Glasgow Math. J. **49** (2007) 411–415. © 2007 Glasgow Mathematical Journal Trust.
doi:10.1017/S0017089507003746. Printed in the United Kingdom

ON THE CENTRALIZER OF AN ELEMENT OF ORDER FOUR IN A LOCALLY FINITE GROUP

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(Received 12 December, 2006; revised 6 March, 2007; accepted 10 March, 2007)

Abstract. We prove that if G is a locally finite group admitting an automorphism ϕ of order four such that $C_G(\phi)$ is Chernikov, then G has a soluble subgroup of finite index.

1. Introduction. The study of centralizers in locally finite groups is an interesting direction of group theory. In particular, the situation that a locally finite group G admits an automorphism ϕ of finite order (or, equivalenly, contains an element ϕ) such that $C_G(\phi)$ is Chernikov has received some attention in the past. Recall that a group C is Chernikov if it has a subgroup of finite index that is a direct product of finitely many groups of type C_{p^∞} for various primes p (quasicyclic p-groups). By a deep result obtained independently by Shunkov [**22**] and also by Kegel and Wehrfritz [**7**] Chernikov groups are precisely the locally finite groups satisfying the minimal condition on subgroups; that is, any non-empty set of subgroups possesses a minimal subgroup. Hartley proved in 1988 that if ϕ is of prime-power order, then G has a locally soluble subgroup of finite index (see [**4**, Theorem 1]). The case of the theorem where ϕ has order two was handled by Asar in [**1**] and for automorphisms of arbitrary prime order the theorem was proved by Turau in [**25**]. Both Hartley's and Turau's works depend on the classification of finite simple groups. We now quote a paragraph from the introduction in Hartley's paper.

"It seems likely that much more remains to be said. Possibly Theorem 1 remains true even if 'locally soluble' is replaced by 'soluble' and 'prime power order' is replaced by 'finite order', and if that is too much to hope for, then at least some progress in that direction might be feasible".

In 1992 Hartley did manage to show that if the automorphism ϕ has arbitrary finite order and $C_G(\phi)$ is finite, then indeed G has a locally soluble subgroup of finite index [**5**]. Hartley's question if this is true with ϕ of arbitrary finite order and $C_G(\phi)$ Chernikov remains unanswered, the main difficulty being that the problem does not reduce to finite groups.

In the present paper we are concerned with the part of Hartley's implicit question that deals with possible replacement of 'locally soluble' by 'soluble'. Trying to figure out a way to handle the question one quickly finds themself faced with the following long-standing problem. Suppose a finite group G admits a fixed-point-free automorphism

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ϕ of order n. Is then G soluble with derived length bounded by a function depending only on n?

The above problem has been open for many years. It is a well-known corollary of the classification of finite simple groups that any finite group G admitting a fixed-point-free automorphism is soluble. However existence of a bound on the derived length of G was established only in the cases that n is a prime or $n = 4$. Thompson proved that if n is prime, then G is nilpotent [**23**] while Higman's work [**6**] provides a bound for the nilpotency class of G. An explicit expression for the bound was given by Kreknin and Kostrikin [**12**, **13**]. Kovács proved that if a finite group G admits a fixed-point-free automorphism of order four, then $G/Z(G)$ is metabelian [**11**]. Kovács' proof uses the famous Feit-Thompson Theorem that any finite group of odd order is soluble [**2**]. Gorenstein's proof of solubility of a finite group admitting a fixed-point-free automorphism of order four that does not use the Feit-Thompson Theorem can be found in [**3**, Theorem 10.4.2]. Since no further progress with respect to the problem when n is not a prime or 4 is in sight, it seems at present Hartley's question can be effectively addressed only in those two cases.

In [**9**] Khukhro described the structure of (locally) finite groups G admitting an automorphism ϕ of prime order p such that $C_G(\phi)$ is finite. It turned out that the groups are nilpotent-by-finite. This result allowed us to handle in [**21**] locally finite groups admitting an automorphism of prime order whose centralizer is Chernikov. Predictably, the groups happen to be nilpotent-by-Chernikov. In particular, it follows that the groups have a soluble subgroup of finite index. Note however that in general there is no obvious way to reduce the situation where $C_G(\phi)$ is Chernikov to that where $C_G(\phi)$ is finite.

