COFINAL TYPES BELOW $\aleph_\omega$

ROY SHALEV

Abstract. It is proved that for every positive integer n, the number of non-Tukey-equivalent directed sets of cardinality $\leq \aleph_n$ is at least c_{n+2}, the $(n+2)$-Catalan number. Moreover, the class $\mathcal{D}_{\aleph_n}$ of directed sets of cardinality $\leq \aleph_n$ contains an isomorphic copy of the poset of Dyck $(n+2)$-paths. Furthermore, we give a complete description whether two successive elements in the copy contain another directed set in between or not.

§1. Introduction. Motivated by problems in general topology, Birkhoff [1], Tukey [15], and Day [2] studied some natural classes of directed sets. Later, Schmidt [9] and Isbell [4, 5] investigated uncountable directed sets under the Tukey order $<_T$. In [12], Todorčević showed that under PFA there are only five cofinal types in the class $\mathcal{D}_{\aleph_1}$ of all cofinal types of size $\leq \aleph_1$ under the Tukey order, namely, $\{1, \omega, \omega_1, \omega \times \omega_1, [\omega_1]^{<\omega}\}$. In the other direction, Todorčević showed that under CH there are $2^{\mathfrak{c}}$ many non-equivalent cofinal types in this class. Later in [14] this was extended to all transitive relations on ω_1. Recently, Kuzeljević and Todorčević [6] initiated the study of the class $\mathcal{D}_{\aleph_2}$. They showed in ZFC that this class contains at least fourteen different cofinal types which can be constructed from two basic types of directed sets and their products: $(\kappa, \in)$ and $([\kappa]^{<\theta}, \subseteq)$, where $\kappa \in \{1, \omega, \omega_1, \omega_2\}$ and $\theta \in \{\omega, \omega_1\}$.

In this paper, we extend the work of Todorčević and his collaborators and uncover a connection between the classes of the $\mathcal{D}_{\aleph_n}$'s and the Catalan numbers. Denote $V_k := \{1, \omega_k, [\omega_k]^{<\omega_m} \mid 0 \leq m < k\}$, $\mathcal{F}_n := \bigcup_{k \leq n} V_k$ and finally let $\mathcal{S}_n$ be the set of all finite products of elements of $\mathcal{F}_n$. Recall (see Section 3) that the n-Catalan number is equal to the cardinality of the set of all Dyck n-paths. The set $\mathcal{K}_n$ of all Dyck n-paths admits a natural ordering $\lhd$, and the connection we uncover is as follows.

THEOREM A. *The posets $(\mathcal{S}_n/{\equiv_T}, <_T)$ and $(\mathcal{K}_{n+2}, \lhd)$ are isomorphic. In particular, the class $\mathcal{D}_{\aleph_n}$ has size at least the $(n+2)$-Catalan number.*

A natural question which arises is whether an interval determined by two successive elements of $(\mathcal{S}_n/{\equiv_T}, <_T)$ forms an empty interval in $(\mathcal{D}_{\aleph_n}, <_T)$. In [6], the authors showed that there are two intervals of $\mathcal{S}_2$ that are indeed empty in $\mathcal{D}_{\aleph_2}$, and they also showed that consistently, under GCH and the existence of a non-reflecting stationary subset of $E^{\omega_2}_{\omega}$, two intervals of $\mathcal{S}_2$ that are nonempty in $\mathcal{D}_{\aleph_2}$.

Received July 31, 2022.
2020 *Mathematics Subject Classification.* 03E04.
Keywords and phrases. cofinal type, directed set, Tukey order, Catalan numbers.

0022-4812/00/0000-0000
DOI:10.1017/jsl.2023.32

In this paper, we prove:

THEOREM B. *Assuming* GCH, *for every positive integer n, all intervals of $\mathcal{S}_n$ that form an empty interval in $\mathcal{D}_{\aleph_n}$ are identified, and counterexamples are constructed to the other cases.*

1.1. Organization of this paper. In Section 2 we analyze the Tukey order of directed sets using characteristics of the ideal of bounded subsets.

In Section 3 we consider the poset $(\mathcal{S}_n/{\equiv_T}, <_T)$ and show it is isomorphic to the poset of good $(n+2)$-paths (Dyck paths) with the natural order. As a corollary we get that the cardinality of $\mathcal{D}_{\aleph_n}$ is greater than or equal to the Catalan number c_{n+2}. Furthermore, we address the basic question of whether a specific interval in the poset $(\mathcal{S}_n/{\equiv_T}, <_T)$ is empty, i.e., considering an element C and a successor of it E, is there a directed set $D \in \mathcal{D}_{\aleph_n}$ such that $C <_T D <_T E$? We answer this question in Theorem 3.5 using results from the next two sections.

In Section 4 we present sufficient conditions on an interval of the poset $(\mathcal{S}_n/{\equiv_T}, <_T)$ which enable us to prove there is no directed set inside.

In Section 5 we present cardinal arithmetic assumptions, enough to construct on specific intervals of the poset $(\mathcal{S}_n/{\equiv_T}, <_T)$ a directed set inside.

In Section 6 we finish with a remark about future research.

In the Appendix diagrams of the posets $(\mathcal{S}_2/{\equiv_T}, <_T)$ and $(\mathcal{S}_3/{\equiv_T}, <_T)$ are presented.

1.2. Notation. For a set of ordinals C, we write $\mathrm{acc}(C) := \{\alpha < \sup(C) \mid \sup(C \cap \alpha) = \alpha > 0\}$. For $\alpha < \gamma$ where α is a regular cardinal, denote $E^{\gamma}_{\alpha} := \{\beta < \gamma \mid \mathrm{cf}(\beta) = \alpha\}$. The set of all infinite (resp. infinite and regular) cardinals below κ is denoted by $\mathrm{Card}(\kappa)$ (resp. $\mathrm{Reg}(\kappa)$). For a cardinal κ we denote by κ^+ the successor cardinal of κ, and by κ^{+n} the nth-successor cardinal. For a function $f : X \to Y$ and a set $A \subseteq X$, we denote $f``A := \{f(x) \mid x \in A\}$. For a set A and a cardinal θ, we write $[A]^{\theta} := \{X \subseteq A \mid |X| = