The Absolute Cesàro Summability of the Successively Derived Allied Series of a Fourier Series

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§ 1. *Introduction.*

We suppose that $f(t)$ is integrable in the Lebesgue sense in $(-\pi, \pi)$, and is periodic with period 2π. We denote its Fourier series by

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=0}^{\infty} A_n(t). \tag{1.1}$$

Then the allied series is

$$\sum_{n=1}^{\infty} (b_n \cos nt - a_n \sin nt) = \sum_{n=1}^{\infty} B_n(t). \tag{1.2}$$

We write

$$\phi(t) = \tfrac{1}{2}\{f(x+t) + f(x-t)\}, \qquad \psi(t) = \tfrac{1}{2}\{f(x+t) - f(x-t)\}, \tag{1.3}$$

so that

$$\phi(t) \sim \Sigma A_n \cos nt, \tag{1.4}$$

where $\qquad A_n = A_n(x),$

and

$$\psi(t) \sim \Sigma B_n \sin nt, \tag{1.5}$$

where $\qquad B_n = B_n(x).$

The series obtained by differentiating the allied series r times at $t = x$ is

$$\Sigma \left(\frac{d}{dx}\right)^r B_n(x) \begin{cases} = \Sigma(-1)^{\frac{1}{2}r} n^r B_n & (r \text{ even}) \\ = \Sigma(-1)^{\frac{1}{2}(r+1)} n^r A_n & (r \text{ odd}). \end{cases} \tag{1.6}$$

In a recent paper [1] Bosanquet has proved the following theorem.

Theorem A. *If $f(t) \epsilon L$ in $(-\pi, \pi)$, a necessary and sufficient condition for the series* (1.6) *to be summable* $(C, \alpha + r)$ *to sum s, where $\alpha \geqq 0$ and r is a positive integer, is that constants α_ν should exist such*

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[1] Referred to as $D_r \bar{F}S$ in the list of references.

that (i) *the odd function* $g(t)$ *is integrable* (CL) *in* $(0, \pi)$ *and its allied series summable* (C, α) *at* $t = 0$, (ii) $g(t)/t$ *is integrable* (CL) *in* $(0, \pi)$ *and*[1]

$$\frac{2}{\pi}\int_0^\infty \frac{g(t)}{t}\,dt = s, \tag{1.7}$$

where, for $0 < t < \pi$,

$$g(t)\begin{cases} = \dfrac{r!}{t^r}\left\{\psi(t) - \displaystyle\sum_{\nu=1}^{\frac{1}{2}r} \frac{a_{2\nu-1}}{(2\nu-1)!}\, t^{2\nu-1}\right\} & (r \text{ even}) \\ = \dfrac{r!}{t^r}\left\{\phi(t) - \displaystyle\sum_{\nu=0}^{\frac{1}{2}(r-1)} \frac{a_{2\nu}}{(2\nu)!}\, t^{2\nu}\right\} & (r \text{ odd}). \end{cases} \tag{1.8}$$

Analogous results concerning the (C) summability of the r-th derived Fourier series and the $|\,C\,|$ summability of the first derived Fourier series have also been given by Bosanquet in D_rFS and $|\,DFS\,|$ respectively. The object of the present paper is to obtain the $|\,C\,|$ analogue of Theorem A.

In Theorem 1 we give a general result concerning the summability $|\,C, \alpha + r\,|$ of the r-th derived allied series, where $\alpha > 1$. In Theorem 2 $f(t)$ is restricted to be a function of bounded variation, and a result is obtained for $\alpha > 0$.

§ 2. *Notation.*

We write

$$S_n^\alpha = \sum_{\mu=0}^{n} A_{n-\mu}^\alpha u_\mu, \qquad s_n^\alpha = S_n^\alpha \big/ A_n^\alpha \qquad (\alpha > -1) \tag{2.1}$$

for the n-th Cesàro partial sum and mean of order α of a series Σu_n,

where $$A_n^\alpha = \binom{n+\alpha}{n}.$$

The series Σu_n is said to be *summable* (C, α) *to* s if $s_n^\alpha \to s$, and to be *summable* $|\,C, \alpha\,|$ *to* s if, in addition, $\Sigma\,|\,\Delta s_n^\alpha\,| < \infty$.

We write $k^\sigma(n, t) + i\bar{k}^\sigma(n, t)$ for the n-th Cesàro mean of order σ of the series $\frac{1}{2} + \Sigma e^{int}$, and require the inequalities[2]

$$\left|\left(\frac{\partial}{\partial t}\right)^k k^\sigma(n, t)\right| \begin{cases} \leqq An^{k+1} \\ \leqq An^{k-\sigma}t^{-\sigma-1} + An^{-1}t^{-k-2} \end{cases} \tag{2.2}$$

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[1] The integral in (1.7) is convergent, and taken in the (CL) sense at the origin.

