SUBORDINATE AND PSEUDO-SUBORDINATE SEMI-ALGEBRAS

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1. Notation and definitions. Throughout this paper E denotes a compact Hausdorff space, which, to avoid trivial complications, is assumed to contain at least two points. $C(E)$, with the uniform norm, is the Banach algebra of all continuous real-valued functions defined on E; $C^+(E)$ is the set of those functions in $C(E)$ which take only non-negative values. A subset of $C(E)$ is a *wedge* if and only if it is closed under addition and multiplication by non-negative scalars; a *semi-algebra* is a wedge closed under (pointwise) multiplication. The set $C^+(E)$ is a semi-algebra, and all semi-algebras considered in this paper are contained in $C^+(E)$. For a subset K of $C^+(E)$, the closed wedge (semi-algebra) *generated by* K is the least closed wedge (semi-algebra) containing K. Note that the closure of a wedge (semi-algebra) is a wedge (semi-algebra) as well. A semi-algebra M is a *maximal-closed* subsemi-algebra of $C^+(E)$ if and only if (i) M is a proper closed subsemi-algebra of $C^+(E)$ and (ii) $M \subseteq A \subseteq C^+(E)$ for a closed semi-algebra A entails that $A = M$ or $A = C^+(E)$.

For $f, g \in C(E)$, define $f \cup g$ for $\xi \in E$ by

$$(f \cup g)(\xi) \equiv \max(f(\xi), g(\xi));$$

for $f \in C^+(E)$, $\lambda > 0$, define f^λ for $\xi \in E$ by

$$f^\lambda(\xi) \equiv \text{principal value } (f(\xi))^\lambda.$$

A subset of $C(E)$ is an *upper semi-lattice* if and only if it contains, along with the functions f and g, the function $f \cup g$. A subset of $C^+(E)$ is *power-closed* if and only if for each positive real λ it contains, along with the function f, the function f^λ. A *cornet* is a closed subsemi-algebra of $C^+(E)$ which is power-closed. For any subset K of $C^+(E)$, K_u and $\sqrt{K}$ denote, respectively, the least closed upper semi-lattice and the least cornet containing K.

The dual space of $C(E)$ is the space $M(E)$ of Radon measures on E **(2)**. The set of positive measures is denoted by $M^+(E)$. For a wedge W, the *dual wedge* W' is the set of measures which are non-negative on W. The Jordan decomposition of a measure μ is given by $\mu = \mu_+ - \mu_-$; the *positive* (*negative*)

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support of μ is the support of μ_+ (μ_-). The support of the measure μ is denoted by $S(\mu)$. For $f \in C(E)$, $\mu \in M(E)$, $f \cdot \mu$ is the measure defined by

$$(f \cdot \mu)(g) \equiv \mu(fg)(g \in C(E)).$$

For $\xi \in E$, the measure with total mass unity all concentrated at ξ is denoted by δ_ξ.

2. Discussion of results. The study of the subsemi-algebras of $C^+(E)$ begun in **(1)** is continued here. Let σ be a probability measure (positive with total mass unity) on E. For $f \in C^+(E)$, the *geometric mean* $GM_\sigma f$ with weight σ is defined to be $\exp \int \log f \, d\sigma$ when the extended real-valued function $\log f$ is σ-integrable, 0 otherwise. A *geometric semi-algebra* is one of the form

$$H_{\sigma,\xi} \equiv \{f : f \in C^+(E), f(\xi) \leqslant GM_\sigma f\}$$

where $\xi \in E$ and σ is a probability measure with no mass at ξ. The following results were established in **(1)**:

(1) The maximal closed subsemi-algebras of $C^+(E)$ are precisely the geometric semi-algebras (Theorems 1.5, 2.5).

(2) A closed semi-algebra containing the identity function 1 is a cornet if and only if it is the intersection of a family of geometric semi-algebras (Theorem 2.1, Corollary 2, and the remark at the top of p. 12).

(3) A cornet without the identity is contained in a proper cornet with identity (Theorem 2.3 and remarks following the proof of Theorem 2.2).

A closed subsemi-algebra of $C^+(E)$ is said to be *subordinate* if and only if it is contained in a geometric semi-algebra. Because of the results (2) and (3) above, every cornet is subordinate. Since $C^+(E)$ contains semi-algebras that are not cornets (e.g. $\{f: 2f(\xi) \leqslant f(\eta)\}$ for fixed $\xi, \eta \in E$), it is natural to ask whether every proper closed subsemi-algebra is subordinate. This is true, almost trivially, when E is finite (§4). However, a class of spaces E such that $C^+(E)$ contains a non-subordinate proper closed subsemi-algebra is given in §6. It is shown in §5 that a semi-algebra is subordinate if its dual contains a measure whose negative support is not a subset of its positive support. Another sufficient condition for a closed semi-algebra to be subordinate was given in **(1)** (Theorem 1.3), namely that it have a non-void interior.

