MEASURE IN SEMIGROUPS

B. R. GELBAUM AND G. K. KALISCH

1. Introduction. The major portion of this paper is devoted to an investigation of the conditions which imply that a semigroup (no identity or commutativity assumed) with a bounded invariant measure is a group. We find in §3 that a weakened form of "shearing" is sufficient and a counter-example (§5) shows that "shearing" may not be dispensed with entirely. In §4 we discuss topological measures in locally compact semigroups and find that shearing may be dropped without affecting the results of the earlier sections (Theorem 2). The next two theorems show that under certain circumstances (shearing or commutativity) the topology of the semigroup (already known to be a group by virtue of earlier results) can be weakened so that the structure becomes a separated compact topological group. The last section treats the problem of extending an invariant measure on a commutative semigroup to an invariant measure on its quotient structure.

2. Measure-theoretic and topological preliminaries. We summarize in this section all definitions, concepts, and general conditions to which reference will be made in the remainder of the paper.

We shall be dealing with semigroups, denoted by S, in which there is a two-sided cancellation law. In general, commutativity and the existence of an identity will not be assumed, unless something to the contrary is stated. Without further comment we shall use the measure-theoretic notations and concepts of [**2**], such as ring, measure on a ring, outer measure, inner measure, completion of a measure, content, etc. We shall consider the preceding on both S and $S \times S$, and we shall distinguish between them by means of the subscripts 1 and 2 for S and $S \times S$ respectively.

A *ring* $\mathfrak{R}_1$ of subsets of S will be called *left-invariant* in case

(A) $x \in S$, $A \in \mathfrak{R}_1$ imply $xA \in \mathfrak{R}_1$.

Similarly, a *measure* m_1 on a left-invariant ring $\mathfrak{R}_1$ will be called *left-invariant* in case

(B) $x \in S$, $A \in \mathfrak{R}_1$ imply $m_1(A) = m_1(xA)$.

We observe that the Conditions A and B for m_1 and $\mathfrak{R}_1$ imply the same for $\bar{m}_1$, $\bar{\mathfrak{S}}_1$, and the corresponding entities of $S \times S$.

In $S \times S$ we shall encounter the following transformations:

$$\text{\textit{shearing}}, \quad \theta(x, y) = (x, xy);$$

$$\text{\textit{reflection}}, \quad \pi(x, y) = (y, x).$$

Received February 26, 1951.

https://doi.org/10.4153/CJM-1952-036-4 Published online by Cambridge University Press

We note that π is a measure preserving transformation for $m_1 \times m_1$ and such of its extensions as we shall have occasion to consider. Concerning θ we shall sometimes make the following assumption:

(C) *$A \in \mathfrak{R}_1$, $B \in \mathfrak{R}_1$ imply $\theta(A \times B)$ is $\bar{m}_2$-measurable, where $\bar{m}_2$ is the completion of $\bar{m}_1 \times \bar{m}_1$.*

We shall be concerned principally with the case where

(D) $\sup\ (m_1(A): A \in \mathfrak{R}_1) = 1$.

We note that this condition implies the same for $\bar{m}_1$, $\bar{m}_2$, etc.

Part of our discussion will revolve around the following cases:

(E) *S is locally compact and T_1; xy is continuous on $S \times S$; $\mathfrak{R}_1 = \mathfrak{R}(\mathfrak{C}_1)$, where $\mathfrak{C}_1$ is the family of compact sets of S; m_1 is regular on $\mathfrak{R}_1$.*

We note that the assumption of regularity for m_1 is not restrictive. For, by **[2]**, there exists a non-trivial regular Borel measure m'_1 associated with any non-trivial Borel measure m_1 such that $m'_1(C) \geqslant m_1(C)$ for all $C \in \mathfrak{C}_1$.

(F) *S is Abelian.*

(G) *$x \in S$, $A \subset S$, $xA \in \mathfrak{R}_1$ imply $A \in \mathfrak{R}_1$.*

3. Semigroups with shearing. We shall be concerned with xS and $xS \times xS$ as well as with the original semigroup S and the rings and measures in the former, which are produced by translation, will be designated by $\mathfrak{R}'_1, \ldots, m'_1, \ldots$, where $\mathfrak{R}'_1$, for instance, consists of all translates by x of the elements of $\mathfrak{R}_1$, and $m'_1(xA) = m_1(A)$, for A in $\mathfrak{R}_1$.