[2] Cf. $D_r\bar{F}S$, 64, $|\,FS_\alpha\,|$, 519, Obrechkoff and Zygmund.

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and
$$\left|\left(\frac{\partial}{\partial t}\right)^k \bar{k}^\sigma(n,t)\right| \begin{cases} \leqq An^{k+1}, \\ \leqq An^{k-\sigma}t^{-\sigma-1} + At^{-k-1}. \end{cases} \tag{2.3}$$

We write

$$\bar{\gamma}_\sigma(t) = \int_0^1 (1-u)^{\sigma-1} \sin tu\, du \qquad (\sigma > 0), \tag{2.4}$$

and require the inequalities [1]

$$| \Delta^k \bar{\gamma}_\sigma^{(\lambda)}(t) | \begin{cases} \leqq At^k \\ \leqq An^{-\sigma}t^{k-\sigma} + An^{-k-\lambda-1}t^{-\lambda-1}. \end{cases} \tag{2.5}$$

Here and elsewhere $\Delta u_n = u_n - u_{n+1}$, and A denotes a positive number, independent of the variables but not necessarily the same at each occurrence.

The Cesàro-Lebesgue integral. Suppose that $g(t)\epsilon L$ in (ϵ, a) for every $0 < \epsilon < a$ (a fixed). If

$$\lim_{\epsilon \to 0} \int_\epsilon^a g(t)dt \tag{2.6}$$

exists, $g(t)$ is said to be *integrable* C_0L *in* $(0, a)$, with the limit (2.6) as the value of the integral. If λ is a positive integer and

(i) $G(t) = \int_t^a g(u)du$ is integrable $C_{\lambda-1}L$ in $(0, a)$,

$$\text{(ii)} \lim_{\epsilon \to 0} \frac{\lambda}{\epsilon^\lambda} \int_0^\epsilon (\epsilon - u)^{\lambda-1} G(u)du \tag{2.7}$$

exists, then $g(t)$ is said to be *integrable* $C_\lambda L$ *in* $(0, a)$, with the limit (2.7) as the value of the integral.

We write

$$\begin{aligned} G_\sigma(t) &= \frac{1}{\Gamma(\sigma)} \int_0^t (t-u)^{\sigma-1} g(u)du \qquad (\sigma > 0) \\ G_0(t) &= g(t), \end{aligned} \tag{2.8}$$

and similar notation is employed with ξ, χ, ..., Ξ_λ, X_λ, ... in place of g, G_λ.

The absolute Cesàro-Lebesgue integral. A function $g(t)$ is said to be *integrable* $| C_\lambda L |$ *in* $(0, a)$, where λ is a non-negative integer, if (i)

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[1] Cf. $D_r\bar{F}S$, 64, and DFS, 273.

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it is integrable $C_\lambda L$ in $(0, a)$, (ii) $t^{-\lambda} G_{\lambda+1}(t)$ is of bounded variation in the interval $0 < t \leqq a$. The common value of the $C_\lambda L$ and $| C_\lambda L |$ integrals is then $\lim_{t \to +0} \lambda ! \, t^{-\lambda} G_{\lambda+1}(t)$.

If
$$t^{-\lambda - p} G_\lambda(t) \to \frac{\Gamma(p+1)}{\Gamma(\lambda+p+1)} s \tag{2.9}$$
as $t \to +0$ we write $g(t) \sim s t^p (C, \lambda)$. If, in addition, $t^{-\lambda - p} G_\lambda(t)$ is of bounded variation in an interval $0 < t \leqq a$ we write $g(t) \sim s t^p \mid C, \lambda \mid$.

Properties of the (CL) and $| CL |$ integrals will be found in *DFS*, $D_r FS$, *CL*, and $| DFS |$.

§ 3. *Preliminary lemmas.*

In Lemmas 1 and 2 we shall suppose that $g(t) \epsilon L$ in every interval (δ, Δ), for $0 < \delta < \Delta < \infty$, and write, for $t > 0$,

$$\xi(t) = t^{\sigma - 1} \int_t^\infty \frac{g(u)}{u^\sigma} du \qquad (\sigma > 0), \tag{3.1}$$

the integral being assumed to be convergent. It is known[1] that if $g(t) \epsilon C_{\lambda+1} L$ in $(0, a)$, then

(i) $\xi(t) \epsilon C_\lambda L$ in $(0, a)$,

(ii) for $a \geqq 0$, $t > 0$,

$$\frac{\Xi_a(t)}{t^a} = t^{\sigma - 1} \int_t^\infty \frac{G_a(u)}{u^{a+\sigma}} du. \tag{3.2}$$

