164
doi:10.1017/S000497270900032X

n-JORDAN HOMOMORPHISMS

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(Received 28 November 2008)

Abstract

Let $n \in \mathbb{N}$ and let A and B be rings. An additive map $h : A \to B$ is called an n-Jordan homomorphism if $h(a^n) = (h(a))^n$ for all $a \in A$. Every Jordan homomorphism is an n-Jordan homomorphism, for all $n \geq 2$, but the converse is false in general. In this paper we investigate the n-Jordan homomorphisms on Banach algebras. Some results related to continuity are given as well.

2000 *Mathematics subject classification*: primary 47B48; secondary 46L05, 46H25.

Keywords and phrases: Jordan homomorphism, n-homomorphism, Banach algebra.

1. Introduction and preliminaries

Let A, B be two rings (algebras). An additive map $h : A \to B$ is called an n-Jordan homomorphism if $h(a^n) = (h(a))^n$ for all $a \in A$; it is an n-ring homomorphism if $h(\prod_{i=1}^{n} a_i) = \prod_{i=1}^{n} h(a_i)$ for all $a_1, a_2, \ldots, a_n \in A$. If $h : A \to B$ is a linear n-ring homomorphism, we say that h is an n-homomorphism. The concept of n-homomorphism was studied for complex algebras by Hejazian, Mirzavaziri, and Moslehian [3] (see also [1, 7]). A 2-Jordan homomorphism is a Jordan homomorphism, in the usual sense, between rings. Every Jordan homomorphism is an n-Jordan homomorphism, for all $n \geq 2$ (see, for example, [6, Lemma 6.3.2]), but the converse is false, in general. For instance, let A be an algebra over $\mathbb{C}$ and let $h : A \to A$ be a nonzero Jordan homomorphism on A. Then $-h$ is a 3-Jordan homomorphism. It is easy to check that $-h$ is not 2-Jordan homomorphism or 4-Jordan homomorphism. The study of ring homomorphisms between Banach algebras A and B is of interest even if $A = B = \mathbb{C}$. For example the zero mapping, the identity and the complex conjugate are ring homomorphisms on $\mathbb{C}$, which are all continuous. On the other hand the existence of a discontinuous ring homomorphism on $\mathbb{C}$ is well known. More explicitly, if G is the set of all surjective ring homomorphisms on $\mathbb{C}$, then $\mathrm{Card}(G) = 2^{\mathrm{Card}(\mathbb{C})}$. In fact, Charnow [2, Theorem 3] proved that there exist $2^{\mathrm{Card}(\mathbb{C})}$ automorphisms for every algebraically closed field K. It is also known that if A is a uniform algebra on a compact metric space, then there are exactly $2^{\mathrm{Card}(\mathbb{C})}$ complex-valued ring homomorphisms on A whose kernels are nonmaximal prime ideals (see [4, Corollary 2.4]). As an example, take

$$\mathcal{A} := \begin{bmatrix} 0 & \mathbb{R} & \mathbb{R} & \mathbb{R} \\ 0 & 0 & \mathbb{R} & \mathbb{R} \\ 0 & 0 & 0 & \mathbb{R} \\ 0 & 0 & 0 & 0 \end{bmatrix};$$

then $\mathcal{A}$ is an algebra equipped with the usual matrix-like operations. It is easy to see that

$$\mathcal{A}^3 \neq 0 = \mathcal{A}^4.$$

So any additive map from $\mathcal{A}$ into itself is a 4-Jordan homomorphism, but its kernel does not need to be an ideal of $\mathcal{A}$. Now let $\mathcal{B}$ be the algebra of all $\mathcal{A}$-valued continuous functions from $[0, 1]$ into $\mathcal{A}$ with supremum norm. Then $\mathcal{B}$ is an infinite-dimensional Banach algebra, and the product of any four elements of $\mathcal{B}$ is 0. Since $\mathcal{B}$ is infinite-dimensional, there are linear discontinuous maps which are 4-Jordan homomorphisms from $\mathcal{B}$ into itself (see [3]). In this paper we study the continuity of linear n-Jordan homomorphisms on C^*-algebras.

2. Main result

By definition, it is obvious that n-ring homomorphisms are n-Jordan homomorphisms. Conversely, under a certain condition, n-Jordan homomorphisms are ring homomorphisms. For example, each Jordan homomorphism h from a commutative Banach algebra A into $\mathbb{C}$ is a ring homomorphism: Fix $a, b \in A$ arbitrarily. Since $h((a+b)^2) = h(a+b)^2$ a simple calculation shows that $h(ab + ba) = 2h(a)h(b)$. The commutativity of A implies that $h(ab) = h(a)h(b)$ and hence h is a ring homomorphism. In 1968, Zelazko [8] proved the following theorem (see also [5, Theorem 1.1]).

