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Estimates for generalized Bohr radii in one and higher dimensions

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Abstract. In this article, we study a generalized Bohr radius $R_{p,q}(X)$, $p,q \in [1,\infty)$ defined for a complex Banach space X. In particular, we determine the exact value of $R_{p,q}(\mathbb{C})$ for the cases (i) $p,q \in [1,2]$, (ii) $p \in (2,\infty)$, $q \in [1,2]$, and (iii) $p,q \in [2,\infty)$. Moreover, we consider an n-variable version $R^n_{p,q}(X)$ of the quantity $R_{p,q}(X)$ and determine (i) $R^n_{p,q}(\mathcal{H})$ for an infinite-dimensional complex Hilbert space $\mathcal{H}$ and (ii) the precise asymptotic value of $R^n_{p,q}(X)$ as $n \to \infty$ for finite-dimensional X. We also study the multidimensional analog of a related concept called the p-Bohr radius. To be specific, we obtain the asymptotic value of the n-dimensional p-Bohr radius for bounded complex-valued functions, and in the vector-valued case, we provide a lower estimate for the same, which is independent of n.

1 Introduction and the main results

The celebrated theorem of Harald Bohr [13] states (in sharp form) that for any holomorphic self-mapping $f(z) = \sum_{n=0}^{\infty} a_n z^n$ of the open unit disk $\mathbb{D}$,

$$\sum_{n=0}^{\infty} |a_n| r^n \leq 1$$

for $|z| = r \leq 1/3$, and this quantity $1/3$ is the best possible. Inequalities of the above type are commonly known as *Bohr inequalities* nowadays, and appearance of any such inequality in a result is generally termed as the occurrence of the *Bohr phenomenon*. This theorem was an outcome of Bohr's investigation on the "absolute convergence problem" of ordinary Dirichlet series of the form $\sum a_n n^{-s}$, and did not receive much attention until it was applied to answer a long-standing question in the realm of operator algebras in 1995 (cf. [19]). Starting there, the Bohr phenomenon continues to be studied from several different aspects for the last two decades, for example, in certain abstract settings (cf. [1]), for ordinary and vector-valued Dirichlet series (see, f.i., [3, 15]), for uniform algebras (see [28]), for free holomorphic functions (cf. [30]), for a Faber–Green condenser (see [26]), for vector-valued functions (cf. [17, 23, 24]), for Hardy space functions (see [5]), and for functions in several variables (see, for example, [2, 8, 12, 21, 29]). We also urge the reader to glance through the

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references of these abovementioned articles to get a more complete picture of the recent developments in this area.

We will now concentrate on a variant of the Bohr inequality, introduced for the first time in [9] in order to investigate the Bohr phenomenon on Banach spaces. Let us start by defining an n-variable analog of this modified inequality. For this purpose, we need to introduce some concepts. Let $\mathbb{D}^n = \{(z_1, z_2, \dots, z_n) \in \mathbb{C}^n : \|z\|_\infty := \max_{1\le k\le n} |z_k| < 1\}$ be the open unit polydisk in the n-dimensional complex plane $\mathbb{C}^n$, and let X be a complex Banach space. Any holomorphic function $f : \mathbb{D}^n \to X$ can be expanded in the power series

$$f(z) = x_0 + \sum_{|\alpha|\in\mathbb{N}} x_\alpha z^\alpha,\ x_\alpha \in X, \tag{1.1}$$

for $z \in \mathbb{D}^n$. Here and hereafter, we will use the standard multi-index notation: α denotes an n-tuple $(\alpha_1, \alpha_2, \dots, \alpha_n)$ of nonnegative integers, $|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_n$, $\alpha! := \alpha_1!\alpha_2!\cdots\alpha_n!$, z denotes an n-tuple $(z_1, z_2, \dots, z_n)$ of complex numbers, and z^α is the product $z_1^{\alpha_1} z_2^{\alpha_2}\cdots z_n^{\alpha_n}$. For $1 \le p, q < \infty$ and for any f as in (1.1) with $\|f\|_{H^\infty(\mathbb{D}^n,X)} \le 1$, we denote

$$R^n_{p,q}(f, X) = \sup\left\{ r \ge 0 : \|x_0\|^p + \left(\sum_{k=1}^{\infty} \sum_{|\alpha|=k} \|x_\alpha z^\alpha\| \right)^q \le 1 \text{ for all } z \in r\mathbb{D}^n \right\},$$

where $H^\infty(\mathbb{D}^n, X)$ is the space of bounded holomorphic functions f from $\mathbb{D}^n$ to X and $\|f\|_{H^\infty(\mathbb{D}^n,X)} = \sup_{z\in\mathbb{D}^n} \|f(z)\|$. We further define

$$R^n_{p,q}(X) = \inf\left\{ R^n_{p,q}(f, X) : \|f\|_{H^\infty(\mathbb{D}^n,X)} \le 1 \right\}.$$

Following the notations of [9], throughout this article, we will use $R_{p,q}(f, X)$ for $R^1_{p,q}(f, X)$ and $R_{p,q}(X)$ for $R^1_{p,q}(X)$. Clearly, $R_{1,1}(\mathbb{C}) = 1/3$. The reason for reshaping the original Bohr inequality in the above fashion becomes clear from [9, Theorem 1.2], which shows that the notion of the classical Bohr phenomenon is not very useful for $\dim(X) \ge 2$. For a given pair of p and q in $[1, \infty)$, it is known from the results of [9] that depending on X, $R_{p,q}(X)$ may or may not be zero. A characterization theorem in this regard has further been established in [6]. However, the question of determination of the exact value of $R_{p,q}(X)$ is challenging, and to the best of our knowledge, there is lack of progress on this problem—even for $X = \mathbb{C}$. In fact, only known optimal result in this direction is the following:

$$R_{p,1}(\mathbb{C}) = \frac{p}{2+p} \tag{1.2}$$

for $1 \le p \le 2$ (cf. [9, Proposition 1.4]), along with rather recent generalizations of (1.2) (see, for example, [27]). This motivates us to address this problem in the first theorem of this article.

Theorem 1.1 *Given $p, q \in [1, \infty)$, let us denote*

$$A_{p,q}(a) = \frac{(1-a^p)^{\frac{1}{q}}}{1 - a^2 + a(1-a^p)^{\frac{1}{q}}},\ a \in [0, 1)$$

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and

$$S_{p,q}(a) = \left(\frac{(1-a^p)^{\frac{2}{q}}}{1-a^2+(1-a^p)^{\frac{2}{q}}} \right)^{\frac{1}{2}}, \quad a \in [0,1).$$

Furthermore, let $\widehat{a}$ *be the unique root in* $(0,1)$ *of the equation*

$$x^p + x^q = 1. \tag{1.3}$$

Then

$$R_{p,q}(\mathbb{C}) = \begin{cases} \inf_{a\in[\widehat{a},1)} A_{p,q}(a) \text{ if } p,q \in [1,2], \\ \min\left\{(1/\sqrt{2}), \inf_{a\in[\widehat{a},1)} A_{p,q}(a)\right\} \quad \text{if } p\in(2,\infty) \text{ and } q \in [1,2], \\ 1/\sqrt{2} \text{ if } p,q\in[2,\infty). \end{cases}$$

For $p \in [1,2]$ *and* $q \in (2,\infty)$, $R_{2,q}(\mathbb{C}) = 1/\sqrt{2}$, $R_{p,q}(\mathbb{C}) = \inf_{a\in[\widehat{a},1)} A_{p,q}(a)$ *if* $p < 2$ *and in addition the inequality*

$$q\widehat{a}^2 + p\widehat{a}^{p+2} \le p\widehat{a}^p + q\widehat{a}^{p+2} \tag{1.4}$$

is satisfied. In all other scenarios, we have, in general,

$$0 < \inf_{a\in[0,1)} S_{p,q}(a) \le R_{p,q}(\mathbb{C}) \le \frac{1}{\sqrt{2}}. \tag{1.5}$$

Remarks 1.2 **(a)** A closer look at the proof of Theorem 1.1 reveals that the conclusions of this theorem remain unchanged if the interval $[1,2]$ is replaced by $(0,2]$ everywhere in its statement. However, doing so includes cases where positive Bohr radius is nonexistent; for example, $R_{p,q}(\mathbb{C}) = \inf_{a\in[\widehat{a},1)} A_{p,q}(a) \le \lim_{a\to1-} A_{p,q}(a) = 0$ if $0 < q < 1$. Therefore, throughout this paper, we stick to the assumption $p,q \ge 1$.