LEMMA 1. *If $g(t) \epsilon \mid C_{\lambda+1} L \mid$ in $(0, a)$, where λ is a non-negative integer, then $\xi(t) \epsilon \mid C_\lambda L \mid$ in $(0, a)$.*

We first prove that $\xi(t) \epsilon \mid C_{\lambda+1} L \mid$ in $(0, a)$. Since $g(t) \epsilon C_{\lambda+1} L$, it follows, by (i), that $\xi(t) \epsilon C_\lambda L$, and hence $\xi(t) \epsilon C_{\lambda+1} L$. Thus in order to prove that $\xi(t) \epsilon \mid C_{\lambda+1} L \mid$ we must prove that $t^{-\lambda-1} \Xi_{\lambda+2}(t)$ is of bounded variation in the interval $0 < t \leqq a$.

We have, by (3.2),[2]

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[1] *CL*, Theorems 20 and 21.

[2] Defining $t^{-\lambda-1} \Xi_{\lambda+2}(t)$ and $t^{-\lambda-1} G_{\lambda+2}(t)$ as zero at $t = 0$. Since $g(t)$ and $\xi(t)$ are integrable $C_{\lambda+1} L$ these functions are then continuous at the origin.

$$\int_0^a \left| d\left\{ t^{-\lambda-1}\Xi_{\lambda+2}(t) \right\} \right| = \int_0^a \left| d\left\{ t^{\sigma} \int_t^{\infty} \frac{G_{\lambda+2}(u)}{u^{\lambda+2+\sigma}}\, du \right\} \right|$$

$$\leqq \int_0^a \left| d\left\{ t^{\sigma} \int_t^{a} \frac{u^{-\lambda-1}G_{\lambda+2}(u)}{u^{1+\sigma}}\, du \right\} \right| + A$$

$$= \int_0^a \left| d\left\{ t^{\sigma} \left[-\frac{1}{\sigma u^{\sigma}} u^{-\lambda-1} G_{\lambda+2}(u) \right]_t^a \right.\right.$$

$$\left.\left. + \frac{t^{\sigma}}{\sigma} \int_t^a \frac{1}{u^{\sigma}} d\left(u^{-\lambda-1} G_{\lambda+2}(u) \right) \right\} \right| + A$$

$$\leqq \frac{1}{\sigma} \int_0^a \left| d\left(t^{-\lambda-1} G_{\lambda+2}(t) \right) \right|$$

$$+ \frac{1}{\sigma} \int_0^a \left| d\left\{ t^{\sigma} \int_t^a \frac{1}{u^{\sigma}} d\left(u^{-\lambda-1} G_{\lambda+2}(u) \right) \right\} \right| + A$$

$$\leqq \frac{2}{\sigma} \int_0^a \left| d\left(t^{-\lambda-1} G_{\lambda+2}(t) \right) \right| + \int_0^a t^{\sigma-1} dt \int_t^a \frac{1}{u^{\sigma}} \left| d\left(u^{-\lambda-1} G_{\lambda+2}(u) \right) \right|$$

$$+ A$$

$$= \frac{2}{\sigma} \int_0^a \left| d\left(t^{-\lambda-1} G_{\lambda+2}(t) \right) \right| + \int_0^a \frac{1}{u^{\sigma}} \left| d\left(u^{-\lambda-1} G_{\lambda+2}(u) \right) \right| \int_0^u t^{\sigma-1} dt$$

$$+ A$$

$$= \frac{3}{\sigma} \int_0^a \left| d\left(t^{-\lambda-1} G_{\lambda+2}(t) \right) \right| + A$$

$$< \infty,$$

since $g(t) \epsilon \mid C_{\lambda+1} L \mid$.

We next prove that $t\xi(t) = o(1) \mid C, \lambda+1 \mid$ as $t \to +0$. Integrating by parts $\lambda + 2$ times, we have, as in the proof of Theorem 20 of CL,

$$t\xi(t) = -\sum_{\rho=1}^{\lambda+2} \frac{\Gamma(\sigma+\rho-1)}{\Gamma(\sigma)} \frac{G_{\rho}(t)}{t^{\rho-1}} + \frac{\Gamma(\sigma+\lambda+2)}{\Gamma(\sigma)} t^{\sigma} \int_t^{\infty} \frac{G_{\lambda+2}(u)}{u^{\lambda+2}} du. \quad (3.3)$$

Since $g(t) \epsilon \mid C_{\lambda+1} L \mid$ in $(0, a)$, it follows[1] that $G_{\rho}(t) = o(t^{\rho-1}) \mid C, \lambda+2-\rho \mid$, for $\rho = 1, 2, \ldots, \lambda+2$, and hence[2] that $G_{\rho}(t)/t^{\rho-1} = o(1) \mid C, \lambda+2-\rho \mid$. Also the integral in (3.3) is $o(1)$ as $t \to +0$, as in the proof of Theorem 20 of CL, and we have just proved that it is of bounded variation in $0 < t \leqq a$. It follows that all the terms in (3.3) are $o(1) \mid C, \lambda+1 \mid$, i.e. that $t\xi(t) = o(1) \mid C, \lambda+1 \mid$.

