Canad. Math. Bull. Vol. **28** (3), 1985

GERMS ASSOCIATED TO REGULAR UNIPOTENT CLASSES IN p-adic SL(n)

BY

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Dedicated to the memory of Robert Arnold Smith

ABSTRACT. For an elliptic torus in SL(n), explicit formulae are given for the germs which are associated to the regular unipotent conjugacy classes. Using them, a formula is found for the germ associated to the "subregular" class, the class whose Jordan canonical form contains a 1×1 and an $(n-1) \times (n-1)$ block.

0. **Introduction**. The extra ingredient in calculations on SL(n) as opposed to GL(n) is the distinction between conjugacy and stable conjugacy. In this light it is reasonable to expect that germs on SL(n) will be related to germs on GL(n) but will distinguish between stably conjugate tori which are not conjugate.

Indeed if T is an elliptic torus and $t \in T'$, it is relatively easy to see that some of the regular germs will vanish at t and others will be multiples of the "stable" germ, i.e. the regular germ for GL(n). The difficulty is that as t varies in T' it may be that different germs are the non-zero ones.

A formula is given in Theorem 6.3 which completely describes this behaviour. In Section 8 it is shown that when n is odd the situation is as simple as possible — some regular germs are multiples of the stable germ and the others vanish identically on T. Examples are presented to show that when n is even things are more complicated.

I wish to thank Jim Arthur, Helaman Ferguson, Paul Gérardin, Robert Langlands, Jon Rogawski and Paul Sally for their help and encouragement.

1. **Notation**. *Let F be a p-adic field, with ring of integers $\mathbf{0} = \mathbf{0}_F$, let $\boldsymbol{p}$ be the maximal ideal of $\mathbf{0}$ and $q = q_F =$* card $(\mathbf{0}/\boldsymbol{p})$. Let $G = \mathrm{SL}(n, F)$, $K = \mathrm{SL}(n, \mathbf{0})$ *and* $K_m = \{k \in K : k \equiv \text{id, mod } \boldsymbol{p}^m\}$. *Let* $\bar{G} = \mathrm{GL}(n,F)$, $\bar{K} = \mathrm{GL}(n, \mathbf{0})$, $\bar{K}_m = \{k \in \bar{K} : k \equiv \text{id, mod } \boldsymbol{p}^m\}$. *Let*

$$(1.1) \qquad u(a) = \begin{bmatrix} 1 & a & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 0 & & \\ 0 & 0 & 1 & 1 & & \\ & & & & \ddots & \\ 0 & & & & 1 & 1 \\ 0 & & & & & 1 \end{bmatrix}$$

Received by the editors January 5, 1984.

AMS (MOS) Subject Classification 22E35.

Research supported by the National Sciences and Engineering Research Council.

For $a \in F^x$, let $d(a) = \mathrm{diag}(a^{-1}, 1, 1, \ldots, 1) \in \bar{G}$. Then $u(a) = d(a)^{-1}u(1)d(a)$.

Let T be an elliptic torus in G, and let $\bar{T} = Z_{\bar{G}}(T)$ be its centralizer in $\bar{G}$. Then $\bar{T} \cong E^x$ for some extension field E/F with $[E:F] = n$.

2. **Unipotent conjugacy classes**. Any regular unipotent conjugacy class in G contains an element of the form $u(a)$ for some $a \in F^x$ (cf. 1.1)). Clearly $u(a)$ and $u(b)$ are conjugate by a diagonal element of G if and only if $a/b \in (F^x)^n$. More is true:

PROPOSITION 2.1. *The regular unipotent conjugacy classes of G are represented by $\{u(a)\}$, where a ranges over a set of representatives for $F^x/(F^x)^n$.*

PROOF. It is necessary to show that if $u(a)$ and $u(b)$ are G-conjugate then $a/b \in (F^x)^n$. Now $u(a)$ is conjugate to $u(b)$ by $d(b/a)$. If they were also conjugate by $g \in G$, then $d(b/a)g^{-1}$ would be in the centralizer in $\mathrm{GL}(n)$ of $u(a)$. The determinant of any such matrix is an nth power. But det $(d(b/a)g^{-1}) = a/b$. □

Similar considerations permit the classification of the other unipotent conjugacy classes. Let $n = n_1 + \ldots + n_r$ and consider unipotent classes whose Jordan canonical form contains blocks of sizes $n_1, \ldots, n_r$. Start with "the" Jordan canonical form, that is with diagonal entries all equal to 1, superdiagonal entries equal to 1 or 0, and zeroes elsewhere. Construct a matrix $v(a)$ by replacing the topmost 1 on the superdiagonal by a.

