

REPRESENTATION THEOREMS FOR NORMED ALGEBRAS

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Abstract

We show that for a normal locally- \mathcal{P} space X (where \mathcal{P} is a topological property subject to some mild requirements) the subset $C_{\mathcal{P}}(X)$ of $C_b(X)$ consisting of those elements whose support has a neighborhood with \mathcal{P} , is a subalgebra of $C_b(X)$ isometrically isomorphic to $C_c(Y)$ for some unique (up to homeomorphism) locally compact Hausdorff space Y . The space Y is explicitly constructed as a subspace of the Stone–Čech compactification βX of X and contains X as a dense subspace. Under certain conditions, $C_{\mathcal{P}}(X)$ coincides with the set of those elements of $C_b(X)$ whose support has \mathcal{P} , it moreover becomes a Banach algebra, and simultaneously, Y satisfies $C_c(Y) = C_0(Y)$. This includes the cases when \mathcal{P} is the Lindelöf property and X is either a locally compact paracompact space or a locally- \mathcal{P} metrizable space. In either of the latter cases, if X is non- \mathcal{P} , then Y is nonnormal and $C_{\mathcal{P}}(X)$ fits properly between $C_0(X)$ and $C_b(X)$; even more, we can fit a chain of ideals of certain length between $C_0(X)$ and $C_b(X)$. The known construction of Y enables us to derive a few further properties of either $C_{\mathcal{P}}(X)$ or Y . Specifically, when \mathcal{P} is the Lindelöf property and X is a locally- \mathcal{P} metrizable space, we show that

$$\dim C_{\mathcal{P}}(X) = \ell(X)^{\aleph_0},$$

where $\ell(X)$ is the Lindelöf number of X , and when \mathcal{P} is countable compactness and X is a normal space, we show that

$$Y = \text{int}_{\beta X} \nu X$$

where νX is the Hewitt realcompactification of X .

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1. Introduction

Throughout this paper the underlying field of scalars (which is fixed throughout each discussion) is assumed to be either the real field \mathbb{R} or the complex field \mathbb{C} , unless specifically stated otherwise. Also, we will use the term *space* to refer only to a topological space; we may assume that spaces are nonempty.

Let X be a space. Denote by $C_b(X)$ the set of all continuous bounded scalar-valued functions on X . If $f \in C_b(X)$, the *zero-set* of f , denoted by $Z(f)$, is $f^{-1}(0)$, the *cozero-set* of f , denoted by $\text{Coz}(f)$, is $X \setminus Z(f)$, and the *support* of f , denoted by $\text{supp}(f)$, is $\text{cl}_X \text{Coz}(f)$. Let

$$Z(X) = \{Z(f) : f \in C_b(X)\}$$

and

$$\text{Coz}(X) = \{\text{Coz}(f) : f \in C_b(X)\}.$$

Denote by $C_0(X)$ the set of all $f \in C_b(X)$ which vanish at infinity (that is, $|f|^{-1}([\epsilon, \infty))$ is compact for each $\epsilon > 0$) and denote by $C_c(X)$ the set of all $f \in C_b(X)$ with compact support.

Let \mathcal{P} be a topological property. Then:

- \mathcal{P} is *closed* (*open*, respectively) *hereditary*, if any closed (open, respectively) subspace of a space with \mathcal{P} , also has \mathcal{P} ;
- \mathcal{P} is *preserved under finite* (*countable*, *locally finite* respectively) *closed sums*, if any space which is expressible as a finite (countable, locally finite, respectively) union of its closed subspaces each having \mathcal{P} , also has \mathcal{P} .

Let X be a space and let \mathcal{P} be a topological property. The space X is called a \mathcal{P} -*space* if it has \mathcal{P} . A \mathcal{P} -*subspace* of X is a subspace of X which has \mathcal{P} . By a \mathcal{P} -*neighborhood* of a point (set, respectively) in X we mean a neighborhood of the point (set, respectively) in X having \mathcal{P} . The space X is called *locally- \mathcal{P}* if each of its points has a \mathcal{P} -neighborhood in X . Note that if X is regular and \mathcal{P} is closed hereditary, then X is locally- \mathcal{P} if and only if each $x \in X$ has an open neighborhood U in X such that $\text{cl}_X U$ has \mathcal{P} .

For other undefined terms and notation we refer to the standard text [8]. (In particular, compact and paracompact spaces are Hausdorff—thus locally compact spaces are completely regular, Lindelöf spaces are regular, and so on.)

The normed subalgebra $C_c(X)$ of $C_b(X)$ consisting of those elements whose support is compact (if X is locally compact, equivalently, consisting of those elements whose support has a compact neighborhood in X) is crucial. Here, motivated by our previous work [16] (in which we have studied the Banach algebra of continuous bounded scalar-valued functions with separable support on a locally separable metrizable space X), we replace compactness by a rather general topological property \mathcal{P} , thus considering the subset $C_{\mathcal{P}}(X)$ of $C_b(X)$ consisting of those elements whose support has a \mathcal{P} -neighborhood in X . Obviously, $C_{\mathcal{P}}(X)$ is identical to $C_c(X)$ if \mathcal{P} is compactness and X is locally compact. We show that for a normal locally- \mathcal{P} space X (with \mathcal{P} subject to some mild requirements) $C_{\mathcal{P}}(X)$ is a subalgebra of $C_b(X)$ isometrically

isomorphic to $C_c(Y)$ for some unique (up to homeomorphism) locally compact space Y . The space Y , which is explicitly constructed as a subspace of the Stone–Čech compactification βX of X , contains X as a dense subspace. Under certain conditions, $C_{\mathcal{P}}(X)$ coincides with the set of those elements of $C_b(X)$ whose support has \mathcal{P} , moreover it becomes a Banach algebra, and at the same time Y satisfies $C_c(Y) = C_0(Y)$; thus, in particular, in such cases Y is countably compact. This includes the cases where \mathcal{P} is the Lindelöf property and X is either a locally compact paracompact space or a locally- \mathcal{P} metrizable space. In either of the latter cases, if X is non- \mathcal{P} , then Y is nonnormal and $C_{\mathcal{P}}(X)$ fits properly between $C_0(X)$ and $C_b(X)$; even more, we can fit a chain of ideals of certain length between $C_0(X)$ and $C_b(X)$. (This shows how differently $C_{\mathcal{P}}(X)$ may behave for different topological properties \mathcal{P} : if \mathcal{P} is compactness, then for any locally- \mathcal{P} metrizable space X , if $C_{\mathcal{P}}(X)$ is a Banach algebra then X has \mathcal{P} , while, if \mathcal{P} is the Lindelöf property, there exist some non- \mathcal{P} locally- \mathcal{P} metrizable spaces X such that $C_{\mathcal{P}}(X)$ is a Banach algebra.) A few further properties of Y or $C_{\mathcal{P}}(X)$ are also derived through the known construction of Y . Specifically, when \mathcal{P} is the Lindelöf property and X is a locally- \mathcal{P} metrizable space we show that

$$\dim C_{\mathcal{P}}(X) = \ell(X)^{\aleph_0},$$

where $\ell(X)$ is the Lindelöf number of X , and when \mathcal{P} is countable compactness and X is a normal space we show that

$$Y = \text{int}_{\beta X} \nu X,$$

where νX is the Hewitt real compactification of X . Some of the results of this paper are further generalized in the follow-up paper [20] (see also [19]).

Let X be a completely regular space. In the recent preprint [23], for a filter base \mathcal{B} of open subspaces of X , the author studies $C_{\mathcal{B}}(X)$ defined as the set of all $f \in C(X)$ with support contained in $X \setminus A$ for some $A \in \mathcal{B}$. Also, if \mathcal{I} is an ideal of closed subspaces of X , in [1], the authors consider $C_{\mathcal{I}}(X)$ defined as the set of all $f \in C(X)$ with support contained in \mathcal{I} . Our approach here is quite different from that of either [23] or [1]. The interested reader may find it useful to compare our results with those obtained in [1, 23]. (See also [3] for related results.)

We now review briefly some known facts from general topology. Additional information on the subject may be found in [8, 10, 21].

1.1. The Stone–Čech compactification. Let X be a completely regular space. The Stone–Čech compactification βX of X is the compactification of X characterized among all compactifications of X by the following property: every continuous $f : X \rightarrow K$, where K is a compact space, is continuously extendable over βX ; denote by f_{β} this continuous extension of f . The Stone–Čech compactification of a completely regular space always exists. Use will be made in what follows of the following properties of βX . (See [8, Sections 3.5 and 3.6].)

- X is locally compact if and only if X is open in βX .
- Any open-closed subspace of X has open-closed closure in βX .

- If $X \subseteq T \subseteq \beta X$ then $\beta T = \beta X$.
- If X is normal then $\beta T = \text{cl}_{\beta X} T$ for any closed subspace T of X .

1.2. The Hewitt realcompactification. A space is called *realcompact* if it is homeomorphic to a closed subspace of some product \mathbb{R}^α . Let X be a completely regular space. A *realcompactification* of X is a realcompact space containing X as a dense subspace. The *Hewitt realcompactification* νX of X is the realcompactification of X characterized among all realcompactifications of X by the following property: every continuous $f : X \rightarrow \mathbb{R}$ is continuously extendable over νX . One may assume that $\nu X \subseteq \beta X$.

1.3. Paracompact spaces and the Lindelöf property. Let X be a space. For open covers \mathcal{U} and \mathcal{V} of X we say that \mathcal{U} is a *refinement* of \mathcal{V} (or \mathcal{U} *refines* \mathcal{V}) if each element of \mathcal{U} is contained in an element of \mathcal{V} . An open cover \mathcal{U} of X is called *locally finite* if each point of X has a neighborhood in X intersecting only a finite number of the elements of \mathcal{U} . The space X is called *paracompact* if it is Hausdorff and for every open cover \mathcal{U} of X there exists a locally finite open cover of X which refines \mathcal{U} . Every metrizable space and every Lindelöf space is paracompact and every paracompact space is normal. Any locally compact paracompact space X can be represented as a disjoint union

$$X = \bigcup_{i \in I} X_i,$$

where I is an index set and the X_i are Lindelöf open-closed subspaces of X . (See [8, Theorem 5.1.27].)

1.4. Metrizable spaces and the Lindelöf property. The *Lindelöf number* of a space X , denoted by $\ell(X)$, is defined by

$$\ell(X) = \min\{n : \text{any open cover of } X \text{ has a subcover of cardinality } \leq n\} + \aleph_0.$$

In particular, a space X is Lindelöf if and only if $\ell(X) = \aleph_0$. By a theorem of Alexandroff, any locally Lindelöf metrizable space X can be represented as a disjoint union

$$X = \bigcup_{i \in I} X_i,$$

where I is an index set and the X_i are nonempty Lindelöf open-closed subspaces of X . (See [8, Problem 4.4.F]; note that in metrizable spaces the two notions of separability and being Lindelöf coincide.) Observe that $\ell(X) = |I|$ if I is infinite.

2. The normed algebra $C_{\mathcal{P}}(X)$

We begin our study by considering the general normed algebra $C_{\mathcal{P}}(X)$ as defined below.

DEFINITION 2.1. Let X be a space and let \mathcal{P} be a topological property. Define

$$C_{\mathcal{P}}(X) = \{f \in C_b(X) : \text{supp}(f) \text{ has a } \mathcal{P}\text{-neighborhood}\}.$$

REMARK. Note that

$$C_{\mathcal{P}}(X) = C_c(X)$$

if \mathcal{P} is compactness and X is a locally compact space. To see this, let $f \in C_{\mathcal{P}}(X)$. Then $\text{supp}(f)$ is compact, as it has a compact neighborhood in X . For the converse, suppose that $\text{supp}(g)$ is compact for some $g \in C_b(X)$. For each $x \in X$ let U_x be an open neighborhood of x in X with compact closure $\text{cl}_X U_x$. By compactness of $\text{supp}(g)$ there exist $x_1, \dots, x_n \in X$ such that

$$\text{supp}(g) \subseteq U_{x_1} \cup \dots \cup U_{x_n} = U.$$

Now

$$\text{cl}_X U = \text{cl}_X U_{x_1} \cup \dots \cup \text{cl}_X U_{x_n}$$

is a neighborhood of $\text{supp}(g)$ in X and it is compact, as it is a finite union of compact subspaces of X . Thus $g \in C_{\mathcal{P}}(X)$.

The following subspace of βX , introduced in [13] (see also [14, 17, 18]), plays a crucial role in what follows.

DEFINITION 2.2. For a completely regular space X and a topological property \mathcal{P} , let

$$\lambda_{\mathcal{P}} X = \bigcup \{ \text{int}_{\beta X} \text{cl}_{\beta X} C : C \in \text{Coz}(X) \text{ and } \text{cl}_X C \text{ has } \mathcal{P} \},$$

considered as a subspace of βX .

REMARK. Note that in Definition 2.2 we have

$$\lambda_{\mathcal{P}} X = \bigcup \{ \text{int}_{\beta X} \text{cl}_{\beta X} Z : Z \in Z(X) \text{ has } \mathcal{P} \},$$

provided that \mathcal{P} is a closed hereditary topological property. (See [14].)

If X is a space and D is a dense subspace of X , then

$$\text{cl}_X U = \text{cl}_X (U \cap D)$$

for every open subspace U of X . This will be used in the following simple observation.

LEMMA 2.3. *Let X be a completely regular space and let $f : X \rightarrow [0, 1]$ be continuous. If $0 < r < 1$ then*

$$f_{\beta}^{-1} [[0, r]] \subseteq \text{int}_{\beta X} \text{cl}_{\beta X} f_{\beta}^{-1} [[0, r]].$$

PROOF. Note that

$$\text{cl}_{\beta X} f_{\beta}^{-1} [[0, r]] = \text{cl}_{\beta X} (X \cap f_{\beta}^{-1} [[0, r]]) = \text{cl}_{\beta X} f_{\beta}^{-1} [[0, r]]$$

and that

$$f_{\beta}^{-1} [[0, r]] \subseteq \text{int}_{\beta X} \text{cl}_{\beta X} f_{\beta}^{-1} [[0, r]].$$

This concludes the proof. □

The following is a slight modification of [13, Lemma 2.10].

LEMMA 2.4. *Let X be a completely regular locally- \mathcal{P} space, where \mathcal{P} is a closed hereditary topological property. Then*

$$X \subseteq \lambda_{\mathcal{P}}X.$$

PROOF. Let $x \in X$ and let U be an open neighborhood of x in X whose closure $\text{cl}_X U$ has \mathcal{P} . Let $f : X \rightarrow [0, 1]$ be continuous with

$$f(x) = 0 \quad \text{and} \quad f|(X \setminus U) \equiv \mathbf{1}.$$

Let

$$C = f^{-1}[[0, 1/2]] \in \text{Coz}(X).$$

Then $C \subseteq U$ and thus $\text{cl}_X C$ has \mathcal{P} , as it is closed in $\text{cl}_X U$. Therefore

$$\text{int}_{\beta X} \text{cl}_{\beta X} C \subseteq \lambda_{\mathcal{P}}X.$$

But then $x \in \lambda_{\mathcal{P}}X$, as $x \in f_{\beta}^{-1}[[0, 1/2]]$ and

$$f_{\beta}^{-1}[[0, 1/2]] \subseteq \text{int}_{\beta X} \text{cl}_{\beta X} C$$

by Lemma 2.3. □

REMARK. Note that in Lemma 2.4 the converse also holds. That is, X is locally- \mathcal{P} whenever $X \subseteq \lambda_{\mathcal{P}}X$. (For a proof, modify the argument given in [13, Lemma 2.10].) However, we will not have any occasion in the following to use the converse statement.

DEFINITION 2.5. Let X be a completely regular locally- \mathcal{P} space, where \mathcal{P} is a closed hereditary topological property. For any $f \in C_b(X)$ denote

$$f_{\lambda} = f_{\beta}|_{\lambda_{\mathcal{P}}X}.$$

Observe that by Lemma 2.4 the function f_{λ} extends f .

LEMMA 2.6. *Let X be a normal locally- \mathcal{P} space, where \mathcal{P} is a closed hereditary topological property preserved under finite closed sums. For any $f \in C_b(X)$ the following are equivalent:*

- (1) $f \in C_{\mathcal{P}}(X)$;
- (2) $f_{\lambda} \in C_c(\lambda_{\mathcal{P}}X)$.

PROOF. To prove that (1) implies (2), let T be a \mathcal{P} -neighborhood of $\text{supp}(f)$ in X . Then $\text{supp}(f) \subseteq \text{int}_X T$. Since X is normal, by the Urysohn lemma, there exists a continuous $g : X \rightarrow [0, 1]$ with

$$g|\text{supp}(f) \equiv \mathbf{0} \quad \text{and} \quad g|(X \setminus \text{int}_X T) \equiv \mathbf{1}.$$

Let

$$C = g^{-1}[[0, 1/2]] \in \text{Coz}(X).$$

Note that

$$\text{cl}_X C \subseteq g^{-1}[[0, 1/2]] \subseteq T,$$

and thus, $\text{cl}_X C$, being closed in T , has \mathcal{P} . Therefore

$$\text{int}_{\beta X} \text{cl}_{\beta X} C \subseteq \lambda_{\mathcal{P}} X.$$

But

$$g_{\beta}^{-1}[[0, 1/2]] \subseteq \text{int}_{\beta X} \text{cl}_{\beta X} C$$

by Lemma 2.3, and thus

$$\text{cl}_{\beta X} \text{Coz}(f) \subseteq Z(g_{\beta}) \subseteq g_{\beta}^{-1}[[0, 1/2]] \subseteq \lambda_{\mathcal{P}} X.$$

This implies that

$$\begin{aligned} \text{supp}(f_{\lambda}) &= \text{cl}_{\lambda_{\mathcal{P}} X} \text{Coz}(f_{\lambda}) \\ &= \text{cl}_{\lambda_{\mathcal{P}} X} (X \cap \text{Coz}(f_{\lambda})) \\ &= \text{cl}_{\lambda_{\mathcal{P}} X} \text{Coz}(f) = \lambda_{\mathcal{P}} X \cap \text{cl}_{\beta X} \text{Coz}(f) = \text{cl}_{\beta X} \text{Coz}(f) \end{aligned}$$

is compact.

To prove that (2) implies (1), let V be an open neighborhood of $\text{supp}(f_{\lambda})$ in βX with

$$\text{cl}_{\beta X} V \subseteq \lambda_{\mathcal{P}} X.$$

(Note that $\lambda_{\mathcal{P}} X$ is open in βX by its definition, and βX , being compact, is normal.) By compactness,

$$\text{cl}_{\beta X} V \subseteq \text{int}_{\beta X} \text{cl}_{\beta X} C_1 \cup \cdots \cup \text{int}_{\beta X} \text{cl}_{\beta X} C_n \quad (2.1)$$

for some $C_1, \dots, C_n \in \text{Coz}(X)$ such that each $\text{cl}_X C_1, \dots, \text{cl}_X C_n$ has \mathcal{P} . Intersecting both sides of (2.1) with X ,

$$\text{cl}_X (X \cap V) \subseteq X \cap \text{cl}_{\beta X} V \subseteq \text{cl}_X C_1 \cup \cdots \cup \text{cl}_X C_n = D.$$

Note that D has \mathcal{P} , as it is a finite union of its closed \mathcal{P} -subspaces. Therefore $\text{cl}_X (X \cap V)$, being closed in D , has \mathcal{P} . But $\text{cl}_X (X \cap V)$ is a neighborhood of $\text{supp}(f)$ in X , as

$$\text{supp}(f) \subseteq X \cap \text{supp}(f_{\lambda}) \subseteq X \cap V.$$

This concludes the proof. \square

A version of the classical Banach–Stone theorem states that for any locally compact spaces X and Y , the rings $C_c(X)$ and $C_c(Y)$ are isomorphic if and only if the spaces X and Y are homeomorphic. (See [2] or [4].) This will be used in the proof of the following theorem.

THEOREM 2.7. *Let X be a normal locally- \mathcal{P} space where \mathcal{P} is a closed hereditary topological property preserved under finite closed sums. Then $C_{\mathcal{P}}(X)$ is a normed subalgebra of $C_b(X)$ isometrically isomorphic to $C_c(Y)$ for some unique (up to homeomorphism) locally compact space Y , namely $Y = \lambda_{\mathcal{P}}X$. Furthermore, $C_{\mathcal{P}}(X)$ is unital if and only if X has \mathcal{P} .*

PROOF. First, we need to show that $C_{\mathcal{P}}(X)$ is a subalgebra of $C_b(X)$. Observe that since X is locally- \mathcal{P} (and nonempty), there exists a \mathcal{P} -subspace of X which constitutes a neighborhood of $\mathbf{0} = \text{supp}(\mathbf{0})$ in X . Thus $\mathbf{0} \in C_{\mathcal{P}}(X)$. To show that $C_{\mathcal{P}}(X)$ is closed under addition, let $f_i \in C_{\mathcal{P}}(X)$ where $i = 1, 2$. For each $i = 1, 2$ let T_i be a \mathcal{P} -neighborhood of $\text{supp}(f_i)$ in X and (using normality of X) let U_i be an open neighborhood of $\text{supp}(f_i)$ in X with $\text{cl}_X U_i \subseteq \text{int}_X T_i$. Then $\text{cl}_X U_1 \cup \text{cl}_X U_2$ has \mathcal{P} , as it is the union of two of its closed subspaces $\text{cl}_X U_1$ and $\text{cl}_X U_2$, and $\text{cl}_X U_i$, for each $i = 1, 2$, being closed in T_i , has \mathcal{P} . Note that $\text{cl}_X U_1 \cup \text{cl}_X U_2$ is a neighborhood of $\text{supp}(f_1 + f_2)$ in X , as

$$\text{supp}(f_1 + f_2) \subseteq \text{supp}(f_1) \cup \text{supp}(f_2) \subseteq U_1 \cup U_2.$$

That $C_{\mathcal{P}}(X)$ is closed under scalar multiplication and multiplication of its elements may be proved analogously.

Let $Y = \lambda_{\mathcal{P}}X$ and define

$$\psi : C_{\mathcal{P}}(X) \rightarrow C_c(Y)$$

by

$$\psi(f) = f_{\lambda}$$

for any $f \in C_{\mathcal{P}}(X)$. By Lemma 2.6 the function ψ is well defined. It is clear that ψ is a homomorphism and that ψ is injective. (Note that $X \subseteq Y$ by Lemma 2.4, and that any two scalar-valued continuous functions on $\lambda_{\mathcal{P}}X$ coincide, provided that they agree on the dense subspace X of Y .) To show that ψ is surjective, let $g \in C_c(Y)$. Then $(g|X)_{\lambda} = g$ and thus $g|X \in C_{\mathcal{P}}(X)$ by Lemma 2.6. Now $\psi(g|X) = g$. To show that ψ is an isometry, let $h \in C_{\mathcal{P}}(X)$. Then

$$\|h_{\lambda}\|[\lambda_{\mathcal{P}}X] = \|h_{\lambda}\|[\text{cl}_{\lambda_{\mathcal{P}}X}X] \subseteq \text{cl}_{\mathbb{R}}(\|h_{\lambda}\|[X]) = \text{cl}_{\mathbb{R}}(\|h\|[X]) \subseteq [0, \|h\|]$$

which yields $\|h_{\lambda}\| \leq \|h\|$. That $\|h\| \leq \|h_{\lambda}\|$ is clear, as h_{λ} extends h .

Note that Y is locally compact, as it is open in the compact space βX .

The uniqueness of Y follows from the fact that for any locally compact space T the ring $C_c(T)$ determines the topology of T .

For the second part of the theorem, suppose that X has \mathcal{P} . Then the function $\mathbf{1}$ is the unit element of $C_{\mathcal{P}}(X)$. To show the converse, suppose that $C_{\mathcal{P}}(X)$ has a unit element u . Let $x \in X$. Let U_x and V_x be open neighborhoods of x in X such that $\text{cl}_X V_x \subseteq U_x$ and $\text{cl}_X U_x$ has \mathcal{P} . Let $f_x : X \rightarrow [0, 1]$ be continuous and such that

$$f_x(x) = 1 \quad \text{and} \quad f_x|_{(X \setminus V_x)} \equiv \mathbf{0}.$$

Then $\text{cl}_X U_x$ is a neighborhood of $\text{supp}(f_x)$ in X , as $\text{supp}(f_x) \subseteq \text{cl}_X V_x$. Therefore $f_x \in C_{\mathcal{P}}(X)$. Since

$$u(x) = u(x)f_x(x) = f_x(x) = 1$$

we have $u = \mathbf{1}$. Thus $X = \text{supp}(u)$ has \mathcal{P} . \square

EXAMPLE 2.8. The list of topological properties satisfying the assumption of Theorem 2.7 is quite long and includes almost all important covering properties (that is, topological properties described in terms of the existence of certain kinds of open subcovers or refinements of a given open cover of a certain type), among them compactness, countable compactness (more generally, $[\theta, \kappa]$ -compactness), the Lindelöf property (more generally, the μ -Lindelöf property), paracompactness, metacompactness, countable paracompactness, subparacompactness, submetacompactness (or θ -refinability), the σ -para-Lindelöf property and also α -boundedness. (See [6, 22] for the definitions. That these topological properties—except for the last one—are closed hereditary and preserved under finite closed sums, follows from [6, Theorems 7.1, 7.3 and 7.4]; for α -boundedness, this directly follows from its definition. Recall that a space X is α -bounded, where α is an infinite cardinal, if every subspace of X of cardinality up to α has compact closure in X .)

REMARK. Let \mathcal{P} be a topological property. Then \mathcal{P} is *finitely additive*, if any space which is expressible as a finite disjoint union of its closed \mathcal{P} -subspaces has \mathcal{P} . Also, \mathcal{P} is *invariant under perfect mappings* (*inverse invariant under perfect mappings*, respectively) if for every perfect surjective mapping $f : X \rightarrow Y$, the space Y (X , respectively) has \mathcal{P} , provided that X (Y , respectively) has \mathcal{P} . If \mathcal{P} is both invariant and inverse invariant under perfect mappings then it is *perfect*. (A closed continuous mapping $f : X \rightarrow Y$ is *perfect*, if each fiber $f^{-1}(y)$, where $y \in Y$, is a compact subspace of X .) Any finitely additive topological property which is invariant under perfect mappings is preserved under finite closed sums. (See [8, Theorem 3.7.22].) Also, any topological property which is hereditary with respect to open-closed subspaces and is inverse invariant under perfect mappings, is hereditary with respect to closed subspaces. (See [8, Theorem 3.7.29].) Therefore, the assumption that ‘ \mathcal{P} is closed hereditary and preserved under finite closed sums’ in Lemma 2.6 and Theorem 2.7 may be replaced by ‘ \mathcal{P} is open-closed hereditary, finitely additive and perfect’.

3. The Banach algebra $C_{\mathcal{P}}(X)$

In this section we turn our attention to the case in which $C_{\mathcal{P}}(X)$ becomes a Banach algebra. It is interesting that in spite of the fact that $C_{\mathcal{P}}(X)$ is required to have a richer structure, it turns out to be better expressible, and at the same time, $\lambda_{\mathcal{P}}X$ reveals nicer properties. These are all more precisely expressed in the statement of our next result.

Let X be a locally compact noncompact space. It is known that $C_0(X) = C_c(X)$ if and only if every σ -compact subspace of X is contained in a compact subspace of X . (See [10, Problem 7G.2].) In particular, $C_0(X) = C_c(X)$ implies that X is countably compact (recall that a space T is countably compact if and only if each countably

infinite subspace of T has an accumulation point; see [8, Theorem 3.10.3]) and thus, non-Lindelöf and nonparacompact, as every countably compact space which is either Lindelöf or paracompact is necessarily compact. (See [8, Theorems 3.11.1 and 5.1.20]; observe that Lindelöf spaces are realcompact and completely regular countably compact spaces are pseudocompact; see [8, Theorems 3.11.12 and 3.10.20].) These will be used in the proof of the following theorem.

THEOREM 3.1. *Let X be a normal locally- \mathcal{P} space where \mathcal{P} is a closed hereditary topological property preserved under countable closed sums. Moreover, suppose that the closure of each \mathcal{P} -subspace of X has a \mathcal{P} -neighborhood. Then $C_{\mathcal{P}}(X)$ is a Banach subalgebra of $C_b(X)$ isometrically isomorphic to $C_c(Y)$ for some unique (up to homeomorphism) locally compact space Y , namely $Y = \lambda_{\mathcal{P}}X$. Moreover:*

- $C_{\mathcal{P}}(X) = \{f \in C_b(X) : \text{supp}(f) \text{ has } \mathcal{P}\}$;
- $C_c(Y) = C_0(Y)$;
- Y is countably compact;
- Y is neither Lindelöf nor paracompact, if X is non- \mathcal{P} .

PROOF. By Theorem 2.7 we know that $C_{\mathcal{P}}(X)$ is a normed subalgebra of $C_b(X)$ isometrically isomorphic to $C_c(Y)$ for some unique locally compact space $Y = \lambda_{\mathcal{P}}X$. To prove that $C_{\mathcal{P}}(X)$ is a Banach algebra it then suffices to show that $C_c(Y) = C_0(Y)$.

Next, note that if $f \in C_{\mathcal{P}}(X)$, then $\text{supp}(f)$ has \mathcal{P} , as it is closed in a \mathcal{P} -neighborhood in X . For the converse, note that if $f \in C_b(X)$ is such that $\text{supp}(f)$ has \mathcal{P} , then $\text{supp}(f)$ has a \mathcal{P} -neighborhood in X by our assumption.

To show that $C_c(Y) = C_0(Y)$, let A be a σ -compact subspace of Y . Then

$$A = A_1 \cup A_2 \cup \dots$$

where each A_1, A_2, \dots is compact. For each $n = 1, 2, \dots$ by compactness of A_n we have

$$A_n \subseteq \text{int}_{\beta X} \text{cl}_{\beta X} C_1^n \cup \dots \cup \text{int}_{\beta X} \text{cl}_{\beta X} C_{k_n}^n \tag{3.1}$$

for some $C_1^n, \dots, C_{k_n}^n \in \text{Coz}(X)$ such that each $\text{cl}_X C_1^n, \dots, \text{cl}_X C_{k_n}^n$ has \mathcal{P} . Note that

$$E = \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{k_n} \text{cl}_X C_i^n$$

has \mathcal{P} , as it is the countable union of its closed \mathcal{P} -subspaces. By our assumption, there exists a \mathcal{P} -neighborhood T of $\text{cl}_X E$ in X . Since X is normal by the Urysohn lemma there exists a continuous $f : X \rightarrow [0, 1]$ with

$$f|_{\text{cl}_X E} \equiv \mathbf{0} \quad \text{and} \quad f|(X \setminus \text{int}_X T) \equiv \mathbf{1}.$$

Let

$$C = f^{-1}[[0, 1/2]] \in \text{Coz}(X).$$

Note that

$$\text{cl}_X C \subseteq f^{-1}[[0, 1/2]] \subseteq T,$$

and thus, $\text{cl}_X C$, being closed in T , has \mathcal{P} . Therefore

$$\text{int}_{\beta X} \text{cl}_{\beta X} C \subseteq \lambda_{\mathcal{P}} X.$$

But

$$f_{\beta}^{-1}[[0, 1/2)] \subseteq \text{int}_{\beta X} \text{cl}_{\beta X} C$$

by Lemma 2.3. Thus

$$\text{cl}_{\beta X} C_i^n \subseteq Z(f_{\beta}) \subseteq f_{\beta}^{-1}[[0, 1/2)] \subseteq \lambda_{\mathcal{P}} X \tag{3.2}$$

for each $n = 1, 2, \dots$ and $i = 1, \dots, k_n$. From (3.1) and (3.2), it then follows that $A_n \subseteq Z(f_{\beta})$ for each $n = 1, 2, \dots$. Therefore $Z(f_{\beta})$ is a compact subspace of $\lambda_{\mathcal{P}} X$ containing A .

To conclude the proof, note that Y is countably compact, as $C_c(Y) = C_0(Y)$. If Y is in addition either Lindelöf or paracompact, then it is compact. Compactness of Y now implies that

$$\lambda_{\mathcal{P}} X = \text{int}_{\beta X} \text{cl}_{\beta X} C_1 \cup \dots \cup \text{int}_{\beta X} \text{cl}_{\beta X} C_n$$

for some $C_1, \dots, C_n \in \text{Coz}(X)$ such that each $\text{cl}_X C_1, \dots, \text{cl}_X C_n$ has \mathcal{P} . Since X is locally- \mathcal{P} , we have $X \subseteq \lambda_{\mathcal{P}} X$ by Lemma 2.4, from which it then follows that

$$X = \text{cl}_X C_1 \cup \dots \cup \text{cl}_X C_n,$$

being the finite union of its closed \mathcal{P} -subspaces, has \mathcal{P} . □

REMARK. Suppose that the underlying field of scalars is \mathbb{C} . In Theorem 3.1 we have proved that $C_{\mathcal{P}}(X)$ is a Banach algebra isometrically isomorphic to $C_0(Y)$ for some locally compact space $Y (= \lambda_{\mathcal{P}} X)$. On the other hand, by the commutative Gelfand–Naimark theorem, we know that $C_{\mathcal{P}}(X)$ is isometrically isomorphic to $C_0(Y')$, with the locally compact space Y' being the spectrum of $C_{\mathcal{P}}(X)$. Thus $C_0(Y)$ and $C_0(Y')$ are isometrically isomorphic, which implies that the spaces Y and Y' are homeomorphic. In particular, this shows that $\lambda_{\mathcal{P}} X$ coincides with the spectrum of $C_{\mathcal{P}}(X)$. (Recall that, by a version of the classical Banach–Stone theorem, for any locally compact spaces X and Y , the Banach algebras $C_0(X)$ and $C_0(Y)$ are isometrically isomorphic if and only if the spaces X and Y are homeomorphic; see [5, Theorem 7.1].)

REMARK. Note that in Theorem 3.1 the space Y is non- \mathcal{P} for any topological property \mathcal{P} such that

$$\mathcal{P} + \text{countable compactness} \rightarrow \text{compactness}.$$

The list of such topological properties is quite long; it includes (in addition to the Lindelöf property and paracompactness themselves) realcompactness, metacompactness, subparacompactness, submetacompactness (or θ -refinability), the meta-Lindelöf property, the submeta-Lindelöf property (or $\delta\theta$ -refinability), weak submetacompactness (or weak θ -refinability) and the weak submeta-Lindelöf property (or weak $\delta\theta$ -refinability), among others. (See [24, Sections 6.1 and 6.2].)

4. The case where \mathcal{P} is the Lindelöf property

In this section we confine ourselves to the case where \mathcal{P} is the Lindelöf property. This consideration leads to some improvements in Theorem 3.1. Part of the results of this section (Lemmas 4.2, 4.3 and 4.5 and part of Theorem 4.6) are slight modifications of certain results from [16]. The proofs are given here for the reader's convenience and completeness of results.

We begin with the following observation.

PROPOSITION 4.1. *Let \mathcal{P} be the Lindelöf property. Let X be a normal locally- \mathcal{P} space. Then*

$$C_{\mathcal{P}}(X) = \{f \in C_b(X) : \text{supp}(f) \text{ has } \mathcal{P}\}.$$

PROOF. Observe that if $f \in C_{\mathcal{P}}(X)$ then $\text{supp}(f)$ has \mathcal{P} , as it is closed in a \mathcal{P} -neighborhood in X .

Next, suppose that $f \in C_b(X)$ and that $\text{supp}(f)$ has \mathcal{P} . For each $x \in \text{supp}(f)$, let U_x be an open neighborhood of x in X such that the closure $\text{cl}_X U_x$ has \mathcal{P} . Since

$$\{U_x : x \in \text{supp}(f)\}$$

is an open cover of $\text{supp}(f)$, there exist some $x_1, x_2, \dots \in \text{supp}(f)$ such that

$$\text{supp}(f) \subseteq U_{x_1} \cup U_{x_2} \cup \dots = W.$$

By normality of X , there exists an open subspace V of X with

$$\text{supp}(f) \subseteq V \subseteq \text{cl}_X V \subseteq W.$$

Now $\text{cl}_X V$ is contained in

$$H = \text{cl}_X U_{x_1} \cup \text{cl}_X U_{x_2} \cup \dots$$

as a closed subspace. Since H has \mathcal{P} , it follows that $\text{cl}_X V$ has \mathcal{P} . That is, $\text{cl}_X V$ is a \mathcal{P} -neighborhood of $\text{supp}(f)$ in X . Therefore $f \in C_{\mathcal{P}}(X)$. \square

LEMMA 4.2. *Let \mathcal{P} be the Lindelöf property. Let X be a completely regular space representable as a disjoint union*

$$X = \bigcup_{i \in I} X_i,$$

such that the X_i are open-closed \mathcal{P} -subspaces of X . Then

$$\lambda_{\mathcal{P}} X = \bigcup \left\{ \text{cl}_{\beta X} \left(\bigcup_{i \in J} X_i \right) : J \subseteq I \text{ is countable} \right\}.$$

PROOF. Let

$$Y = \bigcup \left\{ \text{cl}_{\beta X} \left(\bigcup_{i \in J} X_i \right) : J \subseteq I \text{ is countable} \right\}.$$

To show that $\lambda_{\mathcal{P}} X \subseteq Y$, let $C \in \text{Coz}(X)$ have Lindelöf closure $\text{cl}_X C$. Then

$$\text{cl}_X C \subseteq \bigcup_{i \in J} X_i$$

for some countable $J \subseteq I$. Thus

$$\text{cl}_{\beta X} C \subseteq \text{cl}_{\beta X} \left(\bigcup_{i \in J} X_i \right).$$

We next show that $Y \subseteq \lambda_{\mathcal{P}} X$. Let $J \subseteq I$ be countable. Then

$$D = \bigcup_{i \in J} X_i$$

is a cozero-set of X , as it is open-closed in X , and it is Lindelöf. Since D is open-closed in X , the closure $\text{cl}_{\beta X} D$ in βX is open-closed in βX . Therefore

$$\text{cl}_{\beta X} D = \text{int}_{\beta X} \text{cl}_{\beta X} D \subseteq \lambda_{\mathcal{P}} X.$$

This concludes the proof. □

Let \mathcal{P} be the Lindelöf property. Let D be an uncountable discrete space. Let E be the subspace of $\beta D \setminus D$ consisting of elements in the closure (in βD) of countable subspaces of D . Then

$$E = \lambda_{\mathcal{P}} D \setminus D.$$

(Observe that cozero-sets in D whose closure in D has \mathcal{P} are exactly countable subspaces of D , and that each subspace of D , being open-closed in D , has open closure in βD .) In [25], the author proves the existence of a continuous (two-valued) function $f : E \rightarrow [0, 1]$ which is not continuously extendible over $\beta D \setminus D$. This, in particular, proves that $\lambda_{\mathcal{P}} D$ is not normal. (To see this, suppose, to the contrary, that $\lambda_{\mathcal{P}} D$ is normal. Note that E is closed in $\lambda_{\mathcal{P}} D$, as D , being locally compact, is open in βD . By the Tietze–Urysohn extension theorem, f is extendible to a continuous bounded function over $\lambda_{\mathcal{P}} D$, and thus over $\beta(\lambda_{\mathcal{P}} D)$. Note that $\beta(\lambda_{\mathcal{P}} D) = \beta D$, as $D \subseteq \lambda_{\mathcal{P}} D$ by Lemma 2.4. But this is not possible.) This fact will be used in the following to show that in general $\lambda_{\mathcal{P}} X$ need not be normal. This, in particular, provides us with an example of a locally compact countably compact nonnormal space Y such that $C_c(Y) = C_0(Y)$.

Observe that if X is a space and $D \subseteq X$, then

$$U \cap \text{cl}_X D = \text{cl}_X(U \cap D)$$

for every open-closed subspace U of X . This simple observation will be used below.

LEMMA 4.3. *Let \mathcal{P} be the Lindelöf property. Let X be a completely regular non- \mathcal{P} -space representable as a disjoint union*

$$X = \bigcup_{i \in I} X_i,$$

such that the X_i are open-closed \mathcal{P} -subspaces of X . Then $\lambda_{\mathcal{P}}X$ is nonnormal.

PROOF. Let $x_i \in X_i$ for each $i \in I$. Then

$$D = \{x_i : i \in I\}$$

is a closed discrete subspace of X and, since X is non- \mathcal{P} , is uncountable. Suppose to the contrary that $\lambda_{\mathcal{P}}X$ is normal. Then

$$\lambda_{\mathcal{P}}X \cap \text{cl}_{\beta X}D$$

is normal, as it is closed in $\lambda_{\mathcal{P}}X$. By Lemma 4.2,

$$\lambda_{\mathcal{P}}X \cap \text{cl}_{\beta X}D = \bigcup \left\{ \text{cl}_{\beta X} \left(\bigcup_{i \in J} X_i \right) \cap \text{cl}_{\beta X}D : J \subseteq I \text{ is countable} \right\}.$$

Let $J \subseteq I$ be countable. Since

$$\text{cl}_{\beta X} \left(\bigcup_{i \in J} X_i \right)$$

is open-closed in βX (as $\bigcup_{i \in J} X_i$ is open-closed in X),

$$\begin{aligned} \text{cl}_{\beta X} \left(\bigcup_{i \in J} X_i \right) \cap \text{cl}_{\beta X}D &= \text{cl}_{\beta X} \left(\text{cl}_{\beta X} \left(\bigcup_{i \in J} X_i \right) \cap D \right) \\ &= \text{cl}_{\beta X} \left(\bigcup_{i \in J} X_i \cap D \right) = \text{cl}_{\beta X}(\{x_i : i \in J\}). \end{aligned}$$

But $\text{cl}_{\beta X}D = \beta D$, as D is closed in (the normal space) X . Therefore

$$\text{cl}_{\beta X}(\{x_i : i \in J\}) = \text{cl}_{\beta X}(\{x_i : i \in J\}) \cap \text{cl}_{\beta X}D = \text{cl}_{\beta D}(\{x_i : i \in J\}).$$

Thus

$$\lambda_{\mathcal{P}}X \cap \text{cl}_{\beta X}D = \lambda_{\mathcal{P}}D,$$

contradicting the fact that $\lambda_{\mathcal{P}}D$ is not normal. □

The following corollary of Theorem 3.1, together with Theorem 4.6, constitutes the main result of this section.

THEOREM 4.4. *Let \mathcal{P} be the Lindelöf property. Let X be a paracompact locally- \mathcal{P} space. Then $C_{\mathcal{P}}(X)$ is a Banach subalgebra of $C_b(X)$ isometrically isomorphic to $C_c(Y)$ for some unique (up to homeomorphism) locally compact space Y , namely $Y = \lambda_{\mathcal{P}}X$. Moreover:*

- (1) $C_{\mathcal{P}}(X) = \{f \in C_b(X) : \text{supp}(f) \text{ has } \mathcal{P}\};$
- (2) $C_0(X) \subseteq C_{\mathcal{P}}(X)$, with proper inclusion if X is non- \mathcal{P} ;
- (3) $C_c(Y) = C_0(Y)$;
- (4) Y is countably compact;
- (5) Y is neither Lindelöf nor paracompact, if X is non- \mathcal{P} .

If, moreover, X is locally compact then in addition:

- (6) $C_{\mathcal{P}}(X) = \{f \in C_b(X) : \text{supp}(f) \text{ is } \sigma\text{-compact}\};$
- (7) Y is nonnormal, if X is non- \mathcal{P} .

PROOF. Conditions (1), (3), (4) and (5) follow from Theorem 3.1; we only need to show that the closure in X of each Lindelöf subspace of X has a Lindelöf neighborhood in X . (Note that the Lindelöf property is closed hereditary and is preserved under countable closed sums.)

Let A be a Lindelöf subspace of X . Since paracompactness is closed hereditary (see [8, Theorem 5.1.28]), $\text{cl}_X A$, being closed in X , is paracompact. Since any paracompact space having a dense Lindelöf subspace is itself Lindelöf (see [8, Theorem 5.1.25]), $\text{cl}_X A$ is Lindelöf. For each $x \in \text{cl}_X A$, let U_x be an open neighborhood of x in X with Lindelöf closure $\text{cl}_X U_x$. Then

$$\text{cl}_X A \subseteq U_{x_1} \cup U_{x_2} \cup \dots = U$$

for some $x_1, x_2, \dots \in \text{cl}_X A$. Since X is normal (as it is paracompact) there exists an open neighborhood V of $\text{cl}_X A$ in X such that $\text{cl}_X V \subseteq U$. Observe that $\text{cl}_X V$ is Lindelöf, as it is a closed subspace of the Lindelöf space

$$\text{cl}_X U_{x_1} \cup \text{cl}_X U_{x_2} \cup \dots .$$

To show (2), let $f \in C_0(X)$. Then $|f|^{-1}([1/n, \infty))$ is compact for each $n = 1, 2, \dots$ and therefore

$$\text{Coz}(f) = \bigcup_{n=1}^{\infty} |f|^{-1}([1/n, \infty))$$

is σ -compact and thus Lindelöf. Note that any paracompact space with a dense Lindelöf subspace is Lindelöf. (See [8, Theorem 5.1.25].) Since paracompactness is closed hereditary (see [8, Theorem 5.1.28]), $\text{supp}(f)$ is paracompact, as it is closed in X , and thus it is Lindelöf, as it contains $\text{Coz}(f)$ as a dense subspace. Therefore $f \in C_{\mathcal{P}}(X)$. Now suppose that X is non-Lindelöf. Assume the representation of X given in Section 1.3. We may further assume that the X_i are noncompact. (Otherwise, group together any countable number of X_i .) Then, for the function f which is defined to be identical to 1 on X_i and vanishing elsewhere, we have $f \in C_{\mathcal{P}}(X)$, while trivially $f \notin C_0(X)$.

In the remainder of the proof assume that X is locally compact.

Note that (7) follows from Lemma 4.3 (using the representation of X given in Section 1.3).

To show (6), let $f \in C_b(X)$. If $f \in C_{\mathcal{P}}(X)$, then since X is normal, $\text{supp}(f)$ has a closed Lindelöf neighborhood in X . But then $\text{supp}(f)$ has a closed σ -compact neighborhood in X , as any closed neighborhood of $\text{supp}(f)$ in X (being closed in the locally compact space X) is locally compact, and in the realm of locally compact spaces, the two notions of σ -compactness and being Lindelöf coincide. (See [8, Problem 3.8.C].) Thus $\text{supp}(f)$ is σ -compact. For the converse, note that if $\text{supp}(f)$ is σ -compact, then it is Lindelöf, and therefore $f \in C_{\mathcal{P}}(X)$ by (1). \square

Let I be an infinite set. A theorem of Tarski guarantees the existence of a collection \mathcal{J} of cardinality $|I|^{\aleph_0}$ consisting of countable infinite subsets of I , such that the intersection of any two distinct elements of \mathcal{J} is finite (see [12, Theorem 2.1]); this will be used in the following.