Finally we deduce that $\xi(t) \epsilon C_{\lambda} L$. Writing $\chi(t) = t\xi(t)$, we have[3]

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[1] | *DFS* |, 18. [2] | *DFS* |, Lemma 1. [3] *CL*, 55.

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$$X_{\lambda+1}(t) = t\Xi_{\lambda+1}(t) - (\lambda+1)\Xi_{\lambda+2}(t). \tag{3.4}$$

Hence

$$t^{-\lambda}\Xi_{\lambda+1}(t) = t^{-\lambda-1}X_{\lambda+1}(t) + (\lambda+1)t^{-\lambda-1}\Xi_{\lambda+2}(t). \tag{3.5}$$

But we have just proved that the two expressions on the right of (3.5) are of bounded variation in $0 < t \leqq a$, and hence so is $t^{-\lambda}\Xi_{\lambda+1}(t)$. We have also observed that $\xi(t)\epsilon C_\lambda L$, and thus it follows that $\xi(t)\epsilon \mid C_\lambda L \mid$.

LEMMA 2. *If $g(t)\epsilon \mid CL \mid$ in $(0, a)$, λ is a non-negative integer and $\sigma > 0$, then a necessary and sufficient condition that $g(t)/t^\sigma \epsilon \mid C_\lambda L \mid$ in $(0, a)$ and*

$$\int_0^\infty \frac{g(u)}{u^\sigma} du = l, \tag{3.6}$$

is that $G_\lambda(t)/t^{\lambda+\sigma}$ should be integrable L in $(0, a)$, and

$$\int_0^\infty \frac{G_\lambda(u)}{u^{\lambda+\sigma}} du = \frac{\Gamma(\sigma)}{\Gamma(\lambda+\sigma)} l. \tag{3.7}$$

Necessity. Suppose that $g(t)/t^\sigma\epsilon \mid C_\lambda L \mid$ and that (3.6) holds, i.e. that $\xi(t)/t^{\sigma-1} \to l \mid C, \lambda \mid$ as $t \to +0$. It follows[1] that $\xi(t) \sim lt^{\sigma-1} \mid C, \lambda \mid$ as $t \to +0$, i.e. that

$$\frac{\Xi_\lambda(t)}{t^{\lambda+\sigma-1}} \sim \frac{\Gamma(\sigma)}{\Gamma(\lambda+\sigma)} l \ (C, 0), \tag{3.8}$$

which, by (3.2), is (3.7).

Sufficiency. Since $\xi(t)\epsilon \mid CL \mid$ in $(0, a)$, by Lemma 1, the sufficiency is established by reversing the argument.

LEMMA 3. *If k is a non-negative integer, and Σu_n is summable $\mid C, k \mid$ to s, then*

$$\frac{2\sigma}{\pi}\int_0^\infty \frac{1}{u} \lim_{\rho\to 1-0} \Sigma u_n \bar{\gamma}_\sigma(nu)\rho^n \, du = s \tag{3.9}$$

for $\sigma > k$, where the integral in (3.9) is an absolutely convergent Lebesgue integral.

We have[2]

[1] $\mid DFS \mid$, Lemma 1.

[2] Cf. D_rFS, Lemma 4. We write $s_{-1}^k = 0$.

$$\begin{aligned} S(t) &= \lim_{\rho \to 1-0} \Sigma u_n \bar{\gamma}_\sigma(nt)\rho^n \\ &= \Sigma S_n^k \Delta^{k+1}\bar{\gamma}_\sigma(nt) \\ &= \Sigma s_n^k A_n^k \Delta^{k+1}\bar{\gamma}_\sigma(nt) \\ &= -\Sigma \Delta s_{n-1}^k J(n, t), \end{aligned}$$

where
$$J(n, t) = \sum_{\nu=n}^{\infty} A_\nu^k \Delta^{k+1}\bar{\gamma}_\sigma(\nu t). \tag{3.10}$$

