

ON THE GROWTH OF FUNDAMENTAL GROUPS OF NONPOSITIVE CURVATURE MANIFOLDS

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Milnor's classic result that the fundamental group of a compact Riemannian manifold of negative sectional curvature has exponential growth is generalised to the case of negative Ricci curvature and non-positive sectional curvature.

1. INTRODUCTION AND MAIN RESULTS

In the paper [1], Milnor proved the following:

THEOREM. *If M is a compact Riemannian manifold with sectional curvature less than zero, then the growth function of the fundamental group $\pi_1 M$ is at least exponential:*

$$r(s) \geq a^s,$$

for some constant $a > 1$, which depends on the particular finite set of generators for $\pi_1 M$ which is used to define $r(s)$.

Generally, $r(s)$ is defined as follows. Let G be a finitely generated group and g_1, \dots, g_p a specified set of generators for $\pi_1 M$. Then $r(s)$ associated with G and g_1, \dots, g_p is the number of distinct group elements which can be expressed as the words of length $\leq s$ in the specified generators and their inverses. In addition, Milnor also proved that whether $r(s)$ is of exponential growth does not depend on the particular finite set of generators for $\pi_1 M$.

In the paper, Milnor conjectured that the curvature condition in the Theorem can be weakened to negative Ricci curvature. The purpose of this note is to confirm this conjecture. Namely, we shall show the following:

THEOREM A. *Let N is a compact Riemannian manifold with nonpositive sectional curvature and negative Ricci curvature. Then, the growth function of the fundamental group $\pi_1 N$ is at least exponential.*

The key point of our proof is to give a lower bound estimate for the volume of the geodesic balls of the universal covering for N . This is the following:

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THEOREM B. *Let M be a n -dimensional Cartan-Hadamard manifold, with the Ricci curvature of $M \leq -(n - 1)b^2$, $b > 0$. Then for any fixed $\epsilon > 0$, there exists a positive constant $c(\epsilon, n)$ such that when $r \geq \epsilon$, one has*

$$V_{x_0}(r) \geq c(\epsilon, n)e^{br},$$

where $V_{x_0}(r)$ is the volume of the geodesic ball centred at $x_0 \in M$ with radius r .

2. PROOF OF THEOREM A

The idea of the proof is the same as that of Milnor. For completeness, we write it down. Let M be the universal covering space of N . Since N is of negative Ricci curvature and nonpositive sectional curvature, then M is Cartan-Hadamard manifold and there is a positive constant, say b , such that the Ricci curvature of $M \leq -(n - 1)b^2$. Let d be the diameter of N , then $P : B_{x_0}(d) \subset M \rightarrow N$ is surjective. Here P is the covering map from M to N , and $B_{x_0}(d)$ is the geodesic ball in M centred at x_0 with radius d . Denote $B_{x_0}(d)$ by B in the following. Let $F = \{g \in \pi_1 N \mid gB \cap B \neq \emptyset\}$ and $\mu = \min \text{dist}_M(B, gB) \mid gB \cap B = \emptyset, g \in \pi_1 N$. We claim F generates $\pi_1 N$. Let $\text{dist}_M(x_0, gB) < \mu t + d$. Take $y \in gB$ with $\text{dist}_M(x_0, y) < \mu t + d$. Let $\gamma : [0, \text{dist}_M(x_0, y)] \rightarrow M$ be the geodesic joining x_0 and y . In turn, taking $y_1, \dots, y_t, y_{t+1} = y$ on $\gamma([0, \text{dist}_M(x_0, y)])$ satisfying $\text{dist}_M(x_0, y_1) < d$ and $\text{dist}_M(y_i, y_{i+1}) < \mu$ for arbitrary $i \in \{1, 2, \dots, t\}$, then there exist $f_i \in \pi_1 N (i = 1, \dots, t + 1)$ such that $y_i \in f_i B, f_1 = e$ and $f_{i+1} = g$. Thus, $\text{dist}_M(f_i B, f_{i+1} B) < \mu$. By the definitions of μ and F , we have $f_i^{-1} f_{i+1} \in F$. It is obvious that $g = f_1^{-1} f_2 f_2^{-1} f_3 \dots f_t^{-1} f_{t+1}$, that is, g is generated by F . On the other hand, F is also a finite set of $\pi_1 N$, therefore $\pi_1 N$ is finitely generated. In addition, we also obtain that g can be expressed in at most t elements in F assuming $\text{dist}_M(x_0, gB) < \mu t + d$.