Observing that, when f and g belong to $C^+(E)$, $f \cup g = \lim_{n\to\infty} (f^n + g^n)^{1/n}$, one deduces that every cornet (in particular, every geometric semi-algebra) is an upper semi-lattice. Call a closed subsemi-algebra of $C^+(E)$ *pseudo-subordinate* if and only if it is contained in a proper closed subsemi-algebra that is also an upper semi-lattice. Any subordinate semi-algebra is pseudo-subordinate but not conversely. Indeed, it is not known whether any non-pseudo-subordinate semi-algebras exist. The results of §7 shed a little light on the properties of pseudo-subordinate semi-algebras. In particular, it is shown that a semi-algebra generated by a power-closed set is subordinate if and only if it is pseudo-subordinate. The final section contains a list of the principal open questions, the answers to which have been eluding the author for some time.

This paper is presented as an account of some of the progress made in a very interesting line of research.

3. Preliminary results. For μ a positive Radon measure on E and ξ a point of E, define

$$U_{\mu,\xi} \equiv \{f: f \in C(E), \mu(f) \geqslant f(\xi)\},$$
$$U_\mu \equiv \{f: f \in C(E), \mu(f) \geqslant 0\}.$$

The following proposition is proved in **(3)**.

PROPOSITION 1 (Choquet–Deny). *Let W be a closed wedge contained in $C(E)$ which is an upper semi-lattice. Suppose that $\mathfrak{U}_1$ is the family of all pairs (μ, ξ) with $\xi \in E$, μ a positive Radon measure with no mass at ξ and $\mu - \delta_\xi \in W'$; suppose that $\mathfrak{U}_2$ is the family of all positive Radon measures μ such that $\mu \in W'$. Then*

$$W = (\bigcap \{U_{\mu,\xi}: (\mu, \xi) \in \mathfrak{U}_1\}) \cap (\bigcap \{U_\mu: \mu \in \mathfrak{U}_2\}).$$

The next proposition is Theorem 2.1 in **(1)**. It is obtained from Proposition 1 by considering the logarithms of invertible elements of P. Note that the convention that a void intersection is the whole space is adopted.

PROPOSITION 2. *Let P be a closed subset of $C^+(E)$ which satisfies the conditions:*
(i) *P is closed under multiplication;*
(ii) *$\lambda P \subseteq P$ for each positive real λ;*
(iii) *P is an upper semi-lattice;*
(iv) *P is power-closed;*
(v) *P contains the identity function* 1.
Let $\mathfrak{F}$ be the set of all pairs (σ, ξ) with σ a probability measure and ξ a point of E such that $P \subseteq H_{\sigma,\xi}$. Then $P = \bigcap \{H_{\sigma,\xi}: (\sigma, \xi) \in \mathfrak{F}\}$, so that P is a cornet.

PROPOSITION 3. *Let K be a subset of $C^+(E)$ which is closed under multiplication. Then the closed semi-algebra generated by K is identical with the closed wedge generated by K,* i.e. *the closure of the set*

$$\left\{\sum_{i=1}^{m} \alpha_i q_i: \alpha_i \geqslant 0, q_i \in K, m \text{ a positive integer}\right\}.$$

PROPOSITION 4. *Let A be a proper closed subsemialgebra of $C^+(E)$ and let $A_1 \equiv \{f + \alpha: f \in A, \alpha \geqslant 0\}$. Then A_1 is the closed subsemi-algebra generated by A along with the identity function* 1. *The semi-algebra A_1 is proper. A is subordinate if and only if A_1 is subordinate.*

Proof. See **(1**, Lemmas 1.1, 1.2**)**. To prove the final statement, observe that each geometric semi-algebra contains 1.

PROPOSITION 5. *Let A be a proper closed subsemi-algebra of $C^+(E)$ and f a function belonging to A. Then the closed semi-algebra generated by A and all positive real powers of the particular function f is also proper.*

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Proof. This generalizes (**1**, Lemma 2.3), but the present proof is somewhat different. Two cases are distinguished.

Case (i): *for each* $\mu \in A'$, $\mu_-(f) = 0$. This means that f vanishes almost everywhere (μ_-), hence that f^λ vanishes almost everywhere (μ_-). Thus $\mu(f^\lambda) = \mu_+(f^\lambda) \geqslant 0$ $(\mu \in A', \lambda > 0)$, so that, since A is closed, each positive power of f is in A. Hence the generated semi-algebra is A and the result follows; see (**4**, p. 22, Corollary 3).

Case (ii): *for some measure* $\nu \in A'$, $\nu_-(f) > 0$. With no loss of generality, suppose that $0 \leqslant f \leqslant 1$. Note that, since $f \in A$, all positive integral powers of f belong to A. Thus B is the closure of the set (which is a semi-algebra) of all finite sums of the form

$$(*) \qquad g_0 + \sum_{i=1}^{k} g_i f^{\lambda_i}$$

where $g_i \in A$ $(i = 0, 1, \ldots, k)$ and $0 < \lambda_i < 1$ $(i = 1, \ldots, k)$. Define the measure $\omega \equiv \nu_+ - f\cdot\nu_-$. This measure is not positive. Once it is demonstrated that ω takes a positive value at each sum of the form $(*)$ and hence is a member of B', the result will be proved.