In a series of papers Khukhro and Makarenko studied Lie rings and (locally) finite groups admitting an automorphism of order four whose centralizer is finite and consists of boundedly many, say m, elements [**15**, **16**, **17**, **18**, **19**, **20**]. Recently their study was completed, the main result being that there exist an m-bounded number $i = i(m)$ and a constant c such that if a (locally) finite group G admits an automorphism of order four whose centralizer is of finite order m, then G possesses a characteristic subgroup K with the properties that the index $[G : K]$ is at most i and $\gamma_3(K)$ is nilpotent of class at most c (see [**10**]).

The goal of the present paper is to show how the result of Khukhro and Makarenko can be used to prove that locally finite groups admitting an automorphism of order four with Chernikov centralizer have a soluble subgroup of finite index.

THEOREM 1.1. *Let G be a locally finite group admitting an automorphism ϕ of order four such that $C_G(\phi)$ is Chernikov. Then G has a soluble subgroup of finite index.*

As the reader will see, modulo the Khukhro-Makarenko Theorem the proof of Theorem 1.1 is rather short and elementary. Since locally finite groups admitting an automorphism of prime order with Chernikov centralizer are nilpotent-by-Chernikov, it is natural to conjecture that under the hypothesis of Theorem 1.1 the group G must be metanilpotent-by-Chernikov. At the moment the author is unable to prove that conjecture, though.

2. Proof of the Theorem. Let C be a Chernikov group. The subgroup of finite index that is a direct product of finitely many groups of type C_{p^∞} possibly for various primes p is of course unique in C. Suppose it has index i and is a direct product of precisely j groups of type C_{p^∞}. The ordered pair (j, i) is called the size of C. The set of all pairs (j, i) is endowed with the lexicographic order. It is easy to check that if H is a proper subgroup of C, the size of H is necessarily strictly smaller than that of C. Also, if N is an infinite normal subgroup of C, the size of C/N is strictly smaler than that of C. If N is finite, the size of C/N may be equal to that of C, though.

Given a group G and subsets $X, Y \subseteq G$, we denote by $[X, Y]$ the subgroup generated by all commutators $[x, y]$, where $x \in X, y \in Y$. It is well-known that $[X, Y]$ is always normal in $\langle X, Y\rangle$. We denote $[[X, Y], Y]$ by $[X, Y, Y]$.

If a group G is acted on by a group A, we denote by $[G, A]$ the subgroup of G generated by the elements of the form $g^{-1}g^a$, where $g \in G$ and $a \in A$. It is easy to see that $[G, A]$ is always an A-invariant normal subgroup of G. The following lemma will be very helpful. The proof can be easily deduced from the (well-known) corresponding facts on finite groups.

LEMMA 2.1. *Let A be a finite π-group of automorphisms of a locally finite group G.*

(a) *If N is an A-invariant normal π'-subgroup of G, then $C_{G/N}(A) = C_G(A)N/N$.*
(b) *If G is a π'-group, then $[G, A, A] = [G, A]$.*
(c) *If G is a π'-group, then $G = [G, A]C_G(A)$.*

For a group G let $F(G)$ denote the Hirsch-Plotkin radical of G. This is the product of all locally nilpotent normal subgroups of G. Of course, $F(G)$ is locally nilpotent, too. We define $F_1(G) = F(G)$ and $F_{i+1}(G)/F_i(G) = F(G/F_i(G))$ for $i = 1, 2, \dots$. The group G is said to be of finite Hirsch-Plotkin height m if m is the least integer satisfying the equality $G = F_m(G)$. In this case we write $h(G) = m$. Of course, if G is a finite group, the Hirsch-Plotkin radical $F(G)$ is just the Fitting subgroup of G. There are many results bounding the Fitting height of a finite soluble group. In particular, we will use the classical result of J. G. Thompson [**23**] (see also H. Kurzweil [**14**]) that if G is a finite soluble group admitting a coprime automorphism ϕ, then $h(G)$ is bounded by a function depending only on $|\phi|$ and $h(C_G(\phi))$. The standard inverse limit argument as in [**8**, p. 54] extends this result to periodic locally soluble groups G having no $|\phi|$-torsion. In particular we have:

PROPOSITION 2.2. *If a locally finite $2'$-group G admits an automorphism ϕ of order four such that $C_G(\phi)$ is Chernikov, then G has finite Hirsch-Plotkin height.*

A locally finite group G is said to satisfy min-p if any non-empty set of p-subgroups of G possesses a minimal subgroup. We will require the following helpful proposition [**8**, Corollary 3.2].