Let $A = \{g \in \pi_1 N \mid gB \cap B_{x_0}(\mu t + d) \neq \emptyset\}$. The above conclusion shows any element in A can be expressed in at most t elements in F . So $r(t) \geq$ the number of elements in A , denoted by $|A|$. On the other hand, $\{gB \mid g \in A\}$ covers $B_{x_0}(\mu t + d)$, then $|A| V_{x_0}(d) \geq V_{x_0}(\mu t + d)$. Using Theorem B, we have

$$r(t) \geq \frac{c(\epsilon, n)e^{b(\mu t + d)}}{V_{x_0}(d)} \geq c'a^t, (a > 1).$$

The proof of Theorem A is completed.

3. PROOF OF THEOREM B

We shall use the following Laplacian comparison theorem from [2]. (For a simpler proof, see [3].)

THEOREM C. *Let M, \widetilde{M} be two n -dimensional Cartan-Hadamard manifolds, $x_0 \in M, \widetilde{x}_0 \in \widetilde{M}, \rho(x) = \text{dist}_M(x, x_0), \widetilde{\rho}(\widetilde{x}) = \text{dist}_{\widetilde{M}}(\widetilde{x}, \widetilde{x}_0)$. Assume*

$$\text{Ric} \left(\frac{\partial}{\partial \rho}, \frac{\partial}{\partial \rho} \right) (x) \leq \widetilde{\text{Ric}} \left(\frac{\partial}{\partial \widetilde{\rho}}, \frac{\partial}{\partial \widetilde{\rho}} \right) (\widetilde{x})$$

for $\rho(x) = \widetilde{\rho}(\widetilde{x})$, where $\text{Ric}, \widetilde{\text{Ric}}$ are the Ricci curvatures of M and \widetilde{M} respectively. Then

$$\Delta \rho \geq \frac{1}{n-1} \widetilde{\Delta} \widetilde{\rho},$$

where $\Delta, \widetilde{\Delta}$ are the Laplace-Beltrami operators of M and \widetilde{M} respectively. In particular, if $\text{Ric} \left(\frac{\partial}{\partial \rho}, \frac{\partial}{\partial \rho} \right) \leq -(n-1)b^2$, one has

$$\Delta \rho \geq b \coth(br).$$

Let $x_0 \in M$, a fixed point, with the distance function at x_0 , denoted by ρ . Let $X \in T_{x_0}M$ with $|X| = 1$, and consider the geodesic $\gamma(t) = \text{exp}_{x_0} tX$. Define the function $\phi : [0, \infty) \rightarrow R$ as follows,

$$t \rightarrow \phi(t) = \left| \frac{\text{exp}_{x_0}^* \Omega}{\widetilde{\Omega}} \right|,$$

where $\text{exp}_{x_0}^*$ is the adjoint operator of exp_{x_0} , Ω and $\widetilde{\Omega}$ are the volume elements of M and $T_{x_0}M$ at $\text{exp}_{x_0} tX$ and tX respectively. Similarly, we can define $\phi : T_{x_0}M \rightarrow R$ as follows,

$$X \rightarrow \phi(X) = \left| \frac{\text{exp}_{x_0}^* \Omega}{\widetilde{\Omega}} \right|,$$

where Ω and $\widetilde{\Omega}$ are the volume elements of M and $T_{x_0}M$ at $\text{exp}_{x_0} X$ and X respectively. Let J_1, \dots, J_{n-1} be linearly independent normal Jacobi fields along the geodesic $\gamma(t)$ with $J_i(0) = 0, i = 1, \dots, n-1$. Then, Jacobi field theory tells us

$$\phi(t) = \frac{|J_1(t) \wedge J_2(t) \wedge \dots \wedge J_{n-1}(t)|}{t^{n-1} |J'_1(0) \wedge J'_2(0) \wedge \dots \wedge J'_{n-1}(0)|},$$

where $|A_1 \wedge A_2 \wedge \dots \wedge A_n| = \sqrt{\det(\langle A_i, A_j \rangle)}$. A direct computation shows

$$\frac{\phi'(t)}{\phi(t)} = \left(\Delta \rho - \frac{n-1}{\rho} \right) \circ \gamma(t).$$

Here, we have used the fact $(\Delta \rho) \circ \gamma(t) = \sum_{i=1}^{n-1} \langle J'_i(t), J_i(t) \rangle$ provided for this fixed $t > 0, \langle J_i(t), J_j(t) \rangle = \delta_{ij}$.