Observe that, with λ_i and g_i as specified above, $0 \leqslant f \leqslant f^{\lambda_i} \leqslant 1$ and

$$\sum_{i=0}^{k} g_i f \in A.$$

Using the positivity of ν_+ and ν_- and the fact $\nu \in A'$, one obtains

$$\nu_+\left(g_0 + \sum_{i=1}^{k} g_i f^{\lambda_i}\right) \geqslant \nu_+\left(\sum_{i=0}^{k} g_i f\right) \geqslant \nu_-\left(\sum_{i=0}^{k} g_i f\right);$$

$$\nu_-\left(\sum_{i=0}^{k} g_i f\right) = f\cdot\nu_-\left(\sum_{i=0}^{k} g_i\right) \geqslant f\cdot\nu_-\left(g_0 + \sum_{i=1}^{k} g_i f^{\lambda_i}\right),$$

so that

$$\nu_+\left(g_0 + \sum_{i=1}^{k} g_i f^{\lambda_i}\right) \geqslant f\cdot\nu_-\left(g_0 + \sum_{i=1}^{k} g_i f^{\lambda_i}\right),$$

i.e.

$$\omega\left(g_0 + \sum_{i=1}^{k} g_i f^{\lambda_i}\right) \geqslant 0.$$

PROPOSITION 6. *Let A be a closed subsemi-algebra of $C^+(E)$.*

(a) *$\sqrt{A}$ is the closure of the set*

$$\{f : f \in C^+(E), f^n \in A \text{ for a positive integer } n \equiv n(f)\}.$$

(b) *A is subordinate if and only if $\sqrt{A} \neq C^+(E)$.*

Proof. (a) See (**1**, Theorems 2.2, 2.4).

(b) If $\sqrt{A} \neq C^+(E)$, then $\sqrt{A}$ is a proper cornet that is necessarily contained in a geometric semi-algebra. On the other hand, if A is contained in a geometric semi-algebra, then so also is $\sqrt{A}$.

Remark. Proposition 5 allows one to adjoin to a proper closed semi-algebra all positive powers of the functions belonging to a finite set of its elements and still obtain a proper semi-algebra. That the word "finite" cannot be changed to "infinite" in general is attested to by the fact that non-subordinate semi-algebras do exist.

PROPOSITION 7. *Let K be a subset of $C^+(E)$. Then K_u is the closure of the set $\{f_1 \cup f_2 \cup \ldots \cup f_m : f_i \in K \ (i = 1, \ldots, m), m$ an integer$\}$. If K possesses any of the following properties, then so does K_u: closure under addition, closure under multiplication, closure under multiplication by positive scalars, power-closure.*

If A is a subsemi-algebra of $C^+(E)$, then A_u is the least closed wedge which is an upper semi-lattice and the least closed semi-algebra which is an upper semi-lattice containing A. Finally A is pseudo-subordinate if and only if A_u is proper.

Proof. The set in brackets is clearly an upper semi-lattice; so also is its closure since union is continuous. That this set (and hence its closure) may inherit any of the four listed properties from K is seen from the equalities for $f_i, g_j \in C^+(E)$, $\lambda > 0$, where i and j are indexed over finite sets:

$$(\cup f_i) + (\cup g_j) = \cup (f_i + g_j), \qquad \lambda(\cup f_i) = \cup (\lambda f_i),$$
$$(\cup f_i)(\cup g_j) = \cup (f_i g_j), \qquad (\cup f_i)^\lambda = \cup (f_i^\lambda).$$

The proof of the assertions for A_u, the least closed upper semi-lattice containing A, is straightforward.

4. The situation for E finite.

THEOREM 1. *Let E be a finite compact Hausdorff space. Every proper closed subsemi-algebra of $C^+(E)$ is subordinate and pseudo-subordinate.*

Proof. Since E has the discrete topology, the topology on $C(E)$ is that of pointwise convergence. By Proposition 6, it suffices to show that for a closed semi-algebra A, $\sqrt{A} = C^+(E)$ implies that $A = C^+(E)$. Let

$$E = \{\xi_1, \xi_2, \ldots, \xi_m\}$$

and k_r be the characteristic function of the singleton $\{\xi_r\}$ $(r = 1, 2, \ldots, m)$; $k_r \in C^+(E)$. Suppose $\sqrt{A} = C^+(E)$ and let $0 < \epsilon < 1$. Then there exists a function f in A and a positive integer n such that $0 \leqslant f^{1/n}(\xi_s) < \frac{1}{2}\epsilon$ $(1 \leqslant s \leqslant m;$ $s \neq r)$ and $1 - \frac{1}{2}\epsilon < f^{1/n}(\xi_r) < 1 + \frac{1}{2}\epsilon$. The function $g \equiv (f(\xi_r))^{-1} f$ belongs to A and satisfies $g(\xi_s) < \epsilon^n < \epsilon$ $(s \neq r)$, $g(\xi_r) = 1$, so that

$$k_r = \lim_{p \to \infty} g^p \in A.$$

It follows that A contains the characteristic function of each singleton, and hence contains the closed wedge $C^+(E)$ generated by these functions.