PROPOSITION 2.3. *If a locally finite group G contains elements of order p, then G satisfies min-p if and only if there is a p-element $g \in G$ such that $C_G(g)$ satisfies min-p.*

Let us say that a group almost has certain property if it has a subgroup of finite index with that property. Given a locally finite group G, we denote by $O_{p'}(G)$ the product of all normal p'-subgroups of G. We write $O(G)$ for $O_{2'}(G)$. Theorem 3.17 in [**8**] tells us that if G is a periodic almost locally soluble group satisfying min-p, then $G/O_{p'}(G)$ is Chernikov. Therefore, if G is a periodic almost locally soluble group admitting an automorphism of order four with Chernikov centralizer, then $G/O(G)$ is Chernikov.

We are now ready to prove our main result.

THEOREM 2.4. *Let G be a locally finite group admitting an automorphism ϕ of order four such that $C_G(\phi)$ is Chernikov. Then G is almost soluble.*

Proof. By Hartley's theorem G is almost locally soluble so without loss of generality we can assume that G is a locally soluble group. Then $G/O(G)$ is Chernikov so we can assume that G is a $2'$-group. By Proposition 2.2 G has finite Hirsch-Plotkin height. Hence, a counter-example to the theorem can be found among locally nilpotent sections of G. Therefore without loss of generality we can additionally assume that G is locally nilpotent.

Assume further that G is a counter-example such that the size of $C = C_G(\phi)$ is as small as possible. In view of the Khukhro-Makarenko Theorem it is clear that C is infinite. Let $G^{(i)}$ denote the ith term of the derived series of G. Each one of the groups $G^{(i)}$ provides a counter-example to the theorem. Since the size of C is as small as possible, it follows that $C \leq G^{(i)}$ for each i. We deduce from Lemma 2.1(a) that ϕ acts fixed-point-freely on the quotient $G/G^{(4)}$. By the Kovács Theorem, $G/G^{(4)}$ has derived length at most three. Therefore $G^{(3)} = G^{(4)}$. Thus, considering $G^{(3)}$ in place of G we can make the additional assumption that

(1) $G = G'$.

Next, we can assume that the following holds.

(2) If N is a proper normal ϕ-invariant subgroup of G, the centralizer $C_N(\phi)$ is finite.

Indeed, suppose that $C_N(\phi)$ is infinite. Since the size of C is as small as possible, we deduce that G/N is soluble, a contradiction.

Let A be a quasicyclic p-subgroup of C for some prime p. Let us show that

(3) $[G, A] = G$.

This is because $[G, A]A$ is a normal ϕ-invariant subgroup of G having an infinite intersection with C. Now (2) implies that $[G, A]A = G$. Putting this together with $G = G'$ we conclude that $[G, A] = G$.

We will use the fact that if $x, y \in A$, then necessarily either $\langle x\rangle \leq \langle y\rangle$ or $\langle y\rangle \leq \langle x\rangle$. It follows that

(4) for any finite subset S of A we can choose $a \in S$ such that $[G, S] = [G, a]$.

Let T be a finite subset of G. Since $G = [G, A]$, every element of T can be written as a product of finitely many commutators $[g_i, a_i]$ with $g_i \in G$, $a_i \in A$. In turn, each of the elements g_i can be written as a similar product of $[h_j, b_j]$ with $h_j \in G$ and $b_j \in A$. Let S be the set of all a_i that appear in the commutators $[g_i, a_i]$ united with the set of all b_j that appear in $[h_j, b_j]$. It is obvious that $T \subseteq [G, S, S]$. Thus, we derive from (4) that

(5) for any finite subset T of G there exists $a \in A$ such that $T \subseteq [G, a, a]$.

Since we deal with the locally nilpotent group G, it follows that the subgroup $[G, a]$ is proper for each $a \in A$. Obviously it is also ϕ-invariant so we conclude from (2) that $[G, a]$ has finite intersection with $C_G(\phi)$ for any $a \in A$. Thus, by the Khukhro-Makarenko Theorem, $[G, a]$ contains a characteristic subgroup K_a of finite index such that $\gamma_3(K_a)$ is nilpotent of class at most c. Since $[G, a]/K_a$ is finite and since A has no subgroups of finite index, it follows that A centralizes the factor $[G, a]/K_a$. Thus,

(6) $[[G, a], A] \leq K_a$ for every $a \in A$.