Let \widetilde{M} be hyperbolic space form with curvature $K_{\widetilde{M}} = -b^2$. Similarly, we have

$$\frac{\widetilde{\phi}'(t)}{\widetilde{\phi}(t)} = \left(\widetilde{\Delta\rho} - \frac{n-1}{\widetilde{\rho}} \right) \circ \widetilde{\gamma}(t).$$

Here, the meaning of the notation with tilde on its top is the same as that of the above notation without tilde. Thus, we obtain

$$\frac{\phi'(t)}{\phi(t)} - \frac{\widetilde{\phi}'(t)}{\widetilde{\phi}(t)} = (\Delta\rho) \circ \gamma(t) - (\widetilde{\Delta\rho}) \circ \widetilde{\gamma}(t).$$

Using theorem C, we obtain

$$\frac{\phi'(t)}{\phi(t)} - \frac{\widetilde{\phi}'(t)}{\widetilde{\phi}(t)} \geq -\frac{n-2}{n-1} (\widetilde{\Delta\rho}) \circ \widetilde{\gamma}(t) = -(n-2)b \coth(bt).$$

Integrating the above inequality on $[t_1, t_2]$ with $t_2 > t_1 > 0$, we have

$$\ln \frac{\phi(t_2)}{\phi(t_1)} - \ln \frac{\widetilde{\phi}(t_2)}{\widetilde{\phi}(t_1)} \geq -(n-2) \ln \left(\frac{e^{bt_2} - e^{-bt_2}}{e^{bt_1} - e^{-bt_1}} \right).$$

Therefore, $\frac{\phi(t)}{\widetilde{\phi}(t)}(e^{bt} - e^{-bt})^{n-2}$ is an increasing function. Let $S_{x_0}(r)$, $\widetilde{S}_{\widetilde{x}_0}(r)$ be the geodesic spheres centred at x_0 , \widetilde{x}_0 with radius r in M , \widetilde{M} , respectively, and $A(r)$, $\widetilde{A}(r)$ the derived Riemannian volume of $S_{x_0}(r)$, $\widetilde{S}_{\widetilde{x}_0}(r)$ in M , \widetilde{M} , respectively. Then

$$(e^{br} - e^{-br})^{n-2} \frac{A(r)}{\widetilde{A}(r)}$$

is increasing. Therefore,

$$\frac{\int_0^r (e^{bs} - e^{-bs})^{n-2} A(s) ds}{\int_0^r \widetilde{A}(s) ds}$$

is also increasing. For any $\epsilon > 0$, when $r > \epsilon$, we have

$$\begin{aligned} \int_0^r (e^{bs} - e^{-bs})^{n-2} A(s) ds &\geq \frac{\int_0^\epsilon (e^{bs} - e^{-bs})^{n-2} A(s) ds}{\int_0^\epsilon \widetilde{A}(s) ds} \int_0^r \widetilde{A}(s) ds \\ &= c(\epsilon, n) \widetilde{V}_{\widetilde{x}_0}(r), \quad c(\epsilon, n) > 0. \end{aligned}$$

So, we obtain

$$e^{(n-2)br} V_{x_0}(r) \geq c(\epsilon, n) \widetilde{V}_{\widetilde{x}_0}(r), \quad r > \epsilon.$$

On the other hand, $\widetilde{V}_{\widetilde{x}_0}(r) \approx e^{(n-1)br}$. Finally, we have

$$V_{x_0}(r) \geq c'(\epsilon, n) e^{br}, \quad r > \epsilon.$$

This completes the proof of Theorem B.

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