5. A sufficient condition for subordination. It is already known (**1**, Theorem 1.3) that when a closed semi-algebra A has a non-void interior, then it is subordinate. In this section is given another sufficient condition, this time on A', that A be subordinate.

THEOREM 2. *Let A be a proper closed subsemi-algebra of $C^+(E)$ which contains the identity function* 1. *Suppose that the dual wedge A' of A contains a measure μ whose negative support is not a subset of its positive support. Then for each point ξ of E belonging to $S(\mu_-)\backslash S(\mu_+)$, there exists a probability measure σ, whose support $S(\sigma)$ is contained in $S(\mu_+)$, such that $A \subseteq H_{\sigma,\xi}$.*

Remark. The presence of the identity ensures that $S(\mu_+)$ is non-void. However, the hypothesis that $1 \in A$ can be dropped without essentially altering the conclusion. For, suppose $1 \notin A$ and A' contains a measure as described in the theorem. If $S(\mu_+)$ is void, then each function in A vanishes on the support of μ_- with the result that $A \subseteq H_{\sigma,\xi}$ for any point $\xi \in S(\mu_-)$ and any probability measure σ having no mass at ξ. (Of course the conclusion on $S(\sigma)$ is not valid in this case.) On the other hand, if $S(\mu_+)$ is non-void, then a positive integer M can be chosen so that $M\mu_+(1) \geqslant \mu_-(1)$. Thus $M\mu_+ - \mu_-$ is a member of A'_1 and the theorem above can be applied to A_1. We prepare for the proof of the theorem with the following lemma.

LEMMA 1. *Let A be a proper subsemi-algebra of $C^+(E)$. Suppose that μ belongs to A' and that f belongs to A. If f is strictly less than unity on $S(\mu_+)$, then f is strictly less than unity on $S(\mu)$.*

Proof. The modifications of the following argument are easy to make if $S(\mu_+)$ or $S(\mu_-)$ is void. Since $S(\mu_+)$ is compact, there exists a positive real δ (not exceeding $\frac{1}{2}$) such that $f(\eta) \leqslant 1 - 2\delta$ $(\eta \in S(\mu_+))$. The set

$$U \equiv \{\eta \colon \eta \in E, f(\eta) > 1 - \delta\}$$

is open and does not intersect $S(\mu_+)$. Furthermore, for *each* positive integer n, $f^n \in A$, so that

$$\begin{aligned}\int_U d\mu_- &\leqslant (1-\delta)^{-n}\int_E f^n\, d\mu_- \\ &\leqslant (1-\delta)^{-n}\int_E f^n\, d\mu_+ \\ &\leqslant (1-\delta)^{-n}(1-2\delta)^n\int_E d\mu_+,\end{aligned}$$

whence

$$\int_U d\mu_- = 0 \text{ and } U \cap S(\mu_-) = \emptyset.$$

The lemma follows.

Proof of Theorem 2. Let $\xi \in S(\mu_-)\backslash S(\mu_+)$ and define

$$Z \equiv \{\log f \colon f \in A, f(\xi) = 1, f \text{ bounded away from } 0\}.$$

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Since $1 \in A$, Z is non-void. Furthermore, Z is closed under addition and, by Lemma 1, each function $u \in Z$ has the property: $u(\eta) \geqslant 0$ for some point η, depending on u, belonging to $S(\mu_+)$. Let Y be the closure of the set

$$\bigcup \{n^{-1}Z\colon n = 1, 2, \ldots\}.$$

It is straightforward to verify that Y is the closed wedge generated by Z. Let Z_0, Y_0 be the set of restrictions of functions in Z, Y, respectively, to the set $S(\mu_+)$. Since each function in Z_0 takes a non-negative value on $S(\mu_+)$, $\bigcup \{n^{-1}Z_0\colon n = 1, 2, \ldots\}$, hence its closure containing Y_0 is disjoint from the (non-void) interior of $-C^+(S(\mu_+))$.

There is a Radon measure σ_0 on $S(\mu_+)$ such that

$$\int v d\sigma_0 \leqslant \int u\, d\sigma_0 \qquad \text{if } v \in -C^+(S(\mu_+)),\ u \in Y_0$$

(**4**, p. 22). It is clear that σ_0 is positive and belongs to Y_0'. The measure σ on E such that $\int u\, d\sigma = \int u_0\, d\sigma_0$ for all $u \in C(E)$, u_0 being the restriction of u to $S(\mu_+)$, is then positive, takes non-negative values on Z, is non-trivial and supported by $S(\mu_+)$. Arrange that σ has total mass 1.

It follows that when $f \in A$, $f(\xi) = 1$ and f is bounded away from 0, then

$$f(\xi) = 1 \leqslant \exp \int \log f\, d\sigma = GM_\sigma f.$$

For g an arbitrary element of A and $\lambda > 0$, define $g_\lambda \equiv (g(\xi) + \lambda)^{-1}(g + \lambda)$. Then g_λ satisfies the above conditions on f so that $g(\xi) + \lambda \leqslant GM_\sigma(g + \lambda)$. Since $\lim_{\lambda\to 0^+} GM_\sigma(g + \lambda) = GM_\sigma\, g$ (**1**), the result follows.

Remark. The sufficient condition on the dual given in Theorem 2 is not necessary. Let F be the one-point compactification of the positive integers. A typical element x in $C(F)$ can be represented by the "sequence" $(x_1, x_2, \ldots, x_n, \ldots; x_0)$ where $x_0 \equiv \lim x_k$ is the value of x at the point at infinity, ∞. Let $A = H_{\sigma,\infty}$ where

$$\sigma(x) \equiv \sum_{n=1}^{\infty} 2^{-n} x_n \qquad (x \in C(F)).$$

Suppose, if possible, that A' contains a functional μ with the properties of Theorem 2. Since A contains the characteristic function of every singleton except $\{\infty\}$, $S(\mu_-) = \{\infty\}$. If $S(\mu_-) \subseteq S(\mu_+)$ does not hold, then $S(\mu_+)$ is a finite set containing exactly the integers $k_1, k_2, \ldots, k_m$. There exists an element $z \in A$ such that

$$z_0 = 1,\ z_{k_i} = \tfrac{1}{2}\left(\int d\mu_+\right)^{-1} \qquad (i = 1, 2, \ldots, m).$$

But then $\int z\, d\mu_+ = \frac{1}{2} < 1 = \int z\, d\mu_-$, a contradiction. Hence A satisfies the conclusion but not the hypothesis of Theorem 2.

6. Spaces that admit a non-subordinate semi-algebra. Because of Proposition 4, if $C^+(E)$ contains a non-subordinate proper closed subsemi-algebra, it contains one with the identity. Consequently, there is no harm in restricting attention to semi-algebras containing 1.

LEMMA 2. *Let A be a subsemi-algebra of $C^+(E)$ containing the identity function 1. Suppose that A is contained in a geometric semi-algebra $H_{\sigma,\xi}$. Let G be a closed subset of E with the following property: if $g \in C^+(E)$ and there exists a function $h \in A$ such that g and h coincide on G, then $g \in A$. It is concluded that ξ belongs to G and the support of σ is contained in G.*

Proof. Suppose if possible that $\xi \notin G$. Choose an open neighbourhood U of ξ not intersecting G. Since E is normal, a function e_U can be chosen with $0 \leqslant e_U \leqslant 1$, $e_U(\xi) = 1$, $e_U(\eta) = 0$ $(\eta \notin U)$. Since e_U coincides with the function 0 on G, $e_U \in A$, so that

$$1 \leqslant GM_\sigma e_U \leqslant \int_E e_U \, d\sigma \leqslant \int_U d\sigma = \sigma(U).$$

By the regularity of σ,

$$0 = \sigma(\{\xi\}) = \inf\{\sigma(U) \colon U \text{ open}, U \cap G = \emptyset, \xi \in U\} \geqslant 1,$$

a contradiction. Thus $\xi \in G$.

Now let W be an open neighbourhood of G and f_W be a function such that $f_W(\eta) = 1 (\eta \in G)$ and $f_W(\zeta) = 0$ $(\zeta \notin W)$. Then, since $1 \in A, f_W \in A$, so that

$$1 = f_W(\xi) \leqslant GM_\sigma f_W \leqslant \int_E f_W \, d\sigma \leqslant \int_W d\sigma = \sigma(W) \leqslant \sigma(E) = 1.$$

Again by the regularity of σ, $\sigma(G) = 1$. Since σ has total mass equal to unity, $\sigma(\backslash G) = 0$ and the lemma follows.

THEOREM 3. *Let E be a compact Hausdorff space and G be a closed subspace such that $C^+(G)$ contains a non-subordinate proper closed subsemi-algebra. Then $C^+(E)$ contains a non-subordinate proper closed subsemi-algebra.*

Proof. Let B be a non-subordinate proper closed subsemi-algebra of $C^+(G)$ with $1 \in B$. Let A be the set of those functions in $C^+(E)$ whose restrictions to G belong to B. Observe that by the Tietze extension theorem (**5**, p. 43) every function in B is the restriction of a function in A. The set A is a proper closed subsemi-algebra of $C^+(E)$. If A were subordinate, then A would be contained in a geometric semi-algebra $H_{\sigma,\xi}$. But by Lemma 2, both ξ and $S(\sigma)$ are contained in G so that $\sigma \in M(G)$ and $u(\xi) \leqslant GM_\sigma u$ for each $u \in B$, yielding a contradiction.

Example. Let F be the one-point compactification of the positive integers described in §5 (Remark). For a positive integer m and real $\lambda \geqslant 2^m - 1$, define $e(m, \lambda) \equiv (0, 0, \ldots, 0, \lambda, 1, \ldots, 1, \ldots, ; 1)$, i.e.,

$$e(m, \lambda)_n \equiv \begin{cases} 0 & \text{if } n < m, \\ \lambda & \text{if } n = m, \\ 1 & \text{if } n > m, \end{cases}$$

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and let K be the set of all such elements. K is closed under multiplication so that, by Proposition 3, the closed wedge R generated by K is a semi-algebra. Furthermore, R is proper in $C^+(F)$, for the linear functional ω given by

$$\omega(x) \equiv \left(\sum_{n=1}^{\infty} 2^{-n}x_n\right) - x_0 \qquad (x \in C(F))$$

is a non-positive member of R'. It remains only to show that $\sqrt{R} = C^+(F)$ and to apply Proposition 6(b).

When $\lambda > 1$,

$$\lim_{n\to\infty} [\lambda^{-1}e(m, \lambda)]^n = (0, 0, \ldots, 0, 1, 0, \ldots ; 0)$$

and

$$\lim_{n\to\infty} [e(m, \lambda)]^{1/n} = (0, 0, \ldots, 0, 1, 1, \ldots ; 1)$$

so that $\sqrt{R}$ contains all elements of the forms

$$(0, \ldots, 0, 1, 0, \ldots, ; 0) \quad \text{and} \quad (0, \ldots, 0, 1, 1, \ldots ; 1).$$

Let $z \in C^+(F)$, $\epsilon > 0$. Then there is an integer N such that

$$|z_n - z_0| < \epsilon (n \geqslant N).$$

The element

$$z(\epsilon) \equiv (z_1, z_2, \ldots, z_N, z_0, z_0, \ldots ; z_0)$$

belongs to $\sqrt{R}$ and $||z - z(\epsilon)|| < \epsilon$. Since ϵ was arbitrary and $\sqrt{R}$ is closed, $z \in \sqrt{R}$. Thus R is non-subordinate.

THEOREM 4. *If E is an infinite compact metric space, then $C^+(E)$ contains a non-subordinate proper closed subsemi-algebra.*

Proof. E contains a closed subset G consisting of a convergent sequence of distinct points along with its limit. Since G is homeomorphic to F, the result follows by Theorem 3 and the above example.

7. Considerations involving upper semi-lattices. To begin with, a theorem that describes the semi-algebra A_u formed from the semi-algebra A is proved. In fact, it is shown that A_u can be expressed as the intersection of semi-algebras of a certain elementary type. This representation is of some interest, and will be referred to in §8.

Some notation is established. For a subsemi-algebra A of $C^+(E)$,

$N(A) \equiv \{\eta: \eta \in E, f(\eta) = 0 \text{ for all } f \in A\}$,

$\mathfrak{A}(A) \equiv \{(\mu, \xi): \xi \in E, \mu$ a positive measure with no mass at $\xi, \mu - \delta_\xi \in A'$, i.e. $A \subseteq U_{\mu,\xi}\}$,

$\mathfrak{A}_\xi(A) \equiv \{\mu: \mu$ a positive measure with no mass at ξ, $\mu - \delta_\xi \in A'\}$ for each particular $\xi \in E$,

$B_\xi(A) \equiv \{f: f \in C^+(E), f(\xi) \leqslant \mu(f)$ for each $\mu \in \mathfrak{A}_\xi(A)\}$.

It is understood that $B_\xi(A) = C^+(E)$ when $\mathfrak{A}_\xi(A)$ is void. Observe that $N(A) = N(A_u)$; $\mathfrak{A}(A) = \mathfrak{A}(A_u)$; $0 \in \mathfrak{A}_\xi(A)$ if and only if $\xi \in N(A)$. (0 is considered a positive measure.)

THEOREM 5. *Let E be a compact Hausdorff space and A be a subsemi-algebra of $C^+(E)$. Then:*

(a) *A_u is the set of functions f belonging to $C^+(E)$ and vanishing on $N(A)$ such that $(\mu - \delta_\xi)(f) \geqslant 0$ for all $(\mu, \xi) \in \mathfrak{A}(A)$;*

(b) *each $B_\xi(A)$ is a closed upper semi-lattice semi-algebra and*

$$A = \bigcap \{B_\xi(A) \colon \xi \in E\}.$$

Proof. (a) See Propositions 1, 7. Note that, with $W = A_u$ in Proposition 1, $\mathfrak{U}_1 = \mathfrak{A}(A)$ and $\mathfrak{U}_2 = M^+(E)$.

(b) The only difficulty is in showing that each $B_\xi(A)$ is a semi-algebra. Let $u, v \in B_\xi(A)$. If either u or v vanishes at ξ, then so also does uv; thus $uv \in B_\xi(A)$. Suppose, if possible, that u is non-zero at ξ; define $u_1 \equiv (u(\xi))^{-1}u$. Let $\mu - \delta_\xi \in A'$. Suppose $f \in A$. If $f(\xi) \neq 0$, let $f_1 \equiv (f(\xi))^{-1}f$. Then for $g \in A$, $f_1 g \in A$ and $\mu(f_1 g) - (f_1 g)(\xi) \geqslant 0$. Hence $f_1 \cdot \mu - \delta_\xi \in A'$ so that

$$0 \leqslant (f_1 \cdot \mu)(u_1) - u_1(\xi) = (u_1 \cdot \mu)(f_1) - 1.$$

Thus $(u_1 \cdot \mu)(f) \geqslant f(\xi)$. Since this last inequality holds trivially when $f(\xi) = 0$, it follows that $u_1 \cdot \mu - \delta_\xi \in A'$. Hence $(u_1 \cdot \mu)(v) - v(\xi) \geqslant 0$ with the result that

$$\mu(uv) - u(\xi)v(\xi) = u(\xi)[\mu(u_1 v) - v(\xi)] \geqslant 0$$

whenever $\mu - \delta_\xi \in A'$. It is concluded that $uv \in B_\xi(A)$.

COROLLARY. *For a closed subsemi-algebra A of $C^+(E)$, either one of the following two conditions is necessary and sufficient for A to be pseudo-subordinate:*

(1) *A' contains a measure of the form $\mu - \delta_\xi$ with $\xi \in E$ and μ a positive measure with no mass at ξ;*

(2) *there exists a point $\xi \in E$ and a family M of positive measures each having no mass at ξ such that*

$$J_{M,\xi} \equiv \{f \colon f \in C^+(E), f(\xi) \leqslant \mu(f) \text{ for all } \mu \in M\}$$

is a proper closed subsemi-algebra containing A.

Remark. The semi-algebra $J_{M,\xi}$ can be considered as a weak analogue of a geometric semi-algebra. Indeed, it is shown in the author's thesis (Newcastle-upon-Tyne, 1964) that the geometric semi-algebra $H_{\sigma,\xi}$ is equal to $J_{M,\xi}$ where

$$M = \{(h(\xi))^{-1}h \cdot \sigma \colon h \in H_{\sigma,\xi}, h \text{ bounded away from } 0, h(\xi) = GM_\sigma h\}.$$

THEOREM 6. *Let F be the one-point compactification of the positive integers. Then every proper closed subsemi-algebra of $C^+(F)$ is pseudo-subordinate.*

Proof. Let A be a non-pseudo-subordinate closed semi-algebra and μ be a member of A'. Suppose, if possible, that μ has negative mass at the point ξ.

Then $\mu = \mu_1 - \mu_2 - \beta\delta_\xi$ where μ_1 and μ_2 are positive measures and $\beta > 0$. Also, A' would contain the measure

$$\beta^{-1}\mu_1 - \delta_\xi = \beta^{-1}(\mu_1 - \mu_2 - \beta\delta_\xi) + \beta^{-1}\mu_2,$$

contrary to the Corollary (a) of Theorem 5. The fact that μ has no negative mass at any point entails that no positive integer belongs to the negative support of μ, and hence that the point at infinity does not belong to the support of μ either. Thus μ must be positive.

Theorem 6 and the Example in Section 6 show that a pseudo-subordinate semi-algebra need not be subordinate. However, there is one situation in which the two concepts coincide.

THEOREM 7. *Let A be a closed subsemi-algebra of $C^+(E)$.*

(a) *If $1 \in A$ and A is generated by a power-closed set, then A_u is a cornet.*

(b) *If A is generated by a power-closed set, then A is subordinate if and only if A is pseudo-subordinate.*

Proof. (a) The set $B \equiv \{f: f^\lambda \in A \text{ for all } \lambda > 0\}$ is a subset of A containing the generators of A and satisfying the conditions (i), (ii), (iv), (v) of Proposition 2. Since B_u satisfies all five conditions of Proposition 2, B_u is a cornet. Thus B_u contains A, and hence contains A_u. On the other hand, $B \subseteq A$ so that $B_u \subseteq A_u$. It follows that $A_u = B_u$, a cornet.

(b) If A is subordinate, then A is pseudo-subordinate. On the other hand, suppose that A is pseudo-subordinate. Then there is a point ξ in E and a positive measure μ with no mass at ξ such that $\mu - \delta_\xi \in A'$. For ν a probability measure, $\mu + \nu - \delta_\xi \in A_1'$ so that A_1 (defined in Proposition 4) is pseudo-subordinate. Since the generators of A along with 1 generate A_1, (a) entails that A_1 is subordinate. Thus A, contained in A_1, is subordinate.