(b) Following methods similar to the proof of Theorem 1.1, it is easy to see that for any given complex Hilbert space $\mathcal{H}$ with dimension at least 2, the following statements are true:

(i) For $p,q \in [2,\infty)$, $R_{p,q}(\mathcal{H}) = 1/\sqrt{2}$.
(ii) For $p \in [1,2)$ and $q \in [2,\infty)$, inequalities (1.5) are satisfied with $R_{p,q}(\mathbb{C})$ replaced by $R_{p,q}(\mathcal{H})$.

Note that the assumption $q \ge 2$ is justified by [6, Corollary 4]. Later, in Theorem 1.4, we obtain a more complete result for $\dim(\mathcal{H}) = \infty$.

We now turn our attention to the Bohr radius $R^n_{p,q}(X)$, where X is a complex Banach space. The first question we encounter is the identification of the Banach spaces X with $R^n_{p,q}(X) > 0$, which is in fact equivalent to the one-dimensional version of the same problem.

Proposition 1.3 *For any given* $n \in \mathbb{N}$ *and* $p,q \in [1,\infty)$, $R^n_{p,q}(X) > 0$ *for some complex Banach space* X *if and only if* $R_{p,q}(X) > 0$ *for the same Banach space* X.

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Note that from [6, Theorem 1], it is known that $R_{p,q}(X) > 0$ if and only if there exists a constant C such that

$$\Omega_X(\delta) \le C\left((1+\delta)^q - (1+\delta)^{q-p}\right)^{1/q} \tag{1.6}$$

for all $\delta \ge 0$. We mention here that for any $\delta \ge 0$, $\Omega_X(\delta)$ is defined to be the supremum of $\|y\|$ taken over all $x, y \in X$ such that $\|x\| = 1$ and $\|x + zy\| \le 1 + \delta$ for all $z \in \mathbb{D}$ (see [22]). Now, in view of the above discussion, it looks appropriate to consider the Bohr phenomenon, i.e., studying $R^n_{p,q}(X)$ for particular Banach spaces X. We resolve this problem completely for $X = \mathcal{H}$—a complex Hilbert space of infinite dimension. While this question remains open for $\dim(\mathcal{H}) < \infty$, we succeed in determining the correct asymptotic behavior of $R^n_{p,q}(X)$ as $n \to \infty$ for any finite-dimensional complex Banach space X with $R_{p,q}(X) > 0$.

Theorem 1.4 *For any given $n \in \mathbb{N}$, $p \in [1,\infty)$, $q \in [2,\infty)$ and for any infinite-dimensional complex Hilbert space $\mathcal{H}$,*

$$R^n_{p,q}(\mathcal{H}) = \inf_{a \in [0,1)} \left(1 - (1 - (S_{p,q}(a))^2)^{\frac{1}{n}}\right)^{\frac{1}{2}},$$

$S_{p,q}(a)$ as defined in the statement of Theorem 1.1. For any complex Banach space X with $\dim(X) < \infty$ and with $R_{p,q}(X) > 0$, we have

$$\lim_{n\to\infty} R^n_{p,q}(X)\sqrt{\frac{n}{\log n}} = 1.$$

At this point, we like to discuss another interesting related concept called the p-Bohr radius. First, we pose an n-variable version of the definition of p-Bohr radius given in [10]. For any $p \in [1,\infty)$ and for any complex Banach space X, we denote

$$r^n_p(f, X) = \sup\left\{r \ge 0 : \|x_0\|^p + \sum_{k=1}^{\infty}\sum_{|\alpha|=k} \|x_\alpha z^\alpha\|^p \le 1 \text{ for all } z \in r\mathbb{D}^n\right\},$$

where f is as given in (1.1) with $\|f\|_{H^\infty(\mathbb{D}^n,X)} \le 1$, and then define the n-dimensional p-Bohr radius of X by

$$r^n_p(X) = \inf\left\{r^n_p(f,X) : \|f\|_{H^\infty(\mathbb{D}^n,X)} \le 1\right\}.$$

Again, following the notations of [10], we will write $r_p(f,X)$ for $r^1_p(f,X)$ and $r_p(X)$ for $r^1_p(X)$. Clearly, for $X = \mathbb{C}$, one only needs to consider $p \in [1,2)$, as $r^n_p(\mathbb{C}) = 1$ for all $p \ge 2$ and for any $n \in \mathbb{N}$. The quantities $r_p(\mathbb{C})$ and $r^n_p(\mathbb{C})$ were first considered in [20]. Unlike $R_{p,q}(\mathbb{C})$, a precise value of $r_p(\mathbb{C})$ has already been obtained in [25]. We make further progress by determining the asymptotic behavior of $r^n_p(\mathbb{C})$ for all $p \in (1,2)$ (the case $p = 1$ is already resolved) in the first half of Theorem 1.5.

On the other hand, to get a nonzero value of $r^n_p(X)$ where $\dim(X) \ge 2$, one necessarily has to consider $p \ge 2$ and work with p-uniformly PL-convex complex Banach spaces X. A complex Banach space X is said to be p-uniformly PL-convex

$(2 \leq p < \infty)$ if there exists a constant $\lambda > 0$ such that

$$\|x\|^p + \lambda \|y\|^p \leq \frac{1}{2\pi} \int_0^{2\pi} \|x + e^{i\theta} y\|^p d\theta \tag{1.7}$$

for all $x, y \in X$. Denote by $I_p(X)$ the supremum of all λ satisfying (1.7). Now, if we assume $r_p^n(X) > 0$ for some $n \in \mathbb{N}$, then evidently $r_p(X) > 0$ (as any member of $H^\infty(\mathbb{D}, X)$ can be considered as a member of $H^\infty(\mathbb{D}^n, X)$ as well), and therefore [10, Theorem 1.10] asserts that X is p-uniformly $\mathbb{C}$-convex, which is equivalent to saying that X is p-uniformly PL-convex. The second half of our upcoming theorem shows that for any p-uniformly PL-convex complex Banach space X $(p \geq 2)$ with $\dim(X) \geq 2$, the Bohr radius $r_p^n(X) > 0$ for all $n \in \mathbb{N}$ and unlike $r_p^n(\mathbb{C})$ or $R_{p,q}^n(X)$, $r_p^n(X)$ does not converge to 0 as $n \to \infty$.

Theorem 1.5 *For any $p \in (1,2)$ and $n > 1$, we have*

$$r_p^n(\mathbb{C}) \sim \left(\frac{\log n}{n}\right)^{\frac{2-p}{2p}}.$$

For any p-uniformly PL-convex $(p \geq 2)$ complex Banach space X with $\dim(X) \geq 2$, we have

$$\left(\frac{I_p(X)}{2^p + I_p(X)}\right)^{\frac{2}{p}} \leq r_p^n(X) \leq 1$$

for all $n \in \mathbb{N}$.

We clarify that for any two sequences $\{p_n\}$ and $\{q_n\}$ of positive real numbers, we write $p_n \sim q_n$ if there exist constants $C, D > 0$ such that $Cq_n \leq p_n \leq Dq_n$ for all $n > 1$. In Section 2, we will give the proofs of all the results stated so far.

2 Proofs of the main results

We start by recalling the following result of Bombieri (cf. [14]), which is at the heart of the proof of our Theorem 1.1.

Theorem A *For any holomorphic self-mapping $f(z) = \sum_{n=0}^\infty a_n z^n$ of the open unit disk $\mathbb{D}$,*

$$\sum_{n=1}^\infty |a_n| r^n \leq \begin{cases} \frac{r(1-a^2)}{1-ar} & \text{for } r \leq a, \\ \frac{r\sqrt{1-a^2}}{\sqrt{1-r^2}} & \text{for } r \in [0,1) \text{ in general,} \end{cases}$$

where $|z| = r$ and $|a_0| = a$.

It should be mentioned that the above result is not recorded in the present form in [14]. For a direct derivation of the first inequality in Theorem A, see the proof of Theorem 9 of [7]. The second inequality is an easy consequence of the Cauchy–Schwarz inequality combined with the fact that $\sum_{n=1}^\infty |a_n|^2 \leq 1 - |a_0|^2$.

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Proof of Theorem 1.1 Given a holomorphic function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ mapping $\mathbb{D}$ inside $\mathbb{D}$, a straightforward application of Theorem A yields

$$|a_0|^p + \left(\sum_{n=1}^{\infty} |a_n| r^n\right)^q \le \begin{cases} a^p + (1-a^2)^q \left(\frac{r}{1-ar}\right)^q & \text{for } r \le a, \\ a^p + (1-a^2)^{\frac{q}{2}} \left(\frac{r}{\sqrt{1-r^2}}\right)^q & \text{for } r \in [0,1). \end{cases} \tag{2.1}$$

Now,

$$a^p + (1-a^2)^q \left(\frac{r}{1-ar}\right)^q \le 1$$

whenever $r \le A_{p,q}(a)$. A little calculation reveals that $A_{p,q}(a) \le a$ whenever $a^p + a^q \ge 1$, i.e., whenever $a \ge \widehat{a}$, $\widehat{a}$ being the root of equation (1.3). Thus, from (2.1), it is clear that

$$|a_0|^p + \left(\sum_{n=1}^{\infty} |a_n| r^n\right)^q \le 1 \tag{2.2}$$

for $r \le \inf_{a \in [\widehat{a},1)} A_{p,q}(a)$, provided that $a \ge \widehat{a}$. On the other hand,

$$a^p + (1-a^2)^{\frac{q}{2}} \left(\frac{r}{\sqrt{1-r^2}}\right)^q \le 1$$

for $r \le S_{p,q}(a)$, i.e., inequality (2.2) remains valid for $r \le \inf_{a \in [0,\widehat{a}]} S_{p,q}(a)$, provided that $a \le \widehat{a}$. Therefore, we conclude that for any given $p, q \in [1, \infty)$,

$$R_{p,q}(\mathbb{C}) \ge \min\left\{ \inf_{a \in [0,\widehat{a}]} S_{p,q}(a), \inf_{a \in [\widehat{a},1)} A_{p,q}(a) \right\}. \tag{2.3}$$

We also record some other facts which we will need to use later. Observe that for all $p, q \in [1, \infty)$,

$$S_{p,q}(a) = \sqrt{\frac{T(a)}{1+T(a)}} \text{ where } T(a) = \frac{(1-a^p)^{\frac{2}{q}}}{1-a^2},$$

and therefore

$$S'_{p,q}(a) = \frac{T'(a)}{2\sqrt{T(a)(1+T(a))^3}}$$

for $a \in (0,1)$, where

$$T'(a) = \frac{2a^{p-1} T(a)}{1-a^p} \left(\frac{a^2(1-a^p)}{a^p(1-a^2)} - \frac{p}{q} \right). \tag{2.4}$$

Setting $y = 1/a$ for convenience, we write

$$\frac{a^2(1-a^p)}{a^p(1-a^2)} = \frac{y^p - 1}{y^2 - 1} = P(y)$$

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defined on $(1,\infty)$. Note that

$$\frac{d}{da}P(y) = P'(y)\frac{dy}{da} = -y^3\frac{py^p - py^{p-2} - 2y^p + 2}{(y^2-1)^2}, \tag{2.5}$$

and that

$$Q'(y) = y^{p-3}(y^2-1)p(p-2), \tag{2.6}$$

where $Q(y) = py^p - py^{p-2} - 2y^p + 2$.

Furthermore, observe that for the disk automorphisms $\phi_a(z) = (a-z)/(1-az)$, $z \in \mathbb{D}$, $a \in [\widehat{a},1)$, $R_{p,q}(\phi_a,\mathbb{C}) = A_{p,q}(a)$, and hence $R_{p,q}(\mathbb{C}) \le \inf_{a\in[\widehat{a},1)} A_{p,q}(a)$. Moreover, for $\xi(z) = z\phi_{1/\sqrt{2}}(z)$, $z \in \mathbb{D}$, we have $R_{p,q}(\xi,\mathbb{C}) = 1/\sqrt{2}$. Combining these two facts, we write

$$R_{p,q}(\mathbb{C}) \le \min\left\{(1/\sqrt{2}), \inf_{a\in[\widehat{a},1)} A_{p,q}(a)\right\}. \tag{2.7}$$

We now deal with the problem case by case.

Case $p,q \in [1,2]$: Let us start with $p<2$. From (2.6), it is evident that $Q'(y)<0$ for $p<2$, and hence $Q(y) < Q(1) = 0$ for all $y \in (1,\infty)$. Thus, from (2.5), it is clear that $P(y)$ is strictly increasing in $(0,1)$ with respect to a. Consequently, for all $y \in (1,\infty)$,

$$P(y) < \lim_{a\to 1-} P(y) = \frac{p}{2}, \tag{2.8}$$

and using the above estimate in (2.4) gives, for all $a \in (0,1)$,

$$T'(a) < \frac{2a^{p-1}T(a)}{1-a^p}\left(\frac{p}{2} - \frac{p}{q}\right) \le 0,$$

as $q \le 2$. Therefore, $S_{p,q}(a)$ is strictly decreasing in $(0,1)$, and after some calculations, we have, as a consequence,

$$\inf_{a\in[0,\widehat{a}]} S_{p,q}(a) = S_{p,q}(\widehat{a}) = A_{p,q}(\widehat{a}) \ge \inf_{a\in[\widehat{a},1)} A_{p,q}(a).$$

Hence, from (2.3), we have $R_{p,q}(\mathbb{C}) \ge \inf_{a\in[\widehat{a},1)} A_{p,q}(a)$. For $p=2$, if $q<2$, then $T'(a)<0$ for all $a\in(0,1)$, which (as in the case $p<2$) again gives $R_{2,q}(\mathbb{C}) \ge \inf_{a\in[\widehat{a},1)} A_{2,q}(a)$. Otherwise, if $p=q=2$, then $\widehat{a} = 1/\sqrt{2}$, and for all $a\in[0,1)$, we get

$$S_{2,2}(a) = 1/\sqrt{2} = \inf_{a\in[\widehat{a},1)} A_{2,2}(a).$$

Therefore, for all $p,q\in[1,2]$, we have $R_{p,q}(\mathbb{C}) \ge \inf_{a\in[\widehat{a},1)} A_{p,q}(a)$, and from (2.7), it is known that $R_{p,q}(\mathbb{C}) \le \inf_{a\in[\widehat{a},1)} A_{p,q}(a)$. This completes the proof for this case.

Case $p\in(2,\infty), q\in[1,2]$: From (2.6), it is clear that $Q'(y)>0$ for $p>2$, and therefore $Q(y) > Q(1) = 0$ for all $y\in(1,\infty)$. It follows from (2.5) that $P(y)$ is strictly decreasing in $(0,1)$ with respect to a. Thus, for $q<2$, the value of the quantity

$$P(y) - \frac{p}{q} = \frac{a^2(1-a^p)}{a^p(1-a^2)} - \frac{p}{q}$$

decreases from

$$\lim_{a\to 0+}(P(y)-(p/q)) = +\infty \text{ to } \lim_{a\to 1-}(P(y)-(p/q)) = p((1/2)-(1/q)) < 0,$$

i.e., $P(y)-(p/q) > 0$ in $(0,b_1)$ and $P(y)-(p/q) < 0$ in $(b_1,1)$ for some $b_1 \in (0,1)$, where $P(b_1) = (p/q)$. As a consequence, $T'(a) = 0$ only for $a = 0, b_1$, and $T'(a) > 0$ in $(0,b_1)$, $T'(a) < 0$ in $(b_1,1)$. Hence, $S_{p,q}(a)$ strictly increases in $(0,b_1)$, and then strictly decreases in $(b_1,1)$, which implies that

$$\inf_{a\in[0,\widehat{a}]} S_{p,q}(a) = \min\left\{S_{p,q}(0), S_{p,q}(\widehat{a})\right\} = \min\left\{(1/\sqrt{2}), A_{p,q}(\widehat{a})\right\}.$$

Moreover, from the proof of the case $p,q \in [2,\infty)$, we have $R_{p,2}(\mathbb{C}) = 1/\sqrt{2}$. These two facts combined with (2.3) readily yield

$$R_{p,q}(\mathbb{C}) \geq \min\left\{(1/\sqrt{2}), \inf_{a\in[\widehat{a},1)} A_{p,q}(a)\right\},$$

and making use of (2.7), we arrive at our desired conclusion.

Case $p,q \in [2,\infty)$: Applying (2.7) of this paper, (1.9) of [9], and [10, Remark 1.2] together, the proof follows immediately from the observation:

$$(1/\sqrt{2}) \geq R_{p,q}(\mathbb{C}) \geq R_{2,2}(\mathbb{C}) \geq (1/\sqrt{2})r_2(\mathbb{C}) = 1/\sqrt{2}.$$

Case $p \in [1,2], q \in (2,\infty)$: The fact that $R_{2,q}(\mathbb{C}) = 1/\sqrt{2}$ is evident from the proof of the case $p,q \in [2,\infty)$. Furthermore, as we have already seen, from (2.1) it is clear that inequality (2.2) holds for $r \leq S_{p,q}(a), a \in [0,1)$, and therefore for $r \leq \inf_{a\in[0,1)} S_{p,q}(a)$. From this and (2.7), we have (1.5) as an immediate consequence. The assertion $\inf_{a\in[0,1)} S_{p,q}(a) > 0$ is validated from the fact that $S_{p,q}(a) \neq 0$ for all $a \in [0,1)$ and that $\lim_{a\to 1-} S_{p,q}(a) = 1$. Now, we will show that the imposition of the additional condition (1.4) gives an optimal value for $R_{p,q}(\mathbb{C})$. We know that for $p < 2$, $P(y)$ is strictly increasing in $(0,1)$ with respect to a, and as a result, $P(y)-(p/q)$ increases from

$$\lim_{a\to 0+}(P(y)-(p/q)) = -p/q \text{ to } \lim_{a\to 1-}(P(y)-(p/q)) = p((1/2)-(1/q)) > 0,$$

i.e., $P(y)-(p/q) < 0$ in $(0,b_2)$ and $P(y)-(p/q) > 0$ in $(b_2,1)$ for some $b_2 \in (0,1)$, where $P(b_2) = (p/q)$. As a consequence, $T'(a) = 0$ only for $a = 0, b_2$, and $T'(a) < 0$ in $(0,b_2)$, $T'(a) > 0$ in $(b_2,1)$. Hence, $S_{p,q}(a)$ strictly decreases in $(0,b_2)$, and then strictly increases in $(b_2,1)$. Now, if we assume the condition (1.4) in addition, it is equivalent to saying that $T'(\widehat{a}) \leq 0$, i.e., $\widehat{a} \leq b_2$. Thus, $\inf_{a\in[0,\widehat{a}]} S_{p,q}(a) = S_{p,q}(\widehat{a}) = A_{p,q}(\widehat{a})$. Consequently, from (2.3), we get $R_{p,q}(\mathbb{C}) \geq \inf_{a\in[\widehat{a},1)} A_{p,q}(a)$, which completes our proof for this case. ∎

Proof of Proposition 1.3 As any holomorphic function $f : \mathbb{D} \to X$ can also be considered as a holomorphic function from $\mathbb{D}^n$ to X, it immediately follows that $R^n_{p,q}(X) > 0$ for any $n \in \mathbb{N}$ implies that $R_{p,q}(X) > 0$. Thus, we only need to establish the converse. Any holomorphic $f:\mathbb{D}^n\to X$ with an expansion (1.1) can be written as

$$f(z)=x_0+\sum_{k=1}^{\infty}P_k(z),z\in\mathbb{D}^n, \tag{2.9}$$

where $P_k(z):=\sum_{|\alpha|=k}x_\alpha z^\alpha$. Thus, for any fixed $z_0\in\mathbb{T}^n$ (the n-dimensional torus), we have

$$g(u):=f(uz_0)=x_0+\sum_{k=1}^{\infty}P_k(z_0)u^k:\mathbb{D}\to X \tag{2.10}$$

is holomorphic, and if $\|f\|_{H^\infty(\mathbb{D}^n,X)}\leq 1$, then $\|g\|_{H^\infty(\mathbb{D},X)}\leq 1$. Hence, starting with the assumption $R_{p,q}(X)=R>0$, we have $\|P_k(z_0)\|\leq(1/R^k)(1-\|x_0\|^p)^{1/q}$, and since z_0 is arbitrary, we conclude that $\sup_{z\in\mathbb{T}^n}\|P_k(z)\|\leq(1/R^k)(1-\|x_0\|^p)^{1/q}$ for any $k\in\mathbb{N}$. Therefore, for a given $k\in\mathbb{N}$ and for any α with $|\alpha|=k$, we have

$$\begin{aligned}\|x_\alpha\|&=\left\|\frac{1}{(2\pi i)^n}\int_{|z_1|=1}\int_{|z_2|=1}\cdots\int_{|z_n|=1}\frac{P_k(z)}{z^{\alpha+1}}dz_ndz_{n-1}\cdots dz_1\right\|\\&\leq\sup_{z\in\mathbb{T}^n}\|P_k(z)\|\leq\frac{1}{R^k}(1-\|x_0\|^p)^{\frac{1}{q}}.\end{aligned}$$

As a result, we have, for all $r<R$,

$$\|x_0\|^p+\left(\sum_{k=1}^{\infty}r^k\sum_{|\alpha|=k}\|x_\alpha\|\right)^q\leq\|x_0\|^p+(1-\|x_0\|^p)\left(\left(\frac{R}{R-r}\right)^n-1\right)^q,$$

which is less than or equal to 1 whenever $r\leq R\left(1-(1/2)^{1/n}\right)$, thereby asserting that $R^n_{p,q}(X)>0$. ■

Proof of Theorem 1.4 (i) Before we start proving the first part of this theorem, note that the choice of $q\in[2,\infty)$ is again justified due to Proposition 1.3 and [6, Corollary 4]. Now, given a holomorphic $f:\mathbb{D}^n\to\mathcal{H}$ with an expansion (1.1) and with $\|f(z)\|\leq 1$ for all $z\in\mathbb{D}^n$, we have, for any fixed $R\in(0,1)$,

$$(2\pi)^{-n}\int_{\theta_1=0}^{2\pi}\int_{\theta_2=0}^{2\pi}\cdots\int_{\theta_n=0}^{2\pi}\left\|f\left(Re^{i\theta_1},Re^{i\theta_2},\ldots,Re^{i\theta_n}\right)\right\|^2d\theta_nd\theta_{n-1}\cdots d\theta_1\leq 1,$$

which is the same as saying that

$$\|x_0\|^2+\sum_{|\alpha|\in\mathbb{N}}\|x_\alpha\|^2R^{2|\alpha|}+(2\pi)^{-n}MR^{|\alpha|+|\beta|}\leq 1$$

with $M:=\sum_{\alpha\neq\beta}\langle x_\alpha,x_\beta\rangle\int_{\theta_1=0}^{2\pi}\int_{\theta_2=0}^{2\pi}\cdots\int_{\theta_n=0}^{2\pi}e^{i(\theta_1(\alpha_1-\beta_1)+\cdots+\theta_n(\alpha_n-\beta_n))}d\theta_nd\theta_{n-1}\cdots d\theta_1$. Here, $\langle.,.\rangle$ is the inner product of $\mathcal{H}$, α and β denote as usual n-tuples $(\alpha_1,\alpha_2,\ldots,\alpha_n)$ and $(\beta_1,\beta_2,\ldots,\beta_n)$ of nonnegative integers, respectively. As we know $\int_0^{2\pi}e^{ik\theta}d\theta=0$ for any $k\in\mathbb{Z}\backslash\{0\}$, $M=0$. Letting $R\to 1-$ in the above inequality, we therefore get $\|x_0\|^2+\sum_{k=1}^{\infty}\sum_{|\alpha|=k}\|x_\alpha\|^2\leq 1$. Taking $z\in r\mathbb{D}^n$ and using this inequality, we obtain

$$\|x_0\|^p + \left(\sum_{k=1}^{\infty}\sum_{|\alpha|=k}\|x_\alpha z^\alpha\|\right)^q \le \|x_0\|^p + \left(\sum_{k=1}^{\infty}\sum_{|\alpha|=k}\|x_\alpha\|^2\right)^{\frac{q}{2}}\left(\sum_{k=1}^{\infty}\sum_{|\alpha|=k}|z^\alpha|^2\right)^{\frac{q}{2}}$$
$$\le \|x_0\|^p + (1-\|x_0\|^2)^{\frac{q}{2}}\left(\sum_{k=1}^{\infty}\binom{n+k-1}{k}r^{2k}\right)^{\frac{q}{2}}$$
$$= \|x_0\|^p + (1-\|x_0\|^2)^{\frac{q}{2}}\left(\frac{1}{(1-r^2)^n}-1\right)^{\frac{q}{2}},$$

which is less than or equal to 1 if

$$r \le \left(1-(1-(S_{p,q}(\|x_0\|))^2)^{\frac{1}{n}}\right)^{\frac{1}{2}}, \tag{2.11}$$

and therefore

$$R^n_{p,q}(\mathcal{H}) \ge \inf_{a\in[0,1)}\left(1-(1-(S_{p,q}(a))^2)^{\frac{1}{n}}\right)^{\frac{1}{2}}. \tag{2.12}$$

As the quantity on the right-hand side of inequality (2.11) becomes $\sqrt{1-(1/2)^{1/n}}$ at $x_0 = 0$ and converges to 1 as $\|x_0\| \to 1-$, we conclude that the infimum in inequality (2.12) is attained at some $b_3 \in [0,1)$. Since every Hilbert space $\mathcal{H}$ has an orthonormal basis and, in our case, $\dim(\mathcal{H}) = \infty$, we can choose a countably infinite set $\{e_\alpha\}_{|\alpha|\in\mathbb{N}\cup\{0\}}$ of orthonormal vectors in $\mathcal{H}$. Setting $r_3 = (1-(1-(S_{p,q}(b_3))^2)^{\frac{1}{n}})^{\frac{1}{2}}$, we construct

$$\chi(z) := b_3 e_0 + \frac{1-b_3^2}{(1-b_3^p)^{\frac{1}{q}}}\sum_{k=1}^{\infty} r_3^k\left(\sum_{|\alpha|=k} z^\alpha e_\alpha\right) : \mathbb{D}^n \to \mathcal{H},$$

which satisfies $\|\chi(z)\| \le 1$ for all $z \in \mathbb{D}^n$, and $r_3 = R^n_{p,q}(\chi,\mathcal{H}) \ge R^n_{p,q}(\mathcal{H})$. This completes the proof for the first part of this theorem.

(ii) The proof for this part is rather lengthy, so we break it into a couple of steps. Prior to each step, we will provide some auxiliary information whenever needed.

Background for Step 1 : If $R_{p,q}(X) > 0$, we have

$$\Omega_X(\delta) \le C\left((1+\delta)^q - (1+\delta)^{q-p}\right)^{1/q}, \ \delta \ge 0$$

for some constant C (see (1.6) in the introduction). Given any such X, and given any holomorphic function $G(u) = \sum_{n=0}^{\infty} y_n u^n : \mathbb{D} \to X$ with $\|G(u)\| \le 1$ in $\mathbb{D}$, it is known from the proof of [6, Theorem 1] that

$$\|y_k\| \le 2\Omega_X(1-\|y_0\|) \le 2C\left((2-\|y_0\|)^q - (2-\|y_0\|)^{q-p}\right)^{1/q} \tag{2.13}$$

for all $k \ge 1$.

Step 1 : In our context, for any given holomorphic $f : \mathbb{D}^n \to X$ with an expansion (1.1) and with $\|f\|_{H^\infty(\mathbb{D}^n,X)} \le 1$, we define the holomorphic function $g(u) = x_0 + \sum_{k=1}^{\infty} P_k(z_0)u^k : \mathbb{D} \to X$ as in (2.10), which satisfies $\|g(u)\| \le 1$ for all $u \in \mathbb{D}$, z_0 being any chosen point on $\mathbb{T}^n$. Since $R_{p,q}(X) > 0$, making use of inequality (2.13), we conclude that for any $k \geq 1$,

$$\|P_k(z_0)\| \leq 2C\left((2-\|x_0\|)^q-(2-\|x_0\|)^{q-p}\right)^{1/q}$$

for any $z_0 \in \mathbb{T}^n$. Therefore,

$$\sup_{z\in\mathbb{T}^n}\|P_k(z)\| \leq 2C\left((2-\|x_0\|)^q-(2-\|x_0\|)^{q-p}\right)^{1/q} \tag{2.14}$$

for any $k \in \mathbb{N}$, C being the constant for which (1.6) is satisfied.