By the analogue for ordinary Cesàro summability, (3.9) certainly holds with the integral interpreted as a C_0L integral in $(0, a)$. It will therefore be sufficient to prove that

$$\int_0^\infty t^{-1} \mid S(t) \mid dt < \infty . \tag{3.11}$$

We have

$$\begin{aligned} \int_0^\infty t^{-1} \mid S(t) \mid dt &\leqq \int_0^\infty t^{-1}\Sigma \mid \Delta s_{n-1}^k J(n, t) \mid dt \\ &= \Sigma \mid \Delta s_{n-1}^k \mid \int_0^\infty t^{-1} \mid J(n, t) \mid dt. \end{aligned}$$

Now

$$\mid J(n, t) \mid \begin{cases} \leqq An^{k+1}t^{k+1} \\ \leqq An^{-1}t^{-1} + An^{k-\sigma}t^{k-\sigma}. \end{cases} \tag{3.12}$$

For

$$J(0, t) = \sum_{\nu=0}^{\infty} A_\nu^k \Delta^{k+1}\bar{\gamma}_\sigma(\nu t) = \sum_{\nu=0}^{\infty} \Delta \bar{\gamma}(\nu t) = 0,$$

and hence

$$\begin{aligned} J(n, t) = J(n, t) - J(0, t) &= -\sum_{\nu=0}^{n-1} A_\nu^k \Delta^{k+1}\bar{\gamma}_\sigma(\nu t) \\ &= \sum_{\nu=1}^{n} O(\nu^k) O(t^{k+1}) = O(n^{k+1}t^{k+1}). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \sum_{\nu=n}^{\infty} A_\nu^k \Delta^{k+1}\bar{\gamma}_\sigma(\nu t) &= A_n^k \Delta^k \bar{\gamma}_\sigma(nt) + \sum_{\nu=n+1}^{\infty} A_\nu^{k-1} \Delta^k \bar{\gamma}_\sigma(\nu t) \\ &= O(n^k)\{O(n^{-k-1}t^{-1}) + O(n^{-\sigma}t^{k-\sigma})\} \\ &\quad + \sum_{\nu=n+1}^{\infty} O(\nu^{k-1})\{O(\nu^{-k-1}t^{-1}) + O(\nu^{-\sigma}t^{k-\sigma})\} \\ &= O(n^{-1}t^{-1}) + O(n^{k-\sigma}t^{k-\sigma}) \end{aligned}$$

for $\sigma > k$.

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It follows from (3.12) that, for $n \geqq 0$,

$$\int_0^\infty t^{-1} \mid J(n, t) \mid dt = \int_0^{n^{-1}} O(n^{k+1}t^k)dt + \int_{n^{-1}}^\infty \{O(n^{-1}t^{-2}) + O(n^{k-\sigma}t^{k-\sigma-1})\}dt$$

$$= O(1) + O(1)$$

for $\sigma > k$.

Thus $\displaystyle\int_0^\infty t^{-1} \mid S(t) \mid dt \leqq A\Sigma \mid \Delta s_{n-1}^k \mid < \infty$.

LEMMA 4. *If $g(t)\epsilon \mid C_k L \mid$ in $(0, \pi)$, where k is a non-negative integer, and if*

$$\alpha_n + i\beta_n = \int_0^\pi g(t)e^{int}dt, \tag{3.13}$$

then α_n and β_n are $o(1) \mid C, \sigma \mid$ as $n \to \infty$, for $\sigma > k + 1$.

Writing α_n^σ, $c^\sigma(n, t)$ for the n-th Cesàro means of order σ of the sequences α_n and $\cos nt$ respectively, we have [1]

$$\alpha_n^\sigma = \int_0^\pi g(t)c^\sigma(n, t)dt$$

$$= \Big[\sum_{\rho=0}^k (-1)^\rho G_{\rho+1}(t) \left(\frac{\partial}{\partial t}\right)^\rho c^\sigma(n, t) \Big]_0^\pi + (-1)^{k+1} \int_0^\pi G_{k+1}(t)\left(\frac{\partial}{\partial t}\right)^{k+1} c^\sigma(n, t)dt$$

$$= \sum_{\rho=0}^k (-1)^\rho G_{\rho+1}(\pi)\left\{\left(\frac{\partial}{\partial t}\right)^\rho c^\sigma(n, t)\right\}_{t=\pi} + (-1)^{k+1} \int_0^\pi d\{t^{-k}G_{k+1}(t)\}\int_t^\pi v^k \left(\frac{\partial}{\partial v}\right)^{k+1} c^\sigma(n, v)dv,$$

where $t^{-k}G_{k+1}(t)$ is defined as zero for $t = 0$. Hence it follows that

$$\Sigma \mid \Delta \alpha_n^\sigma \mid \leqq \sum_{\rho=0}^k \mid G_{\rho+1}(\pi) \mid \Sigma \mid \Delta\left(\frac{\partial}{\partial t}\right)^\rho c^\sigma(n, t) \mid_{t=\pi} + \int_0^\pi \mid d\{t^{-k}G_{k+1}(t)\} \mid \Sigma \left| \int_t^\pi v^k \Delta\left(\frac{\partial}{\partial v}\right)^{k+1} c^\sigma(n, v)dv \right|. \tag{3.14}$$

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[1] Cf. $D, \bar{F}S$, Lemma 9.