LEMMA 1. *Conditions* (A B D) *imply that $(x, x)E$ is $\bar{m}'_2$-measurable if E is $\bar{m}_2$-measurable, and that $\bar{m}'_2((x, x)E) = \bar{m}_2(E)$.*

Proof. The lemma is clearly true if E is the rectangular set $A \times B$, where $A \in \mathfrak{R}_1$, $B \in \mathfrak{R}_1$. If $\bar{m}_2(E) = 0$, then $\bar{m}'_2((x, x)E) = 0$. For, given an $\epsilon > 0$, one can find sequences of sets $A_n \in \mathfrak{R}_1$, $B_n \in \mathfrak{R}_1$ such that

$$\bigcup_{n=1}^{\infty} (A_n \times B_n) \supset E$$

and such that

$$\sum_{n=1}^{\infty} \bar{m}_2(A_n \times B_n) < \epsilon.$$

The assertion is now obvious. In the general case E may be expressed as a union of two sets M and N, where M is m_2-measurable and $\bar{m}_2(N) = 0$. Clearly $(x, x)M$ is m'_2-measurable, and the proof is complete.

THEOREM 1. *Conditions* (A B C D) *imply that S is a group.*

Proof. 1. We first show that S has an identity. Condition (D) implies that we may choose an $A \in \mathfrak{R}_1$ such that $\bar{m}_2(A \times A) > 0.9$. Then

$$\bar{m}_2(\theta(A \times A)) = \int_A m_1(xA)dm_1(x) = (m_1(A))^2 > 0.9.$$

Obviously, $\bar{m}_2(\pi\theta(A \times A)) > 0.9$, whence, by (D), $\theta(A \times A) \cap \pi\theta(A \times A)$ is not empty. Hence there exist pairs (x, y) and (u, v) in $A \times A$ such that $(x, xy) = (uv, u)$, i.e., $x = uv$, $xy = u$, $xyv = uv = x$, i.e., yv is a right identity for x, and hence, by a standard algebraic result, a two-sided identity for S.

2. We show that xS, $\mathfrak{R}'_1, \ldots, m'_1, \ldots$, etc., constitute a system satisfying conditions (A B C D). The facts that xS is a semigroup, $\mathfrak{R}'_1$ is a ring with a measure m'_1, as well as their fulfilling conditions (A B D) are easily verified. Condition (C) is verified as follows. Let $A \in \mathfrak{R}_1$, $B \in \mathfrak{R}_1$. Then

$$\theta(xA \times xB) = (x, x)\theta(A \times xB),$$

which by Condition (C) (for S) and Lemma 1 is $\bar{m}'_2$-measurable. This shows that xS has an identity for all $x \in S$, and thus S is a group.

4. Locally compact semigroups. We shall show that if S is locally compact then, in a certain sense, condition (C) (shearing preserves measurability) can be dispensed with, while the conclusion of Theorem 1 remains undisturbed. Our proof will be based upon the fact that if m_1 is a Borel measure on S, then $m_1 \times m_1 = m_2$ can be extended to a Borel measure μ_2 on $S \times S$. This construction is carried out by means of a *partial content* which we shall now define. A partial content is a non-negative, finite-valued, finitely additive, and subadditive monotone set function defined on a class of compact sets which has the following properties:

(i) the union of any two elements of the class is in the class;

(ii) if C is in the class, and if C is contained in the union of two open sets U and V, then there are two sets in the class, $D \subset U$, $E \subset V$, such that $C = D \cup E$. A partial content is very closely analogous to the content defined in [**2**, p. 231]. The development found in [**2**, §53] can be duplicated for a partial content, and one obtains a regular Borel measure μ induced by the given partial content. (Note that it is possible to follow the development of [**2**, §53] without restricting oneself to σ-bounded sets. Unless explicit indication to the contrary is made, we shall not restrict ourselves, in what follows, to the exclusive consideration of σ-bounded sets.)

LEMMA 2. *If m_1 is a Borel measure on a locally compact space X, then $m_1 \times m_1 = m_2$ can be extended to a Borel measure μ_2 on $X \times X$.*

Proof. We observe that the set function $m_2(C)$ on the class of m_2-measurable compact sets of $X \times X$ is a partial content in the sense defined above, and the regular Borel measure μ_2 induced by it is easily seen to be an extension of m_2 on $X \times X$.

The measure μ_2 just defined is used in the proof of the following lemma which, as we shall show, can be used to avoid positing Condition (C) (shearing) in the presence of local compactness.