PROPOSITION 2.2. *The unipotent conjugacy classes with block sizes $n_1, \ldots, n_r$ are represented by the matrices $\{v(a)\}$ described above, as a varies over a set of representatives of $F^x/(F^x)^m$, where $m = g.c.d.\ (n_1, \ldots, n_r)$.*

PROOF. The result is an easy consequence of the fact that $(F^x)^{n_1} \cdot (F^x)^{n_2} \cdot \ldots \cdot (F^x)^{n_r} = (F^x)^m$. □

Next we set up subsets of G which discriminate between the regular unipotent classes.

Fix a positive integer k so that $1 + \boldsymbol{p}^k \subset (F^x)^n$. Let $S(1) = u(1) \cdot K_k = \{g \in K : g \equiv u(1), \bmod \boldsymbol{p}^k\}$. For each $a \in F^x$, let $S(a) = d(a^{-1})S(1)d(a)$, so $S(a)$ contains $u(a)$. Similarly let $\bar{S}(1) = u(1)\bar{K}_k$, $\bar{S}(a) = d(a)^{-1}\bar{S}(1)d(a)$.

Suppose $g \in S(1)$. By conjugating by triangular unipotent matrices it is possible to clear the nth column of g above the superdiagonal, then the $(n - 1)$th, etc., until all entries above the superdiagonal are zero and then to clear the rows to the left of the superdiagonal so that in the first $n - 1$ rows the diagonal entries are all 1, the superdiagonal entries are all in $1 + \boldsymbol{p}^k$ and the other entries are zero (cf. Section 2 of [2]). Because of the choice of k it is then possible to conjugate by a diagonal matrix in G to make the superdiagonal entries all equal to 1.

In other words, every element of $S(1)$ is G-conjugate to a (unique) matrix of the form

$$
(2.3) \qquad \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & & \\ & & & & 1 \\ \alpha_0 & \alpha_1 & \alpha_2 & \cdots & 1+\alpha_{n-1} \end{bmatrix}
$$

Note that each α_i will be in p^k.

PROPOSITION 2.4. *The G-conjugacy class of $u(a)$ meets $S(b)$ if and only if $a/b \in (F^x)^n$.*

PROOF. The "if" part is obvious. Conjugating by $d(b^{-1})$ we can assume $b = 1$. Suppose an element of $S(1)$ is G-conjugate to $u(a)$. Then $u(a)$ is G-conjugate to an element g of the form (2.3). Since g is unipotent, we must have $\alpha_0 = \alpha_1 = \cdots = \alpha_{n-1} = 0$. The proposition follows from Proposition 2.1. □

PROPOSITION 2.5. *Let T be an elliptic torus, with $\bar{T} \cong E^x$. Suppose $t \in \bar{T}'$ is G-conjugate to an element of $S(a)$. Then t is G-conjugate to an element of $S(b)$ if and only if $a/b \in N_{E/F}(E^x)$.*

PROOF. Assume $a = 1$, so t is G-conjugate to a uniquely determined matrix of the form (2.3), say $t^h = g$, with $h \in G$.

Suppose $t^{h'} \in S(b)$, for some $h' \in G$. Then $(t^{h'})^{d(b^{-1})} \in S(1)$, so $(t^{h'})^{d(b^{-1})}$ is G-conjugate to g, say $t^{h'd(b^{-1})h''} = g$. So $h'd(b^{-1})h''h^{-1} \in Z_{\bar{G}}(t) = \bar{T}$, so $b \in \det(\bar{T}) = N_{E/F}(E^x)$.

Conversely, suppose $b = \det t_0$, for some $t_0 \in \bar{T}$. If $t^g \in S(1)$, with $g \in G$, then $S(b)$ contains $t^{g\cdot d(b)} = t^{t_0\cdot g\cdot d(b)}$, and $\det(t_0\cdot g\cdot d(b)) = b\cdot 1\cdot b^{-1} = 1$. □

3. **Elliptic Conjugacy Classes**. Fix an elliptic torus T in G, with $\bar{T} = Z_{\bar{G}}(T) \cong E^x$. Fix a regular element $t \in T'$, so $B = \{1, t, t^2, \ldots, t^{n-1}\}$ is a basis for E/F. Assume $\mathrm{id} + t$ is of the form (2.3).