Note that the collection of all subsets of cardinality at most m in a set of cardinality $n \geq m$ has cardinality at most n^m .

LEMMA 4.5. *Let \mathcal{P} be the Lindelöf property. Let X be a locally- \mathcal{P} non- \mathcal{P} metrizable space. Then*

$$\dim C_{\mathcal{P}}(X) = \ell(X)^{\aleph_0}.$$

PROOF. Assume the representation of X given in Section 1.4. Note that I is infinite, as X is non-Lindelöf, and $\ell(X) = |I|$.

Let \mathcal{J} be a collection of cardinality $|I|^{\aleph_0}$ consisting of countable infinite subsets of I , such that the intersection of any two distinct elements of \mathcal{J} is finite. To simplify the notation denote

$$H_J = \bigcup_{i \in J} X_i$$

for each $J \subseteq I$. Define

$$f_J = \chi_{H_J}$$

for any $J \in \mathcal{J}$. Then no element of

$$\mathcal{F} = \{f_J : J \in \mathcal{J}\}$$

is a linear combination of other elements (since each element of \mathcal{J} is infinite and each pair of distinct elements of \mathcal{J} has finite intersection). Observe that \mathcal{F} is of cardinality $|\mathcal{J}|$. This shows that

$$\dim C_{\mathcal{P}}(X) \geq |\mathcal{J}| = |I|^{\aleph_0} = \ell(X)^{\aleph_0}.$$

Note that if $f \in C_{\mathcal{P}}(X)$, then since $\text{supp}(f)$ is Lindelöf,

$$\text{supp}(f) \subseteq H_J$$

where $J \subseteq I$ is countable, therefore one may assume that $f \in C_b(H_J)$. Conversely, if $J \subseteq I$ is countable, then each element of $C_b(H_J)$ can be extended trivially to an element of $C_{\mathcal{P}}(X)$ (by defining it to be identically 0 elsewhere). Thus $C_{\mathcal{P}}(X)$ may be viewed as the union of all $C_b(H_J)$, where J runs over all countable subsets of I . Note that

if $J \subseteq I$ is countable, then H_J is separable (note that in metrizable spaces separability coincides with being Lindelöf); thus any element of $C_b(H_J)$ is determined by its value on a countable set. This implies that for each countable $J \subseteq I$, the set $C_b(H_J)$ is of cardinality at most $c^{\aleph_0} = 2^{\aleph_0}$. Observe that there exist at most $|I|^{\aleph_0}$ countable $J \subseteq I$.
 Now

$$\begin{aligned} \dim C_{\mathcal{P}}(X) \leq |C_{\mathcal{P}}(X)| &\leq \left| \bigcup \{C_b(H_J) : J \subseteq I \text{ is countable}\} \right| \\ &\leq 2^{\aleph_0} \cdot |I|^{\aleph_0} = |I|^{\aleph_0} = \ell(X)^{\aleph_0}, \end{aligned}$$

which together with the first part proves the lemma. □

The following theorem is similar to Theorem 4.4.

THEOREM 4.6. *Let \mathcal{P} be the Lindelöf property. Let X be a metrizable locally- \mathcal{P} space. Then $C_{\mathcal{P}}(X)$ is a Banach subalgebra of $C_b(X)$ isometrically isomorphic to $C_c(Y)$ for some unique (up to homeomorphism) locally compact space Y , namely $Y = \lambda_{\mathcal{P}}X$. Moreover:*

- (1) $C_{\mathcal{P}}(X) = \{f \in C_b(X) : \text{supp}(f) \text{ has } \mathcal{P}\}$;
- (2) $C_0(X) \subseteq C_{\mathcal{P}}(X)$, with proper inclusion if X is non- \mathcal{P} ;
- (3) $C_c(Y) = C_0(Y)$;
- (4) Y is countably compact;
- (5) Y is nonnormal, if X is non- \mathcal{P} ;
- (6) $\dim C_{\mathcal{P}}(X) = \ell(X)^{\aleph_0}$.

PROOF. The theorem follows from Lemmas 4.3 and 4.5 and Theorem 4.4. Observe that metrizable spaces are paracompact. □

REMARK. Note that in metrizable spaces the notions of second countability and separability coincide with being Lindelöf. Thus, in Theorem 4.6,

$$\begin{aligned} C_{\mathcal{P}}(X) &= \{f \in C_b(X) : \text{supp}(f) \text{ is separable}\} \\ &= \{f \in C_b(X) : \text{supp}(f) \text{ is second countable}\}. \end{aligned}$$

REMARK. Theorems 4.4 and 4.6 highlight how differently $C_{\mathcal{P}}(X)$ may behave by varying the topological property \mathcal{P} for a locally- \mathcal{P} space X . If \mathcal{P} is compactness then of course $C_{\mathcal{P}}(X) = C_c(X)$. Thus, in this case, if $C_{\mathcal{P}}(X)$ is a Banach algebra, then it is closed in $C_0(X)$, and since $C_c(X)$ is dense in $C_0(X)$, it follows that $C_0(X) = C_c(X)$. It is known that for any locally compact space Y we have $C_0(Y) = C_c(Y)$ if and only if every σ -compact subspace of Y is contained in a compact subspace of Y . (See [10, Problem 7G.2].) Thus, in particular, if Y is a locally compact space, then $C_0(Y) = C_c(Y)$ implies that Y is countably compact. From this it follows that X is countably compact, and thus compact, if X is metrizable. In other words, if \mathcal{P} is compactness and X is a locally- \mathcal{P} metrizable space, then $C_{\mathcal{P}}(X)$ being a Banach algebra implies that X has \mathcal{P} . However, if we let \mathcal{P} to be the Lindelöf property, then for any locally- \mathcal{P} metrizable (or even paracompact) space X , it follows that $C_{\mathcal{P}}(X)$ is a Banach algebra, without X necessarily having \mathcal{P} .

REMARK. A regular space X is *linearly Lindelöf* if every linearly ordered (by \subseteq) open cover of X has a countable subcover; equivalently, if every uncountable subspace of X has a complete accumulation point in X ; see[11]. (A point $x \in X$ is a *complete accumulation point* of a subspace A of X , if

$$|U \cap A| = |A|$$

for every neighborhood U of x in X .) Obviously, if X is Lindelöf, then it is linearly Lindelöf. The converse holds if X is either locally compact paracompact or locally Lindelöf metrizable. To show this, note that

$$X = \bigcup_{i \in I} X_i,$$

where the X_i are pairwise disjoint nonempty Lindelöf open-closed subspaces of X . (See Sections 1.3 and 1.4.) Now, if X is non-Lindelöf then I is uncountable, and thus there exists an infinite subspace A of X (choose some $x_i \in X_i$ for each $i \in I$ and let $A = \{x_i : i \in I\}$) without even an accumulation point. That is, X is not linearly Lindelöf. Since paracompactness, local compactness and being locally Lindelöf are all closed hereditary, in Theorem 4.4(4) and Theorem 4.6 (using condition (1) of Theorems 4.4 and 4.6, respectively) we further have

$$C_{\varnothing}(X) = \{f \in C_b(X) : \text{supp}(f) \text{ is linearly Lindelöf}\}.$$

REMARK. The *density* of a space X , denoted by $d(X)$, is defined by

$$d(X) = \min\{|D| : D \text{ is dense in } X\} + \aleph_0.$$

In particular, a space X is separable if and only if $d(X) = \aleph_0$. Note that, if X is a locally Lindelöf metrizable space, then $d(X) = \ell(X)$. (To see this, assume the representation of X given in Section 1.4 and observe that $d(X) = |I| = \ell(X)$ if I is infinite and $d(X) = \aleph_0 = \ell(X)$ otherwise.) Thus in Theorem 4.6(5) we may replace $\ell(X)$ by $d(X)$.