LEMMA 3. *Conditions* (A B D E) *imply that* $(\mu_2)_*(\theta(S \times S)) = 1$.

Proof. Given $\epsilon > 0$, select a compact $C \subset S$ such that $m_1(C) \geqslant 1 - \epsilon$. Then $\theta(C \times C) = D$ is a compact subset of $S \times S$ (by E). Owing to the regularity of μ_2 and m_1 (hence of m_2 and $\bar{m}_2$), there exists a decreasing sequence of open sets U_n such that for all n,

$$U_n \in \mathfrak{R}_1 \times \mathfrak{R}_1, U_n \supset D, \quad m_2(U_n) \downarrow \mu_2(D).$$

Thus if

$$A = \bigcap_{n=1}^{\infty} U_n,$$

then $\bar{m}_2(A) = \mu_2(D)$. Since Fubini's theorem is applicable for the measure $\bar{m}_2$,

$$\bar{m}_2(A) = \int_S \bar{m}_1(A_x) d\bar{m}_1(x).$$

Clearly, $A_x \supset D_x$, whence, whenever A_x is $\bar{m}_1$-measurable (which is true for almost every x), $\bar{m}_1(A_x) \geqslant m_1(D_x)$ (D_x is compact, hence m_1-measurable for every x). But $D_x = xC$, for x in C, and is empty otherwise. Thus

$$\mu_2(D) = \bar{m}_2(A) = \int_S \bar{m}_1(A_x) d\bar{m}_1(x) \geqslant \int_S m_1(D_x) d\bar{m}_1(x) = \int_C m_1(xC) d\bar{m}_1(x)$$
$$= (m_1(C))^2 \geqslant (1 - \epsilon)^2.$$

Since ϵ is arbitrary, the contention of the lemma follows.

THEOREM 2. *Conditions* (A B D E) *imply that S is a group.*

Proof. We show that S has an identity. Lemma 3 shows that $\theta(S \times S) \cap \pi\theta (S \times S)$ is not empty, and the existence of an identity then follows as in the proof of Theorem 1. The theorem will be proved if we show that xS has an identity for all $x \in S$. To this end, we select a compact set C for which $m_1(C) \geqslant 1 - \epsilon$. Then

$$\theta(xC \times xC) = (x, x)\theta(C \times xC)$$

is μ_2-measurable. Reasoning similar to that found in the proof of Lemma 3 reveals that

$$\mu_2(\theta(xC \times xC)) \geqslant (1 - \epsilon)^2,$$

and we conclude that $(\mu_2)_*(\theta(xS \times xS)) = 1$; hence $\theta(xS \times xS) \cap \pi\theta(xS \times xS)$ is not empty; xS has an identity, and the proof is complete.

If, in addition to the conditions of Theorem 2, S also satisfies Condition (F) (commutativity), we can strengthen our results. For these purposes we first prove the following purely measure-theoretic lemma.

LEMMA 4. *Let X be a locally compact space, λ a content defined on the compact subsets of X, and μ the measure engendered by λ. Then every open subset of X is measurable.*

https://doi.org/10.4153/CJM-1952-036-4 Published online by Cambridge University Press

Remarks. Note that we do not restrict ourselves to the σ-bounded sets. When only the σ-bounded sets are considered (as is the case in [**2**]), the conclusion of the lemma should be amended to read that every σ-bounded open subset of X is measurable. We note also that if a Borel measure is given, it is possible to find, via an appropriate content, an extension of the given measure, with respect to which every open set is measurable.

Proof. We first prove that if an open set has finite outer measure, then it is measurable. Thus let U be an open set such that $\mu^*(U)$ is finite. Let $C_n \subset U$ be compact sets such that

$$\lambda(C_n) \uparrow \lambda^*(U) = \mu^*(U).$$

Observe that $\lambda(C_n) \leqslant \mu(C_n) \leqslant \mu^*(U)$, whence $\mu(C_n) \uparrow \mu^*(U)$. Thus if

$$A = \bigcup_{n=1}^{\infty} C_n,$$

A is measurable and $\mu(A) = \mu^*(U)$. Applying the Carathéodory criterion to A, using the set U as the testing set, we find

$$\mu^*(U) \geqslant \mu^*(U \cap A) + \mu^*(U \cap A'),$$

where A' is the complement of A. Since $A \subset U$, and since A is measurable, the last inequality becomes $\mu^*(U) \geqslant \mu(A) + \mu^*(U \cap A')$. Thus, by the finiteness of $\mu^*(U) = \mu(A)$, it follows that $\mu^*(U \cap A') = 0$; hence $U \cap A'$ is measurable; and U, being the union of the measurable sets A and $U \cap A'$, is measurable.