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Now

$$\left|\Delta\left(\frac{\partial}{\partial t}\right)^{\rho} c^{\sigma}(n, t)\right| \begin{cases} \leqq An^{\rho}t \\ \leqq An^{\rho-\sigma}t^{-\sigma+1} + An^{-3}\, t^{-\rho-2}, \end{cases} \tag{3.15}$$

$$\left|\int_0^t v^k \Delta\left(\frac{\partial}{\partial v}\right)^{k+1} c^{\sigma}(n, v)dv\right| \leqq An^{k+1}t^{k+2}, \tag{3.16}$$

$$\left|\int_t^{\pi} v^k \Delta\left(\frac{\partial}{\partial v}\right)^{k+1} c^{\sigma}(n, v)dv\right| \leqq An^{k-\sigma}t^{k-\sigma+1} + An^{-3}t^{-2}, \tag{3.17}$$

and

$$\left|\int_0^{\pi} v^k \Delta\left(\frac{\partial}{\partial v}\right)^{k+1} c^{\sigma}(n, v)dv\right| \leqq An^{k-\sigma} + An^{-3}. \tag{3.18}$$

The case $\rho = 0$ of (3.15) is given in | *DFS* |, and the general case is obtained similarly by use of (2.2).[1] We obtain (3.16) immediately from (3.15) i, while (3.17) follows from (3.15) ii after integration by parts, and (3.18) follows from (3.15) after repeated integrations by parts.

The first term on the right of (3.14) is

$$\sum_{\rho=0}^{k} \Sigma\{O(n^{\rho-\sigma}) + O(n^{-3})\} < \infty .$$

Also
$$\int_0^{\pi} \mid d\{t^{-k}G_{k+1}(t)\} \mid < \infty ,$$

since $g(t) \epsilon \mid C_k L \mid$. It will therefore be sufficient to prove that

$$J(t) = \Sigma\left|\int_t^{\pi} v^k \Delta\left(\frac{\partial}{\partial v}\right)^{k+1} c^{\sigma}(n, v)dv\right|$$

is bounded in $0 \leqq t \leqq \pi$.

Disposing of the case $t = 0$ by (3.18), we write, for $0 < t \leqq \pi$,

$$J(t) = \sum_{n < t^{-1}} + \sum_{n > t^{-1}} = \Sigma_1 + \Sigma_2.$$

Thus, by (3.16) and (3.18),

$$\Sigma_1 = \sum_{n < t^{-1}} \{O(n^{k+1}t^{k+2}) + O(n^{k-\sigma}) + O(n^{-3})\}$$
$$= O(1)$$

for $\sigma > k + 1$, and, by (3.17),

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[1] $c^{\sigma}(n, t) = \frac{\sigma}{n+\sigma}\left\{k^{\sigma-1}(n, t) - \tfrac{1}{2}\right\}.$

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$$\Sigma_2 = \sum_{n > t^{-1}} \{O(n^{k-\sigma} t^{k-\sigma+1}) + O(n^{-3} t^{-2})\}$$
$$= O(1)$$

for $\sigma > k + 1$.

This completes the proof of the result for a_n; that for β_n is similar, (2.3) taking the place of (2.2).

§ 4. *The main theorem.*

THEOREM 1. *If $f(t)$ is integrable L in $(-\pi, \pi)$ and periodic with period 2π, a necessary and sufficient condition that the series* (1.6) *should be summable* $| C, \alpha + r |$ *to the sum s, where $\alpha > 1$ and r is a positive integer, is that constants a_ν should exist such that* (i) *the odd function $g(t)$ is integrable* $| CL |$ *in* $(0, \pi)$ *and its allied series summable* $| C, \alpha |$ *at* $t = 0$, (ii) $g(t)/t$ *is integrable* $| CL |$ *in* $(0, \pi)$ *and* (1.7) *holds,*[1] *where $g(t)$ is defined by* (1.8).

The proof depends on the following lemmas.