Background for Step 2 : For $1 \leq p < \infty$ and for a linear operator $U : X_0 \to Y_0$ between the complex Banach spaces X_0 and Y_0, we say that U is *p-summing* if there exists a constant $c \geq 0$ such that regardless of the natural number m and regardless of the choice of $f_1, f_2, \dots, f_m$ in X_0, we have

$$\left(\sum_{i=1}^m \|U(f_i)\|^p\right)^{1/p} \leq c \sup_{\phi\in B_{X_0^*}}\left(\sum_{i=1}^m |\phi(f_i)|^p\right)^{1/p},$$

where $B_{X_0^*}$ is the open unit ball in the dual space X_0^*. The least c for which the above inequality always holds is denoted by $\pi_p(U)$, and the set of all p-summing operators from X_0 into Y_0 is denoted by $\Pi_p(X_0, Y_0)$. Now, from [18, Proposition 2.3], we know that:

Fact I. If $U : X_0 \to Y_0$ is a bounded linear operator and $\dim(U(X_0)) < \infty$, then U is p-summing for every $p \in [1, \infty)$.

Moreover, [18, Theorem 2.8] states that:

Fact II. If $1 \leq p < q < \infty$, then $\Pi_p(X_0, Y_0) \subset \Pi_q(X_0, Y_0)$. Moreover, for $U \in \Pi_p(X_0, Y_0)$, we have $\pi_q(U) \leq \pi_p(U)$.

Step 2 : Coming back to our proof now, we set $X_0 = Y_0 = X$ and $U = I$—the identity operator on X. As X is finite-dimensional, $\dim(I(X)) < \infty$ in this case and thus using Fact I, we have $I \in \Pi_p(X, X)$ for all $p \geq 1$. Therefore,

$$\left(\sum_{|\alpha|=k}\|x_\alpha\|^{\frac{2k}{k+1}}\right)^{\frac{k+1}{2k}} \leq \pi_{\frac{2k}{k+1}}(I) \sup_{\phi\in B_{X^*}}\left(\sum_{|\alpha|=k}|\phi(x_\alpha)|^{\frac{2k}{k+1}}\right)^{\frac{k+1}{2k}}$$

for all $k \in \mathbb{N}$. Since $2k/(k+1) > 1$ for all $k \geq 2$, Fact II asserts that $\pi_{\frac{2k}{k+1}}(I) \leq \pi_1(I)$. Hence, there exists a constant $D = \pi_1(I)$ (depending only on X) such that

$$\left(\sum_{|\alpha|=k}\|x_\alpha\|^{\frac{2k}{k+1}}\right)^{\frac{k+1}{2k}} \leq D \sup_{\phi\in B_{X^*}}\left(\sum_{|\alpha|=k}|\phi(x_\alpha)|^{\frac{2k}{k+1}}\right)^{\frac{k+1}{2k}} \tag{2.15}$$

for all $k \in \mathbb{N}$.

Background for Step 3 : From [4, Theorem 1.1], we know that for any $\varepsilon > 0$, there exists $\mu > 0$ such that, for any complex k-homogeneous polynomial ($k \geq 1$) $P(z) = \sum_{|\alpha|=k} c_\alpha z^\alpha$ ($c_\alpha \in \mathbb{C}$), we have

$$\left(\sum_{|\alpha|=k}|c_\alpha|^{\frac{2k}{k+1}}\right)^{\frac{k+1}{2k}} \leq \mu(1+\varepsilon)^k \sup_{z\in\mathbb{D}^n}|P(z)|.$$

Step 3 : Recall from (2.9) now that $P_k(z) = \sum_{|\alpha|=k} x_\alpha z^\alpha$, $x_\alpha \in X$, and hence $\phi(P_k(z)) = \sum_{|\alpha|=k} \phi(x_\alpha) z^\alpha$ for any $\phi \in B_{X^*}$. Consequently, using the above inequality, we get that for any $\varepsilon > 0$, there exists $\mu > 0$ such that

$$\sup_{\phi\in B_{X^*}} \left(\sum_{|\alpha|=k} |\phi(x_\alpha)|^{\frac{2k}{k+1}} \right)^{\frac{k+1}{2k}} \le \mu(1+\varepsilon)^k \sup_{\phi\in B_{X^*}} \sup_{z\in\mathbb{D}^n} |\phi(P_k(z))| = \mu(1+\varepsilon)^k \sup_{z\in\mathbb{T}^n} \|P_k(z)\|$$

for all $k \ge 1$. Combining this inequality with inequalities (2.14) and (2.15) appropriately, we get

$$\left(\sum_{|\alpha|=k} \|x_\alpha\|^{\frac{2k}{k+1}} \right)^{\frac{k+1}{2k}} \le 2\mu CD(1+\varepsilon)^k \left((2-\|x_0\|)^q - (2-\|x_0\|)^{q-p}\right)^{1/q}.$$

It follows that

$$\left(\sum_{k=1}^{\infty} r^k \sum_{|\alpha|=k} \|x_\alpha\| \right)^q \le \left(\sum_{k=1}^{\infty} r^k \left(\sum_{|\alpha|=k} \|x_\alpha\|^{\frac{2k}{k+1}} \right)^{\frac{k+1}{2k}} \binom{n+k-1}{k}^{\frac{k-1}{2k}} \right)^q$$
$$\le X \left(\sum_{k=1}^{\infty} r^k (1+\varepsilon)^k \binom{n+k-1}{k}^{\frac{k-1}{2k}} \right)^q,$$

where $X = \mu^q C_1^q \left((2-\|x_0\|)^q - (2-\|x_0\|)^{q-p}\right)$, $C_1 = 2CD$. Hence, for $z \in r\mathbb{D}^n$, the inequality

$$\|x_0\|^p + \left(\sum_{k=1}^{\infty} \sum_{|\alpha|=k} \|x_\alpha z^\alpha\| \right)^q \le 1$$

is satisfied if

$$\left(\frac{X}{1-\|x_0\|^p} \right)^{\frac{1}{q}} \left(\sum_{k=1}^{\infty} r^k (1+\varepsilon)^k \binom{n+k-1}{k}^{\frac{k-1}{2k}} \right) \le 1. \tag{2.16}$$

Now, analyzing the function $f_1(t) = ((2-t)^p - 1)/(1-t^p)$, $t \in [0,1)$, we see that $f_1(t) \le f_1(0) = 2^p - 1$ for all $t \in [0,1)$, and hence

$$\frac{X}{1-\|x_0\|^p} = \mu^q C_1^q (2-\|x_0\|)^{q-p} f_1(\|x_0\|) \le \begin{cases} \mu^q C_1^q 2^{q-p}(2^p-1) & \text{if } q \ge p, \\ \mu^q C_1^q (2^p - 1) & \text{if } q \le p. \end{cases}$$