Next, referring to [**2**, §53, Theorem D] (which is valid even when the σ-boundedness restrictions are removed), we must show that for an arbitrary open set V (which we may clearly assume to have finite outer measure)

$$\mu^*(V) = \mu(V) \geqslant \mu^*(V \cap U) + \mu^*(V \cap U'), \tag{1}$$

where U is the open set whose measurability we seek to establish. Since $U \cap V$ is an open set of finite outer measure, it is measurable. Since V is measurable, $V \cap U'$, which is the relative complement of $U \cap V$ in V, is also measurable, whence (1) is established and the lemma follows.

THEOREM 3. *Conditions* (A B D E F) *imply that the topology of S may be weakened in such a way that S becomes a separated, compact topological group whose Haar measure coincides with the measure $\bar{m}_1$.*

Proof. By Theorem 2 above and by [**1**, Theorem 9] there is a weakening of the topology of S which makes S a (separated) topological group S'. Local compactness of S' is a consequence of [**1**, Theorem 8] because, in the light of this result, S' is the continuous open image of the locally compact space $S \times S$.

Since S' is a continuous image of S, compact sets of S' are closed in S, and hence $\bar{m}_1$-measurable by Lemma 4. This completes the proof of the theorem since a locally compact topological group wtih a bounded Haar measure is compact.

The above shows that the presence of commutativity (F) implies that the topology of S may be weakened so that S becomes a separated, compact topological group. In what follows we shall show that we may replace the commutativity assumption by a shearing (C) assumption without sacrificing the conclusion.

LEMMA 5. *Let S be a group which satisfies* (A B C E). *Then the Weil topology for S is separated and is weaker than the given topology.*

Proof. (i) By virtue of the continuity of multiplication we see that if C is compact and if U is open, $U \supset C$, one can find a neighbourhood of the identity V for which $CV \subset U$: for each x in C, choose a neighbourhood of the identity W such that $xW^2 \subset U$. Then one can find a finite subset of the x's and the associated W's for which

$$C \subset \bigcup_{i=1}^{n} x_i W_i,$$

and if

$$V = \bigcap_{i=1}^{n} W_i,$$

then $CV \subset U$. Similarly, one can find a neighbourhood of the identity V' such that $V'C \subset U$.

(ii) Let $N(\epsilon; E) = \{x: \rho(xE, E) < \epsilon\}$ [**2**, p. 270]. Then, given two measurable sets A and B such that $A \subset B$, and given an $\epsilon > 2\bar{m}_1(B - A)$, there is a $\delta > 0$, such that $N(\delta; A) \subset N(\epsilon; B)$. From the identity

$$\begin{aligned}(A \triangle xA) &\cup ((B - xB) - A) \cup ((xB - B) - xA) \\ &= (B \triangle xB) \cup ((A - xA) \cap xB) \cup ((xA - A) \cap B),\end{aligned}$$

it follows that $\rho(B, xB) \leqslant \rho(A, xA) + 2\eta$, where $\eta \geqslant \bar{m}_1(B - A) \geqslant 0$. Thus if $\delta + 2\eta < \epsilon$, our conclusion follows. Thus if $\mathfrak{F}$ is a family of sets (e.g., $\mathfrak{F} = \mathfrak{C}_1$) such that for $\epsilon > 0$ and $A \in \mathfrak{R}_1$ there is an $F \in \mathfrak{F}$, $F \subset A$, such that $\bar{m}_1(A - F) < \epsilon$, the family $\{N(\epsilon; F)\}$ is a basis at the identity for the Weil topology.

(iii) By the use of the Fubini theorem, we see that the Weil topology may be constructed on the basis of our condition (C) which demands less than is demanded in [**2**, p. 257] ("shearing is a measurable transformation").

(iv) The Weil topology in our case is separated because the topology of S is separated and because we are dealing with a topological measure (Condition (E)).