Suppose $\xi \in \bar{T}$ is regular and $\mathrm{id} + \xi \in G$. Construct $A \in \mathrm{GL}(n, F)$ by letting the ith row of A consist of the coefficients of ξ^{i-1} in terms of the basis B. Note that

$$
(3.1) \qquad A \begin{bmatrix} 1 \\ t \\ t^2 \\ \cdot \\ \cdot \\ t^{n-1} \end{bmatrix} = \begin{bmatrix} 1 \\ \xi \\ \xi^2 \\ \cdot \\ \cdot \\ \xi^{n-1} \end{bmatrix}
$$

Using the basis B, identify E with F^n, written as horizontal n-tuples. As an element of E, t acts on E by multiplication, and the corresponding action on F^n is right multiplication by the matrix t (which is the matrix (2.3) minus id), i.e. $v \to v\cdot t$, for $v \in F^n$. The same is then true for the action on F^n by each power of t and hence for every element of $\bar{T} \cong E^x$.

Since A is a "change-of-basis" matrix, we see that $\xi = A^{-1}gA$, where $\mathrm{id} + g$ is a matrix of the form (2.3) (but with the entries in its last row different from those in the last row of t).

Suppose $\sigma_1, \sigma_2, \ldots, \sigma_n$ are the distinct embeddings of E over F. Write $t_i = t^{\sigma_i}$, $\xi_i = \xi^{\sigma_i}$. From (3.1) we see that

$$A \begin{bmatrix} 1 & 1 & \cdots & 1 \\ t_1 & t_2 & \cdots & t_n \\ t_1^2 & t_2^2 & \cdots & t_n^2 \\ \vdots & \vdots & & \vdots \\ t_1^{n-1} & t_2^{n-1} & \cdots & t_n^{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \xi_1 & \xi_2 & \cdots & \xi_n \\ \xi_1^2 & \xi_2^2 & & \xi_n \\ \vdots & \vdots & & \vdots \\ \xi_1^{n-1} & \xi_2^{n-1} & \cdots & \xi_n^{n-1} \end{bmatrix}$$

So $\det A = \delta(\xi)/\delta(t)$, where, for any $\xi \in E$, $\delta(\xi)$ refers to the "semi-discriminant", the determinant of the Van der Monde matrix on the right side of the above equation.

Note that $\det A$ is independent of the order of the embeddings σ_i.

Since $\delta(\xi + 1) = \delta(\xi)$, we can replace t, ξ with $1 + t$, $1 + \xi$ for the next result.

PROPOSITION 3.2. *Suppose $t \in T'$ is G-conjugate to an element of $S(a)$. Then any $\xi \in T'$ sufficiently close to* id *is G-conjugate to an element of $S(b)$ if and only if $\delta(t)/\delta(\xi)\cdot a/b \in N_{E/F}(E^x)$.*

PROOF. Assume $a = 1$ and $t \in S(1)$. We have just seen that $A\xi A^{-1} \in S(1)$, so ξ is conjugate by $A^{-1}d(b)$ to an element of $S(b)$. It will also be conjugate to an element of $S(b)$ by an element of G if and only if there is an element of $\bar{T}$, the centralizer of ξ, whose determinant equals $\det(A^{-1}d(b)) = \delta(t)/\delta(\xi)\cdot b^{-1}$. □

4. **Normalizations of Measures**. We shall compute orbital integrals in $G = \mathrm{SL}(n)$ by comparing them with known orbital integrals in $\bar{G} = \mathrm{GL}(n)$. As above, T is an elliptic torus in G, $\bar{T} = Z_{\bar{G}}(T)$.

Let $\mathcal{R}$ and $\mathcal{S}$ be sets of representatives for $F^x/(F^x)^n$ and $F^x/N_{E/F}(E^x)$, respectively. We can assume that $q^{-n} < |r| \leqslant 1$ for each $r \in \mathcal{R}$, and that for each $s \in \mathcal{S}$, $1 \geqslant |s| > \max\{|x| : x \in N_{E/F}(E^x), |x| < 1\}$.