THEOREM 2.1. *Suppose that A is a Banach algebra, which need not be commutative, and suppose that B is a semisimple commutative Banach algebra. Then each Jordan homomorphism $h :\rightarrow B$ is a ring homomorphism.*

We prove the following result for 3-Jordan homomorphisms and 4-Jordan homomorphisms on commutative algebras.

THEOREM 2.2. *Let $n \in \{3, 4\}$ be fixed, A, B be two commutative algebras, and let $h : A \to B$ be an n-Jordan homomorphism. Then h is an n-ring homomorphism.*

PROOF. First, let $n = 3$. Recall that h is additive mapping such that $h(a^3) = (h(a))^3$ for all $a \in A$. Replacement of a by $x + y$ results in

$$h(x^2y + xy^2) = h(x)^2h(y) + h(x)h(y)^2. \tag{2.1}$$

https://doi.org/10.1017/S000497270900032X Published online by Cambridge University Press

Hence, for every $x, y, z \in A$,

$$\begin{aligned}
h(xyz) &= \tfrac{1}{2}h\{(x+z)^2y + (x+z)y^2 - (x^2y + xy^2 + z^2y + zy^2)\}\\
&= \tfrac{1}{2}\{h[(x+z)^2y + (x+z)y^2] - h[x^2y + xy^2] - h[z^2y + zy^2]\}\\
&= \tfrac{1}{2}\{[h(x+z)]^2h(y) + (x+z)[h(y)]^2 - [h(x)]^2h(y) + h(x)[h(y)]^2\\
&\qquad - [h(z)]^2h(y) + h(z)[h(y)]^2\}\\
&= h(x)h(y)h(z).
\end{aligned}$$

This means that h is a 3-ring homomorphism. Now suppose that $n = 4$. Then h is additive and $h(a^4) = (h(a))^4$ for all $a \in A$. Replace a by $x + y$ in the equality above to get

$$h(4x^3y + 6x^2y^2 + 4xy^3) = 4h(x)^3h(y) + 6h(x)^2h(y)^2 + 4h(x)h(y)^3. \tag{2.2}$$

Replacing x by $x + z$ in (2.2), we obtain

$$\begin{aligned}
&h\{(4x^3y + 6x^2y^2 + 4xy^3) + (4z^3y + 6z^2y^2 + 4zy^3) + 12(x^2zy + xz^2y + xzy^2)\}\\
&\quad = (4h(x)^3h(y) + 6h(x)^2h(y)^2 + 4h(x)h(y)^3) + (4h(z)^3h(y) + 6h(z)^2h(y)^2\\
&\qquad + 4h(z)h(y)^3) + 12(h(x)^2h(z)h(y) + h(x)h(z)^2h(y)\\
&\qquad + h(x)h(z)h(y)^2).
\end{aligned} \tag{2.3}$$

Combining (2.2) and (2.3) gives

$$h\{(xyz)(x + y + z)\} = (h(x)h(y)h(z))(h(x) + h(y) + h(z)). \tag{2.4}$$

Replace z by $-x$ in (2.4) to obtain

$$h(x^2y^2) = h(x)^2h(y)^2 \tag{2.5}$$

and replace y by $y + w$ in (2.5) to get

$$h(x^2yw) = h(x)^2h(y)h(w). \tag{2.6}$$

Now replace x by $x + t$ to obtain

$$h(xtyw) = h(x)h(t)h(y)h(w).$$

Hence, h is a 4-ring homomorphism. □

By Theorem 2.2 and [1, Theorem 3.2] we deduce the following result.

COROLLARY 2.3. *Let $h : A \to B$ be a linear involution preserving 3-Jordan homomorphism between commutative C^*-algebras. Then h is norm contractive (that is, $\|h\| \le 1$).*

Also, by Theorem 2.2 and [7, Theorem 2.3], we have the following corollary.

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COROLLARY 2.4. *Let $h : A \to B$ be a linear involution preserving 4-Jordan homomorphism between commutative C^*-algebras; then h is completely positive. Thus h is bounded.*

Now we prove our main theorem.

THEOREM 2.5. *Suppose that A is a Banach algebra, which need not be commutative, and suppose that B is a semisimple commutative Banach algebra. Then each 3-Jordan homomorphism $h : A \to B$ is a 3-ring homomorphism.*

PROOF. We prove the theorem in two steps as follows.