Thus, inequality (2.16) is satisfied if

$$C_2 \left(\sum_{k=1}^{\infty} r^k (1+\varepsilon)^k \binom{n+k-1}{k}^{\frac{k-1}{2k}} \right) \le 1,$$

where C_2 is a new constant depending on μ, p, q and the Banach space X. Using the estimate

$$\binom{n+k-1}{k} \le \frac{(n+k-1)^k}{k!} < \left(\frac{e}{k}\right)^k (n+k-1)^k < e^k \left(1+\frac{n}{k}\right)^k,$$

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we get, by setting $r = (1-2\varepsilon)\sqrt{(\log n)/n}$,

$$\sum_{k=1}^{\infty} r^k (1+\varepsilon)^k \binom{n+k-1}{k}^{\frac{k-1}{2k}} \le \sum_{k=1}^{\infty} \left(\sqrt{\frac{\log n}{n}} \sqrt{e}(1-2\varepsilon)(1+\varepsilon) \right)^k \left(1+\frac{n}{k}\right)^{\frac{k-1}{2}}.$$

Hence, inequality (2.16) is satisfied if

$$C_2 \sum_{k=1}^{\infty} \left(\sqrt{\frac{\log n}{n}} \sqrt{e}(1-2\varepsilon)(1+\varepsilon) \right)^k \left(1+\frac{n}{k}\right)^{\frac{k-1}{2}} \le 1. \tag{2.17}$$

Starting here, we will follow the similar lines of argument as in [4, pp. 743–744]. For n large enough,

$$t_n := \frac{\sqrt{\log n}}{n^{1/4}} \sqrt{2e}(1-2\varepsilon)(1+\varepsilon) < 1,$$

and for $k > \sqrt{n}$, observe that

$$\left(1+\frac{n}{k}\right)^{\frac{k-1}{2}} < (2\sqrt{n})^{\frac{k}{2}}.$$

Using both the above facts,

$$\begin{aligned} &\sum_{k>\sqrt{n}} \left(\sqrt{\frac{\log n}{n}} \sqrt{e}(1-2\varepsilon)(1+\varepsilon) \right)^k \left(1+\frac{n}{k}\right)^{\frac{k-1}{2}} \\ &\le \sum_{k>\sqrt{n}} \left(\frac{\sqrt{\log n}}{n^{1/4}} \sqrt{2e}(1-2\varepsilon)(1+\varepsilon) \right)^k \le \frac{t_n}{1-t_n}, \end{aligned}$$

which goes to 0 as $n \to \infty$. For $k \le \sqrt{n}$, we start by making n sufficiently large such that $2 < k_0 \le \log n$ can be chosen for which the inequalities

$$k_0^{\frac{1}{k_0-1}} \le 1 + \frac{\varepsilon}{2}, \quad \sum_{k_0 \le k \le \sqrt{n}} ((1-2\varepsilon)(1+\varepsilon)^{3/2})^k \le \frac{1}{2C_2} \quad \text{and} \quad \left(\frac{1}{n}\right)^{\frac{k_0-2}{2(k_0-1)}} \le \frac{\varepsilon}{2}$$

are satisfied. Observing that $x^{1/(x-1)}$ is decreasing and $(x-2)/2(x-1)$ is increasing in $(1,\infty)$, we obtain, for $k \ge k_0$,

$$\begin{aligned} \left(k^{\frac{k}{k-1}} \left(\frac{1}{n} + \frac{1}{k} \right) \right)^{\frac{k-1}{k}} &\le \left(\left(\frac{1}{n}\right)^{\frac{k-2}{2(k-1)}} + k^{\frac{1}{k-1}} \right)^{\frac{k-1}{k}} \\ &\le \left(\left(\frac{1}{n}\right)^{\frac{k_0-2}{2(k_0-1)}} + k_0^{\frac{1}{k_0-1}} \right)^{\frac{k-1}{k}} \le (1+\varepsilon)^{\frac{k-1}{k}} \le 1+\varepsilon, \end{aligned}$$

which, after a little simplification, gives

$$\left(1+\frac{n}{k}\right)^{\frac{k-1}{2}} \le (1+\varepsilon)^{\frac{k}{2}} \frac{n^{\frac{k}{2}}}{n^{\frac{1}{2}} k^{\frac{k}{2}}}.$$

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Since $x \mapsto n^{1/x}x$ is decreasing up to $x = \log n$ and increasing thereafter, we have $n^{1/k}k \geq e \log n$. Therefore,

$$\sum_{k_0 \leq k \leq \sqrt{n}} \left(\sqrt{\frac{\log n}{n}}\sqrt{e}(1-2\varepsilon)(1+\varepsilon)\right)^k \left(1+\frac{n}{k}\right)^{\frac{k-1}{2}}$$

$$\leq \sum_{k_0 \leq k \leq \sqrt{n}} \left(\sqrt{e\log n}(1-2\varepsilon)(1+\varepsilon)^{3/2}\sqrt{\frac{1}{n^{1/k}k}}\right)^k$$

$$\leq \sum_{k_0 \leq k \leq \sqrt{n}} ((1-2\varepsilon)(1+\varepsilon)^{3/2})^k \leq \frac{1}{2C_2}.$$

It remains to analyze the case $1 \leq k \leq k_0$. In this case, we observe that for n large enough,

$$\frac{k}{n}+1 \leq \frac{k_0}{n}+1 \leq \varepsilon+1,$$

and hence

$$\left(1+\frac{n}{k}\right)^{\frac{k-1}{2}} \leq (1+\varepsilon)^{\frac{k}{2}}\left(\frac{n}{k}\right)^{\frac{k-1}{2}}.$$

Making use of the above inequality and the fact that $x \mapsto n^{1/x}x$ is decreasing in $[1, k_0]$ (i.e., $n^{1/k}k \geq n^{1/k_0}k_0$), it is easily seen that

$$\sum_{k=1}^{k_0} \left(\sqrt{\frac{\log n}{n}}\sqrt{e}(1-2\varepsilon)(1+\varepsilon)\right)^k \left(1+\frac{n}{k}\right)^{\frac{k-1}{2}}$$

$$\leq \sum_{k=1}^{k_0} \left(\sqrt{e\log n}(1-2\varepsilon)(1+\varepsilon)^{3/2}\frac{k^{1/(2k)}}{k_0^{1/2}n^{1/(2k_0)}}\right)^k,$$

which tends to 0 as $n \to \infty$. Combining all the above three estimates, we have

$$\sum_{k=1}^{\infty} \left(\sqrt{\frac{\log n}{n}}\sqrt{e}(1-2\varepsilon)(1+\varepsilon)\right)^k \left(1+\frac{n}{k}\right)^{\frac{k-1}{2}} \leq \frac{1}{2C_2}+o(1)$$

for n large enough. Therefore, inequality (2.17) is satisfied for large enough n. Hence, for any given $\varepsilon > 0$, $R^n_{p,q}(X) \geq (1-2\varepsilon)\sqrt{\log n}/\sqrt{n}$ for sufficiently large n. This yields the following:

$$\liminf_{n\to\infty} R^n_{p,q}(X)\sqrt{n}/\sqrt{\log n} \geq 1.$$

Step 4 : In view of the above, it is only left to show that

$$\limsup_{n\to\infty} R^n_{p,q}(X)\sqrt{n}/\sqrt{\log n} \leq 1. \tag{2.18}$$

As $R^n_{p,q}(X) \leq R^n_{p,q}(\mathbb{C})$, it is sufficient to establish this part for $X = \mathbb{C}$. The proof is exactly the same as the proof for the case $p = q = 1$ given in [12, p. 2977], but for the sake of completeness, we reproduce the argument here. From the Kahane–Salem–Zygmund inequality, it is known that there is a constant B such that for every collection of complex numbers c_α and every integer $k > 1$, there is a choice of plus and minus signs for which the supremum of the modulus of $\sum_{|\alpha|=k} \pm c_\alpha z^\alpha$ in $\mathbb{D}^n$ does not exceed $B\left(n\sum_{|\alpha|=k}|c_\alpha|^2 \log k\right)^{1/2}$. We choose $c_\alpha = k!/\alpha!$. Then $\sum_{|\alpha|=k}|c_\alpha|^2 \le k! n^k$. By the definition of the generalized Bohr inequality in our context, we get

$$\left(\left(R^n_{p,q}(\mathbb{C})\right)^k n^k\right)^q = \left(\sum_{|\alpha|=k} |c_\alpha| \left(R^n_{p,q}(\mathbb{C})\right)^k\right)^q$$
$$\le B^q \left(n \sum_{|\alpha|=k} |c_\alpha|^2 \log k\right)^{q/2} \le B^q \left(n^{\frac{k+1}{2}} (k! \log k)^{1/2}\right)^q,$$

or, equivalently,

$$R^n_{p,q}(\mathbb{C}) \le B^{1/k} n^{\frac{1-k}{2k}} (k! \log k)^{\frac{1}{2k}}.$$

We use Stirling's formula $\lim_{k\to\infty} k!(\sqrt{2\pi k}(k/e)^k)^{-1} = 1$ to conclude that

$$R^n_{p,q}(\mathbb{C}) \le \sqrt{\frac{k}{n}} \left(\frac{B_1^{1/k} n^{\frac{1}{2k}} k^{\frac{1}{4k}} (\log k)^{\frac{1}{2k}}}{\sqrt{e}}\right)$$

for a new constant B_1. Setting $k = \lfloor \log n \rfloor$ ($\lfloor . \rfloor$ is the floor function), we observe

$$\limsup_{n\to\infty} R^n_{p,q}(\mathbb{C}) \sqrt{\frac{n}{\log n}} \le \lim_{n\to\infty} \frac{B_1^{1/\lfloor \log n\rfloor} n^{\frac{1}{2\lfloor \log n\rfloor}} \lfloor \log n\rfloor^{\frac{1}{4\lfloor \log n\rfloor}} (\log\lfloor \log n\rfloor)^{\frac{1}{2\lfloor \log n\rfloor}}}{\sqrt{e}} = 1,$$

which implies our desired inequality (2.18). This completes the proof. ∎

Proof of Theorem 1.5 (i) Given a complex-valued holomorphic function f with an expansion (1.1) in $\mathbb{D}^n$ ("x_α's" are complex numbers in this case) and satisfying $\|f\|_{H^\infty(\mathbb{D}^n,\mathbb{C})} \le 1$, an application of Hölder's inequality yields

$$|x_0|^p + \sum_{k=1}^{\infty} r^{kp} \sum_{|\alpha|=k} |x_\alpha|^p = \sum_{k=0}^{\infty}\sum_{|\alpha|=k} |x_\alpha|^{2-p} r^{kp} |x_\alpha|^{2p-2}$$
$$\le \left(\sum_{k=0}^{\infty} r^{\frac{kp}{2-p}} \sum_{|\alpha|=k} |x_\alpha|\right)^{2-p} \left(\sum_{k=0}^{\infty}\sum_{|\alpha|=k} |x_\alpha|^2\right)^{p-1}$$
$$\le \left(\sum_{k=0}^{\infty} r^{\frac{kp}{2-p}} \sum_{|\alpha|=k} |x_\alpha|\right)^{2-p}.$$

Therefore, $r^n_p(\mathbb{C}) \ge (r^n_1(\mathbb{C}))^{(2-p)/p}$. Since $\lim_{n\to\infty} r^n_1(\mathbb{C}) \left(\sqrt{n}/\sqrt{\log n}\right) = 1$ (cf. [4]), we have

$$\liminf_{n\to\infty} r^n_p(\mathbb{C}) \left(\frac{n}{\log n}\right)^{\frac{2-p}{2p}} \ge \liminf_{n\to\infty} \left(r^n_1(\mathbb{C}) \sqrt{\frac{n}{\log n}}\right)^{\frac{2-p}{p}} = 1,$$

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and thus $r_p^n(\mathbb{C}) \geq C((\log n)/n)^{(2-p)/2p}$ for some constant $C > 0$ and for all $n > 1$. The upper bound $r_p^n(\mathbb{C}) \leq D\left((\log n)/n\right)^{(2-p)/2p}$ for some $D > 0$ has already been established in [20, p. 76]. This completes the proof.

(ii) To handle the second part of this theorem, we first construct $g(u)$ as in (2.10) from a given holomorphic $f : \mathbb{D}^n \to X$ with an expansion (1.1) and satisfying $\|f\|_{H^\infty(\mathbb{D}^n, X)} \leq 1$. Now, since X is p-uniformly PL-convex, from the proof of [11, Proposition 2.1(ii)], we obtain

$$\|P_1(z_0)\| \leq \frac{2}{(I_p(X))^{\frac{1}{p}}}(1 - \|x_0\|^p)^{\frac{1}{p}}$$

for any arbitrary $z_0 \in \mathbb{T}^n$. Using a standard averaging trick (see, f.i., [10, p. 94]), it can be shown that the $P_1(z_0)$ in the above inequality could be replaced by $P_k(z_0)$ for any $k \geq 2$. Thus, we conclude that

$$\sup_{z \in \mathbb{T}^n} \|P_k(z)\| \leq \frac{2}{(I_p(X))^{\frac{1}{p}}}(1 - \|x_0\|^p)^{\frac{1}{p}}. \tag{2.19}$$

Now, from [16, Lemma 25.18], it is known that there exists $R > 0$ such that

$$\left(\sum_{|\alpha|=k} \|x_\alpha\|^p\right) R^{kp} \leq \int_{\mathbb{T}^n} \|P_k(z)\|^p dz.$$

Using inequality (2.19) gives

$$\sum_{|\alpha|=k} \|x_\alpha\|^p \leq \frac{2^p}{I_p(X) R^{kp}}(1 - \|x_0\|^p).$$

Assuming $r < R$, it is easy to see that

$$\begin{aligned}
\|x_0\|^p + \sum_{k=1}^{\infty} r^{kp} \sum_{|\alpha|=k} \|x_\alpha\|^p &\leq \|x_0\|^p + \frac{2^p}{I_p(X)}(1 - \|x_0\|^p) \sum_{k=1}^{\infty} \left(\frac{r}{R}\right)^{kp} \\
&\leq \|x_0\|^p + \frac{2^p}{I_p(X)}(1 - \|x_0\|^p) \frac{r^p}{R^p - r^p},
\end{aligned}$$

which is less than or equal to 1 if

$$r \leq R\left(\frac{I_p(X)}{2^p + I_p(X)}\right)^{\frac{1}{p}} = \left(\frac{I_p(X)}{2^p + I_p(X)}\right)^{\frac{2}{p}},$$

as from the arguments in [16, p. 627], it is clear that we can take $R^p = I_p(X)/(I_p(X) + 2^p)$. This proves the lower estimate for $r_p^n(X)$, and the upper estimate is trivial due to the fact that $r_p^n(X) \leq r_p^n(\mathbb{C}) = 1$ for $p \geq 2$. ■

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