Then $\bar{G} = \bigcup_{s\in\mathcal{S}} \bar{T}\cdot G\cdot d(s^{-1}) = \bigcup_{r\in\mathcal{R}} \bar{Z}\cdot G\cdot d(r^{-1})$, where $\bar{Z} \cong F^x$ is the centre of $\bar{G}$. If $\mathrm{d}t$, $\mathrm{d}g$ and $\mathrm{d}z$ are Haar measures on $\bar{T}$, G and $\bar{Z}$, respectively, it is easily checked that one gets Haar measures on $\bar{G}$ by the following formulae (for $f \in C_c^\infty(\bar{G})$, say)

$$\sum_{s\in\mathcal{S}} \int_{\bar{T}} \int_G f(t\cdot g\cdot d(s^{-1}))\mathrm{d}g\ \mathrm{d}t \tag{4.1}$$

$$\sum_{r\in\mathcal{R}} \int_{\bar{Z}} \int_G f(z\cdot g\cdot d(r^{-1}))\mathrm{d}g\ \mathrm{d}z \tag{4.2}$$

We normalize $\mathrm{d}t$ and $\mathrm{d}z$ by identifying $\bar{T} \cong E^x$ and $\bar{Z} \cong F^x$ and insisting that the measure in $\bar{T}$ of $\mathbf{0}_E^x$ equal $1 - 1/q_E$ and that the measure in $\bar{Z}$ of $\mathbf{0}^x$ be $1 - 1/q$.

In order to compare the measures in (4.1) and (4.2), we compute the measure of $\bar{K}$ for each. Because of our assumptions on $\mathcal{R}$ and $\mathcal{S}$, $z\cdot g\cdot d(r^{-1}) \in \bar{K}$ if and only if

$|\det z| = 1 = |r|$ and $g \in K$. If $\bar{T} \cap \bar{K} = \mathbf{0}_E^x$ (as we can assume after conjugating by an element of $\bar{G}$), then $t \cdot g \cdot d(s^{-1}) \in \bar{K}$ if and only if $|\det t| = 1 = |s|$ and $g \in K$. So the measure of $\bar{K}$ using (4.1) is meas $(\mathbf{0}_E^x) \cdot$ meas $(K) \cdot [\mathbf{0}^x : N_{E/F}(\mathbf{0}_E^x)]$, and the measure of $\bar{K}$ using (4.2) is meas $(\mathbf{0}^x) \cdot$ meas $(K) \cdot [\mathbf{0}^x : (\mathbf{0}^x)^n]$. Henceforth we fix Haar measure on $\bar{G}$ to be the measure in (4.2), which also equals the measure in (4.1) divided by the constant

$$(4.3) \qquad c = (1 - 1/q_E)/(1 - 1/q) \cdot [N_{E/F}(\mathbf{0}_E^x) : (\mathbf{0}^x)^n]^{-1}.$$

5. **Orbital Integrals**. Choice of measures on $\bar{G}$ and $\bar{T}$ defines a measure on $\bar{T} \backslash \bar{G}$; for $f \in C_c^\infty(\bar{G})$ and $t \in \bar{T}'$ we have

$$(5.1) \qquad \int_{\bar{T} \backslash \bar{G}} f(g^{-1}tg)\, d\dot{g} = c^{-1} \sum_{s \in \mathcal{S}} \int_G f(d(s)g^{-1}tgd(s^{-1}))\, dg.$$

In particular if $f = \mathrm{char}_{\bar{S}(1)}$, $f(d(s) \bullet d(s^{-1})) = \mathrm{char}_{\bar{S}(s)}(\bullet)$, so

$$(5.2) \qquad \int_{\bar{T} \backslash \bar{G}} f(g^{-1}tg)\, d\dot{g} = c^{-1} \sum_{s \in \mathcal{S}} \int_G \mathrm{char}_{S(s)}(g^{-1}tg)\, dg.$$

Note that by Proposition 2.5 at most one term on the right side of (5.2) is non-zero. Moreover, for t sufficiently close to id, the left side is known by the result of [1].

In order to calculate integrals over unipotent orbits we need to specify the measures on the centralizers. Now $Z(u(1)) = Z_G(u(1))$ is a finite group (nth roots of unity) times the group of all upper triangular unipotent matrices which are constant along diagonal lines (i.e. $b_{ij} = b_{i+1,j+1}$, for all i, j). We let the finite group have total mass 1 and on the unipotent group take the product of the standard F^+-measure on each coordinate (i.e. $db_{12}db_{13} \cdot \ldots \cdot db_{1n}$). The centralizer of $u(1)$ in $\bar{G}$ is then $\bar{Z} \cdot Z(u(1))$, and the product of the measures on $\bar{Z}$ and $Z(u(1))$ agrees with the natural measure on $Z_{\bar{G}}(u(1))$ given in [1].

Since $u(a) = d(a^{-1})u(1)d(a)$, we see that $Z(u(a)) = Z_G(u(a)) = d(a^{-1}) \cdot Z_G(u(1)) \cdot d(a)$, and we use this relation and the above measure on $Z(u(1))$ to define the measure on $Z(u(a))$.

For $f \in C_c^\infty(\bar{G})$,

$$(5.3) \qquad \int_{Z_{\bar{G}}(u(a)) \backslash \bar{G}} f(u(a)^g)\, d\dot{g} = \sum_{r \in \mathfrak{R}} \int_{Z(u(a)) \backslash G} f(u(a)^{gd(r^{-1})})\, d\dot{g}$$

and if $f = \mathrm{char}_{\bar{S}(a)}$, we have

$$\int_{Z_{\bar{G}}(u(a)) \backslash \bar{G}} f(u(a)^g)\, d\dot{g} = \sum_{r \in \mathfrak{R}} \int_{Z(u(a)) \backslash G} \mathrm{char}_{S(ar)}(u(a)^g)\, d\dot{g}$$

Because of Proposition 2.4, the terms on the right vanish with at most one exception, so

$$(5.4) \qquad \int_{Z_{\bar{G}}(u(a)) \backslash \bar{G}} \mathrm{char}_{\bar{S}(a)}(u(a)^g)\, d\dot{g} = \int_{Z(u(a)) \backslash G} \mathrm{char}_{S(a)}(u(a)^g)\, d\dot{g}$$

https://doi.org/10.4153/CMB-1985-031-x Published online by Cambridge University Press

Moreover this integral is independent of $a \in F^\times$ (since the left side is clearly independent of a).

Now suppose $t \in T'$ is G-conjugate to an element of $S(r)$. Equation (5.2) tells us that

$$(5.5) \qquad c^{-1}\int_G \operatorname{char}_{S(r)}(t^g)\,dg = \int_{\bar{T}\backslash\bar{G}} \operatorname{char}_{\bar{S}(1)}(t^g)\,d\dot{g}.$$

Equation (5.4) and the remark immediately following it tell us

$$(5.6) \qquad \int_{Z(u(r))\backslash G} \operatorname{char}_{S(r)}(u(r)^g)\,d\dot{g} = \int_{Z_{\bar{G}}(u(1))\backslash\bar{G}} \operatorname{char}_{\bar{S}(1)}(u(1)^g)\,d\dot{g}$$

6. **The Regular Germs**. We recall the notation from Sections 1 and 2 and state the main result in Theorem 6.3. In $G = \mathrm{SL}(n, F)$, T is an elliptic torus. Its centralizer in $\mathrm{GL}(n,F)$ is isomorphic to $E^\times$, where $[E:F] = n$. For $t \in T$ we defined the "semi-discriminant" $\delta(t) = \prod_{i>j}(t^{\sigma_i} - t^{\sigma_j})$. We let $u(1)$ be the matrix whose diagonal and superdiagonal entries equal 1 and whose other entries are 0, $d(a) = \operatorname{diag}(a^{-1}, 1, \ldots, 1)$, $u(a) = d(a)^{-1}u(1)d(a)$, $S(1) = u(1)K_k$, $S(a) = d(a)^{-1} \times S(1)d(a)$. Also $\mathcal{R}$ and $\mathcal{S}$ are sets of representatives of $F^\times/(F^\times)^n$ and $F^\times/N_{E/F}(E^\times)$, respectively

If $f \in C_c^\infty(G)$, Shalika's theorem ([4], Theorem 2.1.1) says that for $t \in T'$ sufficiently close to the identity,

$$(6.1) \qquad \int_G f(t^g)\,dg = \sum_i \Gamma_i(t)\int_{Z(u_i)\backslash G} f(u_i^g)\,d\dot{g}$$

where $\{u_i\}$ is a set of representatives for the unipotent conjugacy classes in G and Γ_i is the "germ" associated to u_i. Relabelling the regular germs for convenience, for each $r \in F^\times$ we denote by Γ_r the germ associated to the regular unipotent conjugacy class containing $u(r)$. The measures are normalized as in Section 4.

PROPOSITION 6.2. *Any $t \in T'$ sufficiently close to the identity is G-conjugate to an element of $S(s)$ for a unique $s \in \mathcal{S}$.*

PROOF. Any $t \in T'$ is conjugate by some $h \in \mathrm{GL}(n, F)$ to a matrix (2.3), which will be in $S(1)$ provided t is sufficiently near the identity. Suppose $\det h = \det(\tau)\cdot s$, with $\tau \in \bar{T}$, $s \in \mathcal{S}$. Then $\tau^{-1}hd(s) \in G$, and t is conjugate by $\tau^{-1}hd(s)$ to an element of $S(1)^{d(s)} = S(s)$. The uniqueness of s is Proposition 2.5. □

THEOREM 6.3. (i) *For $t \in T'$ sufficiently close to the identity, there is a unique coset of $F^\times/N_{E/F}(E^\times)$ so that $\Gamma_a(t) \neq 0$ if and only if a is in this coset.*

(ii) *Fix $t_0 \in T'$ sufficiently close to the identity, and choose s_0 so that $\Gamma_{s_0}(t_0) \neq 0$. The regular germs Γ_r are given by the following formulae:*

$$\Gamma_r(t) = \begin{cases} c\cdot|D_{E/F}|^{1/2}\cdot|D(t)|^{-1/2}, & \text{if } r\cdot\delta(t)/(s_0\delta(t_0)) \in N_{E/F}(E^\times) \\ 0, & \text{otherwise} \end{cases}$$

($D_{E/F}$ is the discriminant of E/F and $c = (1 - 1/q_E)/(1 - 1/q)\cdot[N_{E/F}(\mathbf{0}_E^\times):(\mathbf{0}^\times)^n]^{-1}$).

PROOF. We may assume $r \in \mathfrak{R}$. To calculate $\Gamma_r(t)$, let $f = \mathrm{char}_{S(r)} \in C_c^\infty(G)$. By Proposition 2.4 and Proposition 1 of [1], the only unipotent class meeting $S(r)$ is that of $u(r)$, so the right side of (6.1) has only one non-zero term, and $\Gamma_r(t) = \int_G f(t^g)\,\mathrm{d}g / \int_{Z(u(r))\backslash G} f(u(r)^g)\,\mathrm{d}\dot{g}$. By Proposition 6.2, the numerator vanishes unless r lies in a certain coset of $F^\times/N_{E/F}(E^\times)$, in which case it doesn't. This proves (i).

With t_0 and s_0 as stated, and $t \in T'$ close to the identity, let $r \in \mathfrak{R}$ and again consider the above expression for $\Gamma_r(t)$. By Proposition 3.2, the numerator vanishes unless $r \cdot \delta(t)/(s_0 \cdot \delta(t_0)) \in N_{E/F}(E^\times)$. When it is not zero, $\Gamma_r(t)$ is c times the quotient of the left sides of (5.5) and (5.6). The quotient of the right sides of those equations is the regular germ for $\mathrm{GL}(n, F)$, as normalized in [1], which is $|D_{E/F}|^{1/2} \cdot |D(t)|^{-1/2}$. □

COROLLARY 6.4. *If r and r' are in the same coset of $F^\times/N_{E/F}(E^\times)$, then $\Gamma_r \equiv \Gamma_{r'}$ on T.*

COROLLARY 6.5. *The sum of all the regular germs is a constant multiple of the stable germ (i.e. the regular germ for* $\mathrm{GL}(n, F)$*). Indeed*

$$\sum_{r \in \mathfrak{R}} \Gamma_r(t) = c \cdot [N_{E/F}(E^\times) : (F^\times)^n] \cdot |D_{E/F}|^{1/2} \cdot D(t)|^{-1/2}.$$

7. **The Subregular Germ**. Let $u = u(0)$ be the matrix whose diagonal and super-diagonal entries are all equal to 1 except for the topmost entry of the superdiagonal, the $(1, 2)$-entry, which is 0, as are all remaining entries. Write $\Gamma(t)$ for the germ associated to the conjugacy class of u, the "subregular" class.

By Proposition 2.2, the subregular class is "stable" (its conjugacy class in $\mathrm{GL}(n)$ contains only one $\mathrm{SL}(n)$ conjugacy class), so it is not surprising that Γ will be a constant multiple of the corresponding germ for $\mathrm{GL}(n)$. We sketch below an argument which proves this and also evaluates the constant.

Let $S = u \cdot K_k$, $f = \sum_{r \in \mathfrak{R}} \mathrm{char}_{d(r^{-1})Sd(r)}$, which is in $C_c^\infty(G)$, and let $\bar{f}$ be the corresponding function on $\bar{G}$ (replace S with $\bar{S} = u \cdot \bar{K}_k$). Then the orbital integrals of f are "stable", i.e.

$$\int_G f(t^g)\,\mathrm{d}g = \int_G f((t^h)^g)\,\mathrm{d}g$$

for any $t \in T'$, $h \in \mathrm{GL}(n, F)$, and also

$$\int_{Z(u(a))\backslash G} f(u(a)^{gh})\,\mathrm{d}\dot{g}$$

is independent of $a \in F^\times$ and $h \in \mathrm{GL}(n)$.

By Proposition 1 of [2], the only non-zero unipotent orbital integrals of f are the regular and subregular ones, so equation (6.1) becomes

$$(7.1)\quad \int_G f(t^g)\,\mathrm{d}g = \sum_{r \in \mathfrak{R}} \Gamma_r(t) \int_{Z(u(r))\backslash G} f(u(r)^g)\,\mathrm{d}\dot{g} + \Gamma(t) \int_{Z(u)\backslash G} f(u^g)\,\mathrm{d}\dot{g}$$

By the "stability" of f and (5.1), the left side of (7.1) equals $c \cdot [F^\times : N_{E/F}(E^\times)]^{-1} \cdot$

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$\int_{\bar{T}/\bar{G}} \bar{f}(t^g)\,d\dot{g}$. Moreover, the summation on the right side of (7.1) equals $\Sigma_{r\in\Re}\Gamma_r(t)\cdot\int_{Z(u(1))\backslash G} f(u(1)^g)\,d\dot{g}$ which, by Corollary 6.5, is $c\cdot[N_{E/F}(E^x):(F^x)^n]\cdot|D_{E/F}]^{1/2}\cdot|D(t)|^{-1/2}\cdot\int_{Z(u(1))\backslash G} f(u(1)^g)\,d\dot{g}$. This last integral equals $[F^x:(F^x)^n]^{-1}\cdot\int_{Z_{\bar{G}}(u(t))\backslash\bar{G}} \bar{f}(u(1)^g)\,d\dot{g}$, by (5.3) and the stability of f.

So (7.1) becomes

$$(7.2)\qquad c\cdot[F^x:N_{E/F}(E^x)]^{-1}\int_{\bar{T}\backslash\bar{G}} \bar{f}(t^g)\,d\dot{g} = c[N_{E/F}(E^x):(F^x)^n]\cdot|D_{E/F}|^{1/2}$$
$$\cdot|D(t)|^{-1/2}\cdot[F^x:(F^x)^n]^{-1}\cdot\int_{Z_{\bar{G}}(u(1))\backslash\bar{G}} \bar{f}(u(1)^g)\,d\dot{g} + \Gamma(t)\int_{Z(u)\backslash G} f(u^g)\,d\dot{g}.$$

The first term on the right side of (7.2) exactly equals the contribution to the left side from the germ associated to the regular unipotent class of $GL(n)$. The remaining term on the right side is, in the first place, the difference of two stable terms, hence stable, and, in the second place, equal to the only other contribution to the left side, namely the term containing the subregular germ for $GL(n)$, which we denote $\bar{\Gamma}$.

In other words,

$$(7.3)\quad c\cdot[F^x:N_{E/F}(E^x)]^{-1}\int_{Z_{\bar{G}}(u)\backslash\bar{G}} \bar{f}(u^g)\,d\dot{g}\cdot\bar{\Gamma}(t) = \Gamma(t)\int_{Z(u)\backslash G} f(u^g)\,d\dot{g}.$$

We normalize the measure on $Z(u)$ in the natural way: $Z(u)$ is the product of a compact group (the nth roots of unity), a diagonal group $(\{\mathrm{diag}(a^{1-n}, a, a, \ldots, a):a\in F^x\})$, and a unipotent group. We take the measure of total mass 1 on the compact group, times the natural F^x measure $d^x a$ on the diagonal group ($\mathrm{meas}(\mathbf{0}^x) = 1 - 1/q$), times the product of the natural F^+-measures ($\mathrm{meas}(\mathbf{0}) = 1$) on each non-trivial parameter of the unipotent group.

This measure is compatible with the measure on $Z_{\bar{G}}(u)$ given in [2] and the measure on $\bar{Z}$, so the analogue of (5.3) holds with u in place of $u(a)$. Moreover, u and the measure on $Z(u)$ are both invariant under conjugation by $d(r^{-1})$, so the integrals on the right side of that equation are all equal. This shows that the integral on the left side of (7.3) equals $[F^x:(F^x)^n]$ times the integral on the right side. So (7.3) becomes

$$c\cdot[N_{E/F}(E^x):(F^x)^n]\cdot\bar{\Gamma}(t) = \Gamma(t).$$

We have showed the following.