(v) We now show that the Weil topology is weaker than the given one. Indeed, if $N(\epsilon; E)$ is given, there exists by virtue of (ii) a compact set $C \subset E$, and a positive δ such that $N(\delta; C) \subset N(\epsilon; E)$. As in (ii) we may choose δ and a positive η such that $\delta + 2\eta < \epsilon$. Choose an open set $U \supset C$ such that $\bar{m}_1(U - C) < \frac{1}{2}\delta$, and then by (i) find V such that $VC \subset U$. Then if $x \in V$,

$$\rho(C, xC) = \bar{m}_1(C - xC) + \bar{m}_1(xC - C) < \delta,$$

since $\bar{m}_1(U - xC) = \bar{m}_1(U - C)$ which is less than $\frac{1}{2}\delta$. Thus

$$V \subset N(\delta;\, C) \subset N(\epsilon;\, E).$$

THEOREM 4. *Conditions* (A B C D E) *imply that the topology of* S *may be weakened in such a way that* S *becomes a separated, compact topological group whose Haar measure coincides with the measure* $\bar{m}_1$.

Proof. Theorem 1 or Theorem 2 implies that S is a group. Lemma 5 shows that the Weil topology for S is separated, and is weaker than the given topology. Owing to (D) (boundedness of the measure) we may employ the technique of [**4**, p. 38] to show that if S is a topological group with a bounded invariant topological measure, and if there exists a compact set of positive measure, then S is compact. Since S is of the character described in the preceding sentence, the theorem is proved.

Remarks. 1. It is clear from the results of Montgomery [**3**] that if a semigroup S satisfies the conditions of Theorem 2 as well as the second axiom of countability, then S is a compact topological group (it is regular, hence metrizable; the compactness follows from the boundedness of the Haar measure).

2. Completion regularity [**2**, p. 230] in $S \times S$ relative to m_2, in addition to Conditions (A B D E), implies the conclusion of Theorem 4.

5. A counter-example. We now present a counter-example which shows that Condition (C) in the hypothesis of Theorem 1 cannot be completely eliminated. The semigroup S which we shall construct consists of the non-negative elements of the following ordered group G. Let A be a linearly ordered set which contains no countable co-final subset, e.g., the ordinals of the first and second classes. Let G be the group of all weak[1] mappings of A into the reals, where the mappings are linearly ordered as follows: let $x = x(a)$ and $y = y(a)$ be two distinct weak mappings, and let $a_0 = \sup\{a : x(a) \neq y(a)\}$ (which exists since the mappings are weak and A is linearly ordered). We say $x > y$ in case $x(a_0) > y(a_0)$. In this way, G becomes an ordered group when addition is defined vectorially. Let $\mathfrak{R}_1$ be the ring generated by the bounded and unbounded, open, half-open, or closed intervals of S; m_1 is defined to be zero for all bounded intervals and finite unions thereof and to be one for a finite union of intervals at least one of which is unbounded. One verifies easily that m_1 is a measure on the ring $\mathfrak{R}_1$, and that m_1 and $\mathfrak{R}_1$ jointly satisfy (A B D).

6. Commutative semigroups. Thus far we have investigated semigroups with bounded invariant measures. We now turn to the consideration of commutative semigroups on which there are invariant, not necessarily bounded, measures. In the following we shall consider the Cartesian product $S \times S$, the equivalence relation R defined in $S \times S$ by: $(a, b)R(c, d)$ if and only if $ad = bc$, and the canonical mapping ϕ of $S \times S$ on the set of R-equivalence classes $Q(S) = S \times S/R = G$ (see [**1**]).

[1]We say that a mapping $x(a)$ is *weak* in case $x(a) \neq 0$ for at most a finite number of a's. The set of these mappings is sometimes called the weak direct product of the reals over the index set A.

THEOREM 5. *Conditions* (A B F G) *imply that there is in* $G = Q(S)$ *a translation invariant measure* μ_1. *The measure induced by* μ_1 *on* S *considered as a subset of* G *coincides with the given measure.*

Proof. It is clear that the set $\phi(xS, x)$ is the same for all x and is a sub-semigroup of G isomorphic to S. Thus we shall always consider S as this sub-semigroup of G. Consider the family $\mathfrak{A}_1$ of subsets of G consisting of all sets of the form gE, where $g \in G$, and $E \in \mathfrak{R}_1$. We show that $\mathfrak{A}_1$ is a ring of sets in G (obviously invariant in G). Indeed let

$$g_iE_i \in \mathfrak{A}_1, \quad g_i = a_ib_i^{-1}, \quad a_i \in S, \quad b_i \in S \qquad (i = 1, 2).$$