In our next result we fit a certain type of ideals between $C_0(X)$ and $C_b(X)$. Let μ be an infinite cardinal. A regular space X is called μ -Lindelöf if every open cover of X has a subcover of cardinality up to μ . Note that the μ -Lindelöf property turns weaker as μ increases. Since the \aleph_0 -Lindelöf property coincides with the Lindelöf property, it then follows that every Lindelöf space is μ -Lindelöf.

THEOREM 4.7. *Let X be a non-Lindelöf space which is either locally compact paracompact or locally Lindelöf metrizable. Then there exists a chain*

$$C_0(X) \subsetneq H_0 \subsetneq H_1 \subsetneq \dots \subsetneq H_\lambda = C_b(X)$$

of Banach subalgebras of $C_b(X)$ such that H_μ , for each $\mu \leq \lambda$, is an ideal of $C_b(X)$ isometrically isomorphic to

$$C_0(Y_\mu) = C_c(Y_\mu)$$

for some locally compact space Y_μ . Furthermore, \aleph_λ equals the Lindelöf number $\ell(X)$ of X .

PROOF. For each ordinal μ , let \mathcal{P}_μ denote the \aleph_μ -Lindelöf property, and let

$$H_\mu = C_{\mathcal{P}_\mu}(X).$$

Let μ be an ordinal. Note that X is normal, and it is locally \aleph_μ -Lindelöf, as it is locally Lindelöf. Also, the \aleph_μ -Lindelöf property, by its definition, is closed hereditary and preserved under countable closed sums. Thus, to use Theorem 3.1, we only need to show that the closure in X of each \aleph_μ -Lindelöf subspace of X has a \aleph_μ -Lindelöf neighborhood in X . Assume the representation of X given in Sections 1.3 and 1.4 and note that $\ell(X) = |I|$. Suppose that A is a \aleph_μ -Lindelöf subspace of X . Since

$$\{X_i : i \in I\}$$

is an open cover of A , there exists some $J \subseteq I$ with $|J| \leq \aleph_\mu$ such that

$$A \subseteq \bigcup_{i \in J} X_i.$$

If we let

$$U = \bigcup_{i \in J} X_i,$$

then U is a neighborhood of $\text{cl}_X A$ in X , and it is \aleph_μ -Lindelöf, as it is the union of \aleph_μ number of its Lindelöf subspaces. By Theorem 3.1 we then know that H_μ is a Banach subalgebra of $C_b(X)$ isometrically isomorphic to

$$C_0(Y_\mu) = C_c(Y_\mu)$$

for some locally compact space Y_μ ; furthermore,

$$H_\mu = \{h \in C_b(X) : \text{supp}(h) \text{ is } \aleph_\mu\text{-Lindelöf}\}. \tag{4.1}$$

That H_μ is an ideal of $C_b(X)$ follows easily, as if $h \in H_\mu$, then $\text{supp}(fh)$, for any $f \in C_b(X)$, is \aleph_μ -Lindelöf, as it is closed in $\text{supp}(h)$ and the latter is \aleph_μ -Lindelöf, and thus $fh \in H_\mu$.

Note that if $\mu \leq \kappa$ then $H_\mu \subseteq H_\kappa$ by (4.1). Let λ be such that $\aleph_\lambda = \ell(X)$. Note that X is \aleph_λ -Lindelöf (as it is the union of \aleph_λ of its Lindelöf subspaces). This implies that $H_\lambda = C_b(X)$, as if $f \in C_b(X)$, then $\text{supp}(f)$ is \aleph_λ -Lindelöf, as it is closed in X . We now show that the inclusions in the chain are all proper. First, note that by Theorems 4.4 and 4.6, we have $C_0(X) \subsetneq H_0$. Now, let $\mu < \kappa \leq \lambda$. Let $J \subseteq I$ be of cardinality \aleph_κ . Then, for the function f which is identical to 1 on $\bigcup_{i \in J} X_i$ and vanishing elsewhere, we have $f \in H_\kappa$, while $f \notin H_\mu$. □

REMARK. Uncountable limit regular cardinals are referred to as *weakly inaccessible cardinals*. Weakly inaccessible cardinals cannot be proved to exist within ZFC, though their existence is not known to be inconsistent with ZFC. The existence of weakly inaccessible cardinals is sometimes taken as an additional axiom. Note that weakly inaccessible cardinals are necessarily aleph function fixed points, that is, if λ is a weakly inaccessible cardinal, then $\aleph_\lambda = \lambda$. It is worth noting that in Theorem 4.7, if the Lindelöf number $\ell(X)$ of X is weakly inaccessible, then the chain is of length $\ell(X)$.

5. The case where \mathcal{P} is countable compactness

In this section we determine $\lambda_{\mathcal{P}}X$ in the case where X is normal and \mathcal{P} is countable compactness. This may also be deduced from [14, Lemma 2.17] (see also [15]), observing that normality is hereditary with respect to closed subspaces and in the realm of normal spaces countable compactness and pseudocompactness coincide; see [8, Theorems 3.10.20 and 3.10.21]. (Recall that a completely regular space X is *pseudocompact*, if every continuous $f : X \rightarrow \mathbb{R}$ is bounded.) We include the proof here for the reader's convenience and completeness of results.

The following result is due to Hager and Johnson in [9]; a direct proof may be found in [7]. (See also [26, Theorem 11.24].)

LEMMA 5.1 (Hager and Johnson [9]). *Let U be an open subspace of a completely regular space X . If $\text{cl}_{\nu X}U$ is compact then $\text{cl}_X U$ is pseudocompact.*

Observe, in the proof of the following, that realcompactness is closed hereditary, a space having a pseudocompact dense subspace is pseudocompact, and that realcompact pseudocompact spaces are compact; see [8, Theorems 3.11.1 and 3.11.4].

LEMMA 5.2. *Let U be an open subspace of a completely regular space X . Then $\text{cl}_{\beta X}U \subseteq \nu X$ if and only if $\text{cl}_X U$ is pseudocompact.*

PROOF. The first half of the lemma follows from Lemma 5.1. For the second half, note that if $A = \text{cl}_X U$ is pseudocompact then so is its closure $\text{cl}_{\nu X}A$. But $\text{cl}_{\nu X}A$, being closed in νX , is also realcompact, and thus compact. Therefore $\text{cl}_{\beta X}A \subseteq \text{cl}_{\nu X}A$. \square

THEOREM 5.3. *Let \mathcal{P} be countable compactness. Let X be a normal space. Then*

$$\lambda_{\mathcal{P}}X = \text{int}_{\beta X}\nu X.$$

PROOF. If $C \in \text{Coz}(X)$ has countably compact (and thus pseudocompact) closure in X , then $\text{cl}_{\beta X}C \subseteq \nu X$, by Lemma 5.2, and then

$$\text{int}_{\beta X}\text{cl}_{\beta X}C \subseteq \text{int}_{\beta X}\nu X.$$

For the reverse inclusion, let $t \in \text{int}_{\beta X}\nu X$. Let $f : \beta X \rightarrow [0, 1]$ be continuous with

$$f(t) = 0 \quad \text{and} \quad f|(\beta X \setminus \text{int}_{\beta X}\nu X) \equiv \mathbf{1}.$$

Then

$$C = X \cap f^{-1}[[0, 1/2]] \in \text{Coz}(X)$$

and $t \in \text{int}_{\beta X}\text{cl}_{\beta X}C$ by Lemma 2.3. (Note that $(f|X)_{\beta} = f$, as they coincide on the dense subspace X of βX .) Also, $\text{cl}_X C$ is pseudocompact by Lemma 5.2, as

$$\text{cl}_{\beta X}C \subseteq f^{-1}[[0, 1/2]] \subseteq \nu X,$$

and thus it is countably compact, since (being closed in X) it is normal. \square

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