THEOREM 7.4. *With measures normalized as above, the "subregular" germ, i.e. the germ associated to the class of u, the unipotent class whose Jordan canonical form has a 1×1 block and an $(n-1)\times(n-1)$ block, is*

$$\Gamma(t) = c\cdot[N_{E/F}(E^x):(F^x)^n]\cdot\bar{\Gamma}(t),$$

where $\bar{\Gamma}(t)$ is the subregular germ for $GL(n, F)$.

(Recall from [2] that $\bar{\Gamma}(t) = -q^{-2}|D_{E/F}|^{1/2}\cdot|D(t)|^{-1/2}q^{-[r/n]}(nq - (q-1)(r - n[r/n]))$ where r is defined by $q^{-r/n} = d(t, F) = \min\{|t-y|:y\in F\}$, with the absolute value on E extending the normalized absolute value on F. The constant c is given by (4.3)).

8. **Complements**. 1. In certain cases, the situation can be described more simply. If $t_0 \in T'$ is fixed and $t \in T'$ is arbitrary, the behaviour of the germs is controlled by $\delta(t)/\delta(t_0)$.

LEMMA 8.1. *Let T be an elliptic torus, with $\bar{T} \cong E^x$. Then $\delta(t)^2 \in N_{E/F}(E^x)$ for all $t \in T$ or $-\delta(t)^2 \in N_{E/F}(E^x)$ for all $t \in T$, or both.* □

PROOF. $\delta(t)^2 = \pm N((t^{\sigma_1} - t^{\sigma_2})(t^{\sigma_1} - t^{\sigma_3}) \cdot \cdots \cdot (t^{\sigma_1} - t^{\sigma_n}))$, and the sign is independent of $t \in T$. □

THEOREM 8.2. *Suppose n is odd, T an elliptic torus in* $\mathrm{SL}(n, F)$. *Suppose $\Gamma_r(t_0) \neq 0$ for some $r \in F^x$, $t_0 \in T'$ sufficiently close to* id. *Then for any $t \in T'$, sufficiently close to* id,

$$\Gamma_{r'}(t) = \begin{cases} \Gamma_r(t) \neq 0, & \text{if} \quad r/r' \in N_{E/F}(E^x) \\ 0 & \text{otherwise} \end{cases}$$

PROOF. Lemma 8.1 implies $\delta(t)/\delta(t_0)$ is an element of order 1 or 2 in $F^x/N_{E/F}(E^x)$. Since n is odd, it must be of order 1, which says $\delta(t)$ and $\delta(t_0)$ are in the same coset of $F^x/N_{E/F}(E^x)$, which says that the non-zero germs are the same ones for every $t \in T'$. □

2. Suppose n is even, and E/F is cyclic. Then there is exactly one element of order 2 in $F^x/N_{E/F}(E^x)$, which implies that Γ_r is non-zero for r in at most two cosets of $F^x/N_{E/F}(E^x)$.

In particular, if E/F is unramified, then there are two distinct cosets for which the corresponding germs are not identically zero on T'. Indeed, if $t = 1 + x$, $t' = \lambda + \pi x$, with $\lambda \in F$, $|\pi| = 1/q$, then $\delta(t') = \pi^{n(n-1)/2}\delta(t)$. Since $n(n$-$1)/2$ is not an integer multiple of n, $\delta(t')$ is not in the same coset as $\delta(t)$ modulo $N_{E/F}(E^x)$.

3. To illustrate that the situation is not always so simple, consider SL(4), suppose $p \neq 2$, and suppose E/F is biquadraic, i.e. $\mathrm{Gal}(E/F) = (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$. An easy computation shows that as t ranges over T', $\delta(t)$ meets three of the four cosets of F^x modulo $N_{E/F}(E^x) = (F^x)^2$. So one class of germs is identically zero for a given T, and the three others take turns being non-zero.

4. Note that the equivalences of Corollary 6.4 depend on T (or at least on the stable conjugacy class of T). It is possible to have $\Gamma_r = \Gamma_{r'}$, on one torus but $\Gamma_r \neq \Gamma_{r'}$, on another.

5. The measures on $Z(u(r))$ could be normalized differently, but the justification for this choice is that it makes Corollary 6.5 work.

6. The results of Section 6 and Section 7, together with Rogawski's result ([3]) give all the germs for $\mathrm{SL}(3, F)$.

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