STEP I. Suppose $B = \mathbb{C}$. We have $h(a^3) = h(a)^3$ for all $a \in A$. Replace a by $x + y$ to obtain

$$h(xyx + yx^2 + y^2x + x^2y + xy^2 + yxy) = 3(h(x)^2h(y) + h(x)h(y)^2) \tag{2.7}$$

and replace y by $-y$ in (2.7) to get

$$h(-xyx - yx^2 + y^2x - x^2y + xy^2 + yxy) = 3(-h(x)^2h(y) + h(x)h(y)^2). \tag{2.8}$$

By (2.7) and (2.8) we obtain the relation

$$h(xy^2 + y^2x + yxy) = 3(h(x)h(y)^2). \tag{2.9}$$

Replacing y by $y - z$ in (2.9), we get

$$\begin{aligned} &h(xy^2 + xz^2 - 2xyz + yxy - yxz - zxy + zxz + z^2x + y^2x - 2yzx) \\ &\quad = 3(h(x)^2h(y) + h(x)h(y)^2) - 6h(x)h(y)h(z). \end{aligned} \tag{2.10}$$

By (2.9) and (2.10), we obtain

$$h(yxz + zxy + 2xyz + 2yzx) = 6h(x)h(y)h(z). \tag{2.11}$$

Replacing z by x in (2.11), we get

$$h(3yx^2 + x^2y + 2xyx = 6h(x)^2h(y), \tag{2.12}$$

and combining (2.9) and (2.12), we obtain

$$h(xyx + 2yx^2) = 3h(x)^2h(y). \tag{2.13}$$

From (2.8) and (2.13), we conclude that

$$h(yx^2 - x^2y) = 0. \tag{2.14}$$

Replacing x by $x + z$ in (2.14), we get

$$h(yx^2 + yz^2 + 2yxz - x^2y - z^2y - 2xzy) = 0,$$

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and from this equality and (2.14) it follows that

$$h(yxz - xzy) = 0. \tag{2.15}$$

Combining (2.11) and (2.15) gives

$$h(yxz + 3xyz + 2yzx) = 6h(x)h(y)h(z), \tag{2.16}$$

and then replacing z by x in (2.16) leads to

$$h(xyx + yx^2) = 2h(x)^2h(y). \tag{2.17}$$

Finally, combining (2.13) and (2.17) to obtain

$$h(yx^2) = h(y)h(x)^2 \tag{2.18}$$

and then replacing x by $x + z$ in (2.18), we conclude that

$$h(yxz) = h(y)h(x)h(z);$$

hence, h is a 3-ring homomorphism.

STEP II. B is arbitrary semisimple and commutative. Let M_B be the maximal ideal space of B. We associate with each $f \in M_B$ a function $h_f : A \to \mathbb{C}$ defined by

$$h_f(a) := f(h(a))$$

for all $a \in A$. It is easy to see that h_f is additive and $h_f(a^3) = (h_f(a))^3$ for all $a \in A$. So step I applied to h_f implies that h_f is a 3-ring homomorphism. By the definition of h_f, we obtain that

$$f(h(abc)) = f(h(a))f(h(b))f(h(c)) = f(h(a)h(b)h(c)).$$

Hence

$$h(abc) - h(a)h(b)h(c) \in \operatorname{Ker}(f)$$

for all $a, b, c \in A$ and all $f \in M_B$. Since B is assumed to be semisimple, we get $h(abc) = h(a)h(b)h(c)$ for all $a, b, c \in A$. We thus conclude that h is a 3-ring homomorphism, and the proof is complete. □

From now on we consider such n-Jordan homomorphisms as are linear.

COROLLARY 2.6. *Suppose that A, B are C^*-algebras, where A need not be commutative, and suppose that B is semisimple and commutative. Then every involution preserving 3-Jordan homomorphism $h : A \to B$ is norm contractive (that is, $\|h\| \leq 1$).*

PROOF. It follows from Theorem 2.5 and [1, Theorem 2.1].

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THEOREM 2.7. *Let $h : A \to B$ be a bounded involution preserving k-Jordan homomorphism between C^*-algebras such that $h(a^*a) = h(a)^*h(a)$ for all $a \in A$. Then h is norm contractive (that is, $\|h\| \le 1$).*

PROOF. From [7, Lemma 2.4],

$$\begin{aligned}
\|h(a)\|^{4k+2} &= \|(h(a)^*h(a))^{2k+1}\| = \|(h(a)^*h(a))^k(h(a)^*h(a))(h(a)^*h(a))^k\| \\
&= \|[h(a)(h(a)^*h(a))^k]^*[h(a)(h(a)^*h(a))^k]\| \\
&= \|h(a)(h(a)^*h(a))^k\|^2 = \|h(a)(h(a^*a))^k\|^2 \\
&= \|h(a)(h(a^*a)^k)\|^2 \le \|h(a)\|^2\|h((a^*a)^k)\|^2 \\
&\le \|h\|^2\|a\|^2\|h\|^2\|(a^*a)^k\|^2 \\
&\le \|h\|^4\|a\|^{4k+2},
\end{aligned}$$

for all $a \in A$, which implies that $\|h\| \le 1$ by taking $(4k+2)$th roots. □

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