Combining this with (5), we deduce that for an arbitrary finite subset T of G there exists $a \in A$ such that $T \subseteq K_a$. It follows that $\gamma_3(G)$ is nilpotent of class at most c. This contradicts (1). The proof is complete. □

REFERENCES

1. A. O. Asar, The solution of a problem of Kegel and Wehrfritz, *Proc. London Math. Soc.* (3) **45** (1982), 337–364.

2. W. Feit and J. Thompson, Solvability of groups of odd order, *Pacific J. Math.* **13** (1963), 773–1029.

3. D. Gorenstein, *Finite groups* (Harper and Row, New York, 1968).

4. B. Hartley, Fixed points of automorphisms of certain locally finite groups and Chevalley groups, *J. London Math. Soc. (2)* **37** (1988), 421–436.

5. B. Hartley, A general Brauer-Fowler theorem and centralizers in locally finite groups, *Pacific J. Math.* **152** (1992), 101–117.

6. G. Higman, Groups and Lie rings having automorphisms without non-trivial fixed points, *J. London Math. Soc.* **32** (1957), 321–334.

7. O. H. Kegel and B. A. F. Wehrfritz, Strong finiteness conditions in locally groups, *Math. Z.* **117** (1970), 309–324.

8. O. H. Kegel and B. A. F. Wehrfritz, *Locally finite groups* (North Holland, Amsterdam, 1973).

9. E. I. Khukhro, Groups and Lie rings admitting an almost regular automorphism of prime order, *Math. USSR Sbornik* **71** (1992), 51–63.

10. E. I. Khukhro and N. Yu. Makarenko, Large characteristic subgroups satisfying multilinear commutator identities, *J. London Math. Soc.* to appear.

11. L. Kovács, Groups with regular automorphisms of order four, *Math. Z.* **75** (1961), 277–294.

12. V. A. Kreknin, Solvability of Lie algebras with a regular automorphism of finite period, *Soviet Math. Dokl.* **4** (1963), 683–685.

13. V. A. Kreknin and A. I. Kostrikin, Lie algebras with regular automorphisms, *Soviet Math. Dokl.* **4** (1963), 355–358.

14. H. Kurzweil, p-Automorphismen von auflösbaren p'-Gruppen, *Math. Z.* **120** (1971), 326–354.

15. N. Yu. Makarenko, Finite 2-groups admitting an automorphism of order 4 with few fixed points, *Algebra and Logic* **32** (1993), 215–230.

16. N. Yu. Makarenko, Finite 2-groups with an automorphism of order 4, *Algebra and Logic* **40** (2001), 47–54.

17. N. Yu. Makarenko and E. I. Khukhro, Lie rings with automorphisms of degree 4 with small number of fixed points, *Algebra and Logic* **35** (1996), 21–43.

18. N. Yu. Makarenko and E. I. Khukhro, Nilpotent groups admitting an almost regular automorphism of order 4 *Algebra and Logic* **35** (1996), 176–187.

19. N. Yu. Makarenko and E. I. Khukhro, Lie rings admitting an automorphism of order 4 with few fixed points II, *Algebra and Logic* **37** (1998), 78–91.

20. N. Yu. Makarenko and E. I. Khukhro, Finite groups with an almost regular automorphism of order four, *Algebra and Logic* **45** (2006), 326–343.

21. P. Shumyatsky, On locally finite groups admitting an automorphism of prime order whose centralizer is Chernikov, *Houston J. Math.* **20** (1994), 187–192.

22. V. P. Shunkov, On the minimality problem for locally finite groups, *Algebra and Logic* **9** (1970), 137–151.

23. J. G. Thompson, Finite groups with fixed-point-free automorphisms of prime order, *Proc. Nat. Acad. Sci. USA* **45** (1959), 578–581.

24. J. G. Thompson, Automorphisms of solvable groups, *J. Algebra* **1** (1964), 259–267.

25. V. Turau, Zentralisatoren in lokal-endlichen Gruppen vom Chevalley-Typ, *Arch. Math. (Basel)* **44** (1985), 297–308.