Then

$$g_1E_1 \cup g_2E_2 = (b_1b_2)^{-1}(b_2a_1E_1 \cup b_1a_2E_2),$$
$$g_1E_1 - g_2E_2 = (b_1b_2)^{-1}(b_2a_1E_1 - b_1a_2E_2),$$

whence $\mathfrak{A}_1$ is a ring. We now define an invariant measure μ_1 on $\mathfrak{A}_1$: $\mu_1(gE) = m_1(E)$. This number is well defined, for if

$$g_1E_1 = g_2E_2, \quad g_1 = a_1b_1^{-1}, \quad g_2 = a_2b_2^{-1} \qquad (a_i \text{ and } b_i \text{ in } S),$$

then

$$b_2a_1E_1 = b_1a_2E_2$$

whence

$$\mu_1(g_1E_1) = m_1(E_1) = m_1(b_2a_1E_1) = m_1(b_1a_2E_2) = m_1(E_2) = \mu_1(g_2E_2).$$

Now we show that μ_1 is an invariant measure on $\mathfrak{A}_1$—and we need only verify the countable additivity of μ_1. Let g_iE_i be disjoint sets of $\mathfrak{A}_1$ whose union g_0E_0 is also in $\mathfrak{A}_1$, where $g_i = a_ib_i^{-1}$ (a_i and b_i in S). The sets $b_0g_iE_i$ are disjoint sets whose union a_0E_0 is in S; hence each $b_0g_iE_i$ is a subset of S which is also in $\mathfrak{R}_1$: for, $b_ib_0g_iE_i = b_0a_iE_i$ is in $\mathfrak{R}_1$, and hence, by Condition (G), $b_0g_iE_i$ is in $\mathfrak{R}_1$ too. Thus,

$$m_1(a_0E_0) = \sum_{i=1}^{\infty} m_1(b_0g_iE_i) = \sum_{i=1}^{\infty} m_1(E_i) = m_1(E_0),$$

whence $\mu_1(g_0E_0) = \sum \mu_1(g_iE_i)$.

Our next objective is to show that the boundedness of the measure (Condition (D)) allows us to dispense with Condition (G) in the preceding theorem. To this end we prove a preliminary lemma.

LEMMA 6. *Conditions* (A B D F) *imply the truth of the following statements:*

(a) *Let* $\mathfrak{R}'_1 = \mathfrak{R}_1 \cup \{S - A : A \in \mathfrak{R}_1\}$; $m'_1(S - A) = 1 - m_1(A)$, $m'_1(A) = m_1(A)$, $A \in \mathfrak{R}_1$. $\mathfrak{R}'_1$ *is a ring of sets;* m'_1 *a measure on* $\mathfrak{R}'_1$, *and* m'_1 *and* $\mathfrak{R}'_1$ *satisfy Conditions* (A) *and* (B).

(b) *Let* $\mathfrak{R}''_1 = \{E : E \subset S;$ *for some* $x \in S$, $xE \in \mathfrak{R}_1\}$; $m''_1(E) = m_1(xE)$, *where* $E \in \mathfrak{R}''_1$, $xE \in \mathfrak{R}_1$. *Then* $\mathfrak{R}''_1$ *is a ring of sets,* m''_1 *is a measure on* $\mathfrak{R}''_1$, *and* m''_1 *and* $\mathfrak{R}''_1$ *satisfy Conditions* (A) *and* (B).

https://doi.org/10.4153/CJM-1952-036-4 Published online by Cambridge University Press

(c) *The class* $\mathfrak{A}_1$ *introduced in the proof of Theorem* 5 *is a translation invariant ring of subsets of* G *and the function* μ_1 *of Theorem* 5 *is a translation invariant measure on* $\mathfrak{A}_1$.

Proof. 1. (a) *implies* (b). The fact that $\mathfrak{R}''_1$ is a ring is easily verified. We now show that m''_1 is uniquely defined on $\mathfrak{R}''_1$. Indeed, if $E \in \mathfrak{R}''_1$, and if both xE and yE belong to $\mathfrak{R}''_1$, then xyE and yxE are in $\mathfrak{R}_1$ and

$$m_1(xE) = m_1(yxE) = m_1(xyE) = m_1(yE).$$

Next assume that $E_i \in \mathfrak{R}''_1$, E_i disjoint $(i = 1, 2, \ldots)$,

$$\bigcup_{i=1}^{\infty} E_i = E_0 \in \mathfrak{R}''_1, \quad x_i E_i \in \mathfrak{R}_1 \qquad (i = 0, 1, 2, \ldots).$$