Example. A semi-algebra generated by a power-closed set need not be a cornet. Let $E = [0, 1]$, K be the cornet generated by the function $\chi(x) \equiv x$. Then K_1, obtained by adjoining the identity to K, is generated by the power-closed set $K \cup \{1\}$. Let σ be any probability measure with $GM_\sigma\, \chi \neq 0$, whose mass is not concentrated at a single point, and let ξ be the number $GM_\sigma\, \chi \in (0, 1]$. Clearly, $K_1 \subseteq H_{\sigma,\xi}$. Consider the function $(1 + \chi)^{\frac{1}{2}}$; this function belongs to the *cornet* generated by $K \cup \{1\}$. Supposing that $(1 + \chi)^{\frac{1}{2}}$ is a member of K_1, there would be a function $h \in K$ and a positive real α such that $(1 + \chi)^{\frac{1}{2}} = h + \alpha$. Since χ vanishes at 0, so does every function in K, including h; therefore $\alpha = 1$ and $h = (1 + \chi)^{\frac{1}{2}} - 1$. Let $g \equiv (1 + \chi)^{\frac{1}{2}} + 1$; observe that $gh = \chi$. Making use of the properties of the geometric mean (**1**, Section 0.2), in particular superadditivity and the fact that $GM_\sigma\, 1 = 1$, one finds that

$$\begin{aligned} g(\xi) &= (1 + \chi)^{\frac{1}{2}}(\xi) + 1 = (1 + \xi)^{\frac{1}{2}} + 1 \\ &= (1 + GM_\sigma\, \chi)^{\frac{1}{2}} + 1 < GM_\sigma(1 + \chi)^{\frac{1}{2}} + 1 \\ &< GM_\sigma((1 + \chi)^{\frac{1}{2}} + 1) = GM_\sigma\, g. \end{aligned}$$

The inequalities are strict since χ is not equal to a constant multiple of 1 a.e. (σ). Then

$$h(\xi) = \chi(\xi)/g(\xi) > \xi/GM_\sigma g = GM_\sigma \chi/GM_\sigma g = GM_\sigma h,$$

so that h does not belong to $H_{\sigma,\xi}$; a fortiori, h does not belong to K. Thus, a function $(1 + \chi)^{\frac{1}{2}}$ has been found belonging to the cornet but not to the closed semi-algebra generated by $K \cup \{1\}$.

THEOREM 8. *For the following three properties of the compact Hausdorff space E, the chain* (1) $\Rightarrow$ (2) $\Rightarrow$ (3) *of implications holds:*

(1) *$C^+(E)$ contains no non-pseudo-subordinate proper closed subsemi-algebra;*

(2) *every proper closed subsemi-algebra of $C^+(E)$ generated by a power-closed set is subordinate;*

(3) *every proper closed subsemi-algebra of $C^+(E)$ generated by a finite set of functions is subordinate.*

Proof. (1) $\Rightarrow$ (2). Use Theorem 7(b).

(2) $\Rightarrow$ (3). If A be a proper closed subsemi-algebra generated by a finite set F, then, by Proposition 5 applied to each generator in turn, one sees that the closed semi-algebra generated by all powers of elements in F is proper and contains A. By (2), this semi-algebra is subordinate.

Remark. The statements in Theorem 8 hold for the space F described earlier.

8. Some open questions. The results of this paper have arisen out of an attempt to resolve the following two conjectures:

CONJECTURE 1. *If E is an infinite compact Hausdorff space, then $C^+(E)$ contains a non-subordinate proper closed subsemi-algebra.*

CONJECTURE 2. *For any compact Hausdorff space E, every proper closed subsemi-algebra of $C^+(E)$ is pseudo-subordinate.*

A few open questions are listed:

1. Either prove or disprove each conjecture. In the case of disproof, find those compact Hausdorff spaces for which the conjecture holds. Because of §6, Conjecture 1 is open only for compact spaces E which contain no converging sequence of distinct points. Does Conjecture 2 hold when $E = [0, 1]$?

2. Give simple necessary and sufficient conditions that a finite set of functions generates the closed semi-algebra $C^+(E)$. Such conditions have been given for the case of two generators by Jurkat and Lorentz **(6)**. Find an alternative proof of this case in the context of the above theory. If Conjecture 2 is true (probably only required for the closed unit interval), then Theorem 8, along with a proof that the Jurkat–Lorentz condition ensures the existence of no geometric semi-algebra containing both generators, will yield the result.

3. Give characteristic properties of the dual wedge of a semi-algebra. In particular, determine those closed semi-algebras which are upper semi-lattices.

This may be done by using Theorem 5 and finding, for a given point, all families $M \subseteq M^+(E)$ such that $J_{M,\xi}$ is a semi-algebra (these semi-algebras can be called *pseudo-geometric*). (In the case of lower semi-lattices, there is an analogue of Theorem 5. However, it is hoped to publish elsewhere a more complete analysis of this case, which is simpler and more satisfactory than the upper semi-lattice case. The reader may observe that the smallest lower semi-lattice containing each geometric semi-algebra $H_{\sigma,\xi}$ with σ not a point measure is $C^+(E)$, so that the analogue of Conjecture 2 is false.)

4. Suppose instead of being compact, E is merely locally compact. Replace $C^+(E)$ by $C_0^+(E)$ (non-negative continuous functions vanishing at infinity). What are the maximal closed subsemi-algebras now? Are geometric semi-algebras still maximal?

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