Canad. Math. Bull. Vol. 17 (1), 1974

MATHEMATICAL NOTES

Manuscripts for this Department should be sent to R. D. Bercov and A. Meir, Editors-in-Chief Canadian Mathematical Bulletin, Department of Mathematics, University of Alberta, Edmonton, 7, Alberta.

ON A THEOREM OF NIVEN

BY
ROBERT E. DRESSLER

Introduction. In [3], Niven proved that for any positive integer k, the density of the set of positive integers n for which $(n, (\varphi(n)) \le k$ is zero (where φ is the Euler totient function). In this paper, we prove a related result—namely if k and j are any positive integers, then the density of the set of positive integers n for which $(n, \sigma_j(n)) \le k$ is zero (where $\sigma_j(n)$ is the sum of the jth powers of the positive divisors of n). We will borrow from Niven's technique, but we must make some crucial modifications.

Before we prove the theorem, we recall the following formula.

$$(\#) \quad \sigma_j(n) = \prod_{p^e \| n} (p^{ej} + p^{(e-1)j} + \cdots + p^j + 1)$$

THEOREM 1. *For any positive integers k and j, the density of the set of positive integers n for which $(n, \sigma_j(n)) \le k$ is zero. That is, if $A_{j,k}(m)$ is the number of positive integers n not exceeding m for which $(n, \sigma_j(n)) \le k$, then* $\lim_{m\to\infty} A_{j,k}(m)/m = 0$.

Proof. We use the following two results of Niven [3].

(1) For any fixed positive integer b, if $\{p_i\}$ is a set of primes for which $\sum p_i^{-1} = \infty$, and if T is any sequence whose members are divisible by at most b of these primes only to the first degree, then $d(T)=0$ (where $d(T)$ denotes the density of T).

(2) For a sequence T of positive integers, let T_p be the set of elements of T which are divisible by p but not by p^2. If for a set of primes $\{p_i\}$ we have $d(T_{p_i})=0$ for for every i and if $\sum p_i^{-1} = \infty$, then $d(T)=0$.

Since finite unions of sets of density zero are also of density zero, it suffices to prove that the density of the set T of positive integers n such that $(n, \sigma_j(n)) = k$ is zero. With r defined so that $2^r \| j$, we define a set of primes $\{p_i\}$ by

$$\{p_i\} = \{p; p \nmid k \text{ and } p \equiv 1 \pmod{2^{r+1}}\}$$

By Dirichlet's Theorem, $\sum p_i^{-1} = \infty$ and so, by (2), it will suffice to show that $d(T_{p_i})=0$ for each i.

Choose an i, and recall (cf. [2, Theorem 4-13]) that the congruence $x^j \equiv -1 \pmod{p_i}$ is solvable if and only if $(-1)^{(p_i-1)/d} \equiv 1 \pmod{p_i}$, where $d=(j, p_i-1)$.

Since $2 \mid (p_i - 1)/d$, it follows that the congruence $x^j \equiv -1 \pmod{p_i}$ has a solution, t_i. Let $\{q_s\}$ be the sequence of primes of the form $yp_i + t_i$.

Now any member n of T_{p_i} can be written $n = mp_i$, where $(m, p_i) = 1$. Also, $(\sigma_j(m), p_i) = 1$ because otherwise $p_i \mid \sigma_j(m)$, whence $p_i \mid \sigma_j(n)$ and so $p_i \mid (n, \sigma_j(n))$, which is a contradiction of the definition of p_i. Thus any prime divisor of m which divides m only to the first degree cannot, by (#), be a member of $\{q_s\}$. For if so, $\sigma_j(m)$ would have a factor $q_s^j + 1 \equiv (yp_i + t_i)^j + 1 \equiv t_i^j + 1 \equiv 0 \pmod{p_i}$, whereas we know $(\sigma_j(m), p_i) = 1$. Since by Dirichlet's Theorem $\sum q_s^{-1} = \infty$, it follows from (1) that the set of permissible values for m has density zero. Thus $d(T_{p_i}) = 0$ and we are done.

Finally, we have the following result.

THEOREM 2. *For any positive integers k and j, the density of the set of positive integers n for which $(\varphi(n), \sigma_j(n)) \leq k$ is zero.*

Proof. If $\omega(m)$ is the number of distinct prime divisors of the positive integer m and $\Omega(m)$ is the total number of prime divisors of m, then it is known that the density of the set of positive integers m satisfying both

$$\tfrac{4}{5} \log\log m < \omega(m) < \tfrac{6}{5} \log\log m \tag{1}$$

and

$$\tfrac{4}{5} \log\log m < \Omega(m) < \tfrac{6}{5} \log\log m \tag{2}$$

is 1. (cf. [1, Theorem 431].

From (1) we see that the density of the set of positive integers m satisfying

$$\omega(m) > 2k \tag{3}$$

is 1. Also from (1) and (2) we see that the density of the set of positive integers m such that

$$1 \leq \frac{\Omega(m)}{\omega(m)} \leq \frac{3}{2} \tag{4}$$

is 1.

Since the density of the set of positive integers m satisfying both (3) and (4) is 1, it follows that the density of the set of positive integers m having at least k odd prime divisors which divide m only to the first degree is 1. (If $\omega(m) > 2k$ and m has less than k odd prime divisors which divide m only to the first degree then $\Omega(m)/\omega(m) > 3/2$.) For these m, $2^k \mid \varphi(m)$ and $2^k \mid \sigma_j(m)$ and so $(\varphi(m), \sigma_j(m)) > k$. Thus the density of the set of positive integers n for which $(\varphi(n), \sigma_j(n)) \leq k$ is zero.

BIBLIOGRAPHY

1. Hardy, G. H. and Wright, E. M. *An Introduction to the theory of numbers*. Oxford University Press, Oxford, Fourth Edition (1960).

2. Le Veque, W. J., *Topics in number theory*, Vol. I, Addison-Wesley Publishing Co. (1958).

3. Niven, I., *The asymptotic density of sequences*, Bull. A.M.S., **57** (1951), pp. 420–434.