Note that for each y in x_iS, $yE_i \in \mathfrak{R}_1$. Now (a) implies that each x_iS is m'_1-measurable and $m'_1(x_iS) = 1$. Thus, passing to $\bar{\mathfrak{S}}_1$ and $\bar{m}'_1$,

$$\bigcap_{i=0}^{\infty} x_iS \in \bar{\mathfrak{S}}_1, \quad \bar{m}'_1\left(\bigcap_{i=0}^{\infty} x_iS\right) = 1,$$

whence $\bigcap x_iS$ is not empty. Thus let $y \in \bigcap x_iS$. Then the sets yE_i are disjoint and in $\mathfrak{R}_1$ and

$$\bigcup_{i=1}^{\infty} yE_i = yE_0,$$

whence

$$m''_1(E_0) = m_1(yE_0) = \sum_{i=1}^{\infty} m_1(yE_i) = \sum_{i=1}^{\infty} m''_1(E_i).$$

2. (b) *implies* (c). Again, the fact that $\mathfrak{A}_1$ is a translation invariant ring is easily verified and we have already shown, in the proof of Theorem 5, that μ_1 is uniquely defined. Note that if E is such that there is some y for which $yE \in \mathfrak{R}''_1$, then $E \in \mathfrak{R}''_1$. Let μ''_1 be the function corresponding to m''_1, and let $\mathfrak{A}''_1$ be the associated ring in G. By what has just been noted, $\mathfrak{R}''_1$ fulfils Condition (G) and thus μ''_1 is a measure on $\mathfrak{A}''_1$. Since $\mathfrak{A}''_1 \supset \mathfrak{A}_1$, the assertion follows.

3. (c) *implies* (a). Let $\mathfrak{A}'_1 = \mathfrak{A}_1 \cup \{G - A : A \in \mathfrak{A}_1\}$. In a straightforward manner one can show that $\mathfrak{A}'_1$ is a ring. Condition (D) implies that $\sup\{\mu_1(gE) : g \in G, E \in \mathfrak{R}_1\} = 1$, that is, μ_1 is a bounded measure on $\mathfrak{A}_1$. Let

$$\mu'_1(G - A) = 1 - \mu_1(A), \quad \mu'_1(A) = \mu_1(A), \quad A \in \mathfrak{A}_1.$$

We verify that μ'_1 is a measure on $\mathfrak{A}'_1$. Let $E_i \in \mathfrak{A}'_1$, E_i disjoint and

$$\bigcup_{i=1}^{\infty} E_i = E \in \mathfrak{A}'_1.$$

We assert that if $G \notin \mathfrak{A}_1$ then at most one of the sets E_i is of the form $G - A$, $A \in \mathfrak{A}_1$. For if $(G - A) \cap (G - B)$ is empty, where A and B are in $\mathfrak{A}_1$, then $G - (A \cup B)$ is empty, that is, $A \cup B = G$, $G \in \mathfrak{A}_1$, a contradiction. Thus either G is in $\mathfrak{A}_1$ or at most one of the E_i is of the form $G - A$, $A \in \mathfrak{A}_1$. In the former case $(G, \mathfrak{A}'_1, \mu'_1) = (G, \mathfrak{A}_1, \mu_1)$. In the latter, all that need be considered is the case where E_1, say, is of the form $G - A$, $A \in \mathfrak{A}_1$, and then

$$\sum_{i=1}^{\infty} \mu'_1(E_i) = (1 - \mu_1(A)) + \sum_{i=2}^{\infty} \mu_1(E_i).$$

Furthermore,

$$\bigcup_{i=1}^{\infty} E_i = (G - A) \cup (\bigcup_{i=2}^{\infty} E_i) = (G - A) \cup B = E,$$

and since $G - A$ and B are disjoint,

$$B = E - (G - A) = E \cap A \in \mathfrak{A}'_1.$$

If $B \notin \mathfrak{A}_1$ then $B = G - C$, but then as already shown, B and $G - A$ are not disjoint, whence $B \in \mathfrak{A}_1$. Thus

$$\mu_1(B) = \sum_{i=2}^{\infty} \mu_1(E_i).$$

But $(G - A) \cup B = G - (A - B)$, and we observe $B \subset A$, whence

$$\mu'_1(G - (A - B)) = 1 - \mu_1(A - B) = 1 - (\mu_1(A) - \mu_1(B))$$

$$= 1 - \mu_1(A) + \sum_{i=2}^{\infty} \mu_1(E_i).$$

Thus μ'_1 is a measure on $\mathfrak{A}'_1$.