LEMMA 5. *If $f(t) \epsilon L$ and the series* (1.6) *is summable* $| C |$ *to s, then there exist constants a_ν such that* (i) $g(t)/t \,\epsilon\, | CL |$ *in* $(0, \pi)$, *where $g(t)$ is given by* (1.8), (ii) (1.7) *is satisfied.*

The proof is similar to that of Lemma 10 of $D_r\bar{F}S$, but with Lemmas 2 and 3 of the present paper taking the place of Lemmas 5 and 8 respectively of $D_r\bar{F}S$.

LEMMA 6. *If $g(t)$ is odd and* $g(t)/t \,\epsilon\, | CL |$ *in* $(0, \pi)$, *then* $g(t) \,\epsilon\, | CL |$ *in* $(0, \pi)$ *and its allied series is summable* $| C |$ *at* $t = 0$.

The proof is similar to that of the sufficiency part of Lemma 11 of $D_r\bar{F}S$, but with Lemma 2 of $| DFS |$ and Lemma 4 of the present paper taking the place of Lemmas 2 and 9 respectively of $D_r\bar{F}S$, and with Lemma 3 of $D_r\bar{F}S$ replaced by its $| C |$ analogue.[2]

LEMMA 7. *If constants a_ν exist such that the odd function $g(t)$, defined by* (1.8) *in* $(0, \pi)$, *is integrable* $| CL |$ *in* $(0, \pi)$, *then constants* β_ν

[1] The integral in (1.7) being convergent and existing as a | CL | integral at the origin.

[2] The proof of this is similar to that of Lemma 2 of | *DFS* |.

exist such that the odd function $G(t)$ *is integrable* $|\,CL\,|$ *in* $(0, \pi)$, *where, for* $0 < t < \pi$,

$$G(t) \begin{cases} = \dfrac{r!}{(2\sin\frac{1}{2}t)^r}\left\{\psi(t) - \sum_{\nu=1}^{\frac{1}{2}r} \dfrac{\beta_{2\nu-1}}{(2\nu-1)!}\sin^{2\nu-1} t\right\} & (r \text{ even}) \\ = \dfrac{r!}{(2\sin\frac{1}{2}t)^r}\left\{\phi(t) - \sum_{\nu=0}^{\frac{1}{2}(r-1)} \dfrac{\beta_{2\nu}}{(2\nu)!}\sin^{2\nu} t\right\} & (r \text{ odd}); \end{cases} \qquad (4.1)$$

and if the allied series of either $g(t)$ *or* $G(t)$ *is summable* $|\,C, \alpha\,|$ *at* $t = 0$, *where* $\alpha > 0$, *then so is that of the other.*

The proof is similar to that of Lemma 12 of $D_r\bar{F}S$, but with Lemma 2 of $|\,DFS\,|$ and the case $k = 0$ of Lemma 4 of the present paper taking the place of Lemma 2 of $D_r\bar{F}S$ and the Riemann-Lebesgue theorem respectively.

LEMMA 8. *If* (i) $G(t)\epsilon\,|\,CL\,|$ *in* $(0, \pi)$, (ii)$t^rG(t)\epsilon L$ *in* $(0, \pi)$, *where* r *is a positive integer, and if*

$$\beta(\mu) = \frac{2}{\pi}\int_0^\pi G(t)\sin\mu t\,dt, \qquad (4.2)$$

then if one of the series $\Sigma\beta(n)$ *and* $\Sigma\beta(n - \frac{1}{2}r)$ *is summable* $|\,C, \alpha\,|$, *where* $\alpha > 1$, *so is the other.*

The proof is similar to that of Lemma 13 of $D_r\bar{F}S$, but with Lemma 2 of $|\,DFS\,|$ and the case $k = 0$ of Lemma 4 of the present paper taking the place of Lemma 2 of $D_r\bar{F}S$ and the Riemann-Lebesgue theorem respectively.

LEMMA 9. *If* $G(t)$ *is defined by* (4.1) *and* $\beta(\mu)$ *by* (4.2), *then*

$$n^r\,\Delta^r\beta(n - \tfrac{1}{2}r) = r!(d/dx)^r(b_n\cos nx - a_n\sin nx) \qquad \textit{for } n \geqq r.$$

This is Lemma 14 of $D_r\bar{F}S$.

LEMMA 10. *If* Σu_n *is summable* $|\,C\,|$, *then a necessary and sufficient condition for* $\Sigma n^r\Delta^r u_n$ *to be summable* $|\,C, \alpha + r\,|$, *where* $\alpha > -1$ *and* r *is a positive integer, is that* Σu_n *be summable* $|\,C, \alpha\,|$.

The proof is similar to that of Lemma 14 of $D_r\bar{F}S$.[1]

Proof of Theorem 1. *Necessity.* Suppose that the series (1.6) is summable $|\,C, \alpha + r\,|$ to sum s, where $\alpha > 1$. Then, by Lemma 5, constants α_ν exist such that $g(t)/t\,\epsilon\,|\,CL\,|$ in $(0, \pi)$, where $g(t)$ is given

[1] The case $r = 1$ was given in $|\,DFS\,|$ (Lemma 8). See also Chow, and Bosanquet and Chow.

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by (1.8), and (1.7) holds. It follows, by Lemma 6, that $g(t)\epsilon \mid CL \mid$ in $(0, \pi)$ and its allied series is summable $\mid C \mid$ at $t = 0$.

Now, by Lemma 7, $g(t)$ defines a function $G(t)$, given by (4.1), which is integrable $\mid CL \mid$ in $(0, \pi)$, and its allied series is also summable $\mid C \mid$ at $t = 0$, i.e. $\Sigma\beta(n)$ is summable $\mid C \mid$, where $\beta(n)$ is defined by (4·2). It follows, by Lemma 8, that $\Sigma\beta(n - \frac{1}{2}r)$ is summable $\mid C \mid$. Now, by Lemma 9, our hypothesis is that $\Sigma n^r \Delta^r\beta(n - \frac{1}{2}r)$ is summable $\mid C, a+r \mid$. Therefore, by the necessity part of Lemma 10, $\Sigma\beta(n-\frac{1}{2}r)$ is summable $\mid C, a \mid$, and, by Lemma 8, so also is $\Sigma\beta(n)$, i.e. the allied series of $G(t)$ is summable $\mid C, a \mid$ at $t = 0$. It follows, by Lemma 7, that the allied series of $g(t)$ is summable $\mid C, a \mid$ at $t = 0$.

Thus the conditions are necessary.

Sufficiency. Suppose that constants a_ν exist such that the odd function $g(t)$, given by (1.8), is integrable $\mid CL \mid$ in $(0, \pi)$, that its allied series is summable $\mid C, a \mid$ at $t = 0$, where $a > 1$, and that (1.7) holds, the function $g(t)/t$ being integrable $\mid CL \mid$ in $(0, \pi)$. Then $g(t)$ defines $G(t)$, given by (4.1), which is also integrable $\mid CL \mid$, and its allied series is summable $\mid C,a \mid$ at $t = 0$, i.e. $\Sigma\beta(n)$ is summable $\mid C, a \mid$. Then by Lemma 8, $\Sigma\beta(n - \frac{1}{2}r)$ is summable $\mid C, a \mid$ and, by Lemma 9 and the sufficiency part of Lemma 10, the series (1.6) is summable $\mid C, a + r \mid$. Finally, by (1.7) and the necessity part of the theorem, the sum is s.

This completes the proof of the theorem.

§ 5. *Additional result.*

THEOREM 2. *If the function $f(t)$ in Theorem 1 is of bounded variation in $(-\pi, \pi)$, then the result of Theorem 1 holds for $a > 0$.*

The proof of Theorem 2 is similar to that of Theorem 1 except that at the points in the proof where we used the case $k = 0$ of Lemma 4 we now use the following lemma.

LEMMA 11. *If $g(t)$ is of bounded variation in $(0, \pi)$, and a_n, β_n are given by (3.13), then a_n and β_n are $o(1) \mid C, \sigma \mid$ as $n \to \infty$, for $\sigma > 0$.*

To prove the result for a_n we suppose, as we may, that $g(t)$ is even and let

$$g(t) \sim \frac{2}{\pi}\Sigma a_n \cos nt.$$

Since $g(t)$ is of bounded variation in $(0, \pi)$ it follows from a theorem of Bosanquet[1] that Σa_n is summable $\mid C, \sigma \mid$, for $\sigma > 0$, and hence $a_n = o(1) \mid C, \sigma \mid$.

[1] | FS. |. Theorem 1.

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To prove the result for β_n we write

$$\beta_n - \beta_{n-1} = \int_0^\pi g(t)\{\sin nt - \sin(n-1)t\}dt$$

$$= \int_0^\pi g(t) \sin t \cos nt\, dt + \int_0^\pi g(t)(1 - \cos t) \sin nt\, dt$$

$$= \gamma_n + \delta_n.$$

Then, by the same theorem of Bosanquet and a theorem of Bosanquet and Hyslop,[1] $\Sigma\gamma_n$ and $\Sigma\delta_n$ are both summable $|C, \sigma|$, for $\sigma > 0$. Since $\beta_n = o(1)$, by the Riemann-Lebesgue theorem, the result follows.

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[1] Bosanquet and Hyslop, Theorem, 4.

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