Next, if $G - A \in \mathfrak{A}'_1$ and if $x \in G$, then

$$x(G - A) = G - xA \in \mathfrak{A}'_1,$$

$$\mu'_1(x(G - A)) = \mu'_1(G - xA) = 1 - \mu_1(xA) = 1 - \mu_1(A)$$

and so $\mathfrak{A}'_1$ and μ'_1 satisfy Conditions (A) and (B). Note that $(G, \mathfrak{A}'_1, \mu'_1)$ is an extension of $(G, \mathfrak{A}_1, \mu_1)$ which in turn is an extension of $(G, \mathfrak{R}_1, m_1)$. We assert that, in terms of the completion $\bar{\mu}'_1$ of μ'_1, the set S is measurable. In fact, let

$$A_n \in \mathfrak{R}_1, \quad m_1(A_n) = \mu'_1(A_n) \to 1.$$

If

$$A = \bigcup_{n=1}^{\infty} A_n,$$

then A is $\bar{\mu}'_1$-measurable and $\bar{\mu}'_1(A) = 1$, whence $\bar{\mu}'_1(G - A) = 0$. But $G - S \subset G - A$ and thus $\bar{\mu}'_1(G - S) = 0$; whence S is $\bar{\mu}'_1$-measurable and, for $g \in G$, gS is $\bar{\mu}'_1$-measurable and $\bar{\mu}'_1(gS) = \bar{\mu}'_1(S)$. Clearly then, the contraction of $\bar{\mu}'_1$ to the ring $\mathfrak{R}'_1$ is the measure m'_1 of assertion (a) of the lemma and $(S, \mathfrak{R}'_1, m'_1)$ satisfies Conditions (A) and (B).

4. (b) *is true*. As indicated above (part 1), we need only show that if $E_i \in \mathfrak{R}''_1$, E_i disjoint $(i = 1, 2, \ldots)$,

$$\bigcup_{i=1}^{\infty} E_i = E_0 \in \mathfrak{R}''_1,$$

then

https://doi.org/10.4153/CJM-1952-036-4 Published online by Cambridge University Press

$$m''_1(E_0) = \sum_{i=1}^{\infty} m''_1(E_i).$$

To this end we choose a sequence of sets $A_i \in \mathfrak{R}_1$ such that

$$m_1(A_i) \geqslant 1 - 10^{-i-1} \qquad (i = 0, 1, 2, \ldots).$$

Assume $x_iE_i \in \mathfrak{R}_1 (i = 0, 1, 2, \ldots)$. Then

$$x_iA_i \in \mathfrak{R}_1, \quad m_1(x_iA_i) \geqslant 1 - 10^{-i-1}.$$

We consider $m_1(\bigcap x_iA_i)$. A simple induction shows

$$m_1(\bigcap_{i=0}^{n} x_iA_i) \geqslant 0.88 \ldots 89$$

where the number of 8's preceding 9 is n. Thus, passing to $\bar{\mathfrak{S}}_1$ and $\bar{m}_1$,

$$\bar{m}_1(\bigcap_{i=0}^{\infty} x_iA_i) \geqslant 0.8,$$

that is, $\bigcap_{i=0}^{\infty} x_iA_i$ is not empty. Let

$$y \in \bigcap_{i=0}^{\infty} x_iA_i.$$

Then $yE_i \in \mathfrak{R}_1$ $(i = 0, 1, 2, \ldots)$, and the remainder of the proof proceeds as in part 1.

THEOREM 6. *Conditions* (A B D F) *imply that there is in* $G = Q(S)$ *a translation invariant bounded measure* μ_1. *The measure induced on* S *considered as a subset of* G *coincides with the given measure. The class* $\mathfrak{A}_1$ *introduced in the proof of Theorem* 5 *is in fact the minimal translation invariant extension in* G *of* $\mathfrak{R}_1$.

Proof. This is an immediate consequence of Lemma 6.

REFERENCES

1. B. R. Gelbaum, G. K. Kalisch, and J.M.H. Olmsted, *On the embedding of topological semigroups and integral domains*, Proc. Amer. Math. Soc., vol. 2 (1951), 807-821.
2. P. R. Halmos, *Measure theory* (New York, 1950).
3. D. Montgomery, *Continuity in topological groups*, Bull. Amer. Math. Soc., vol. 42 (1936), 879-882.
4. A. Weil, *L'intégration dans les groupes topologiques et ses applications* (Paris, 1938).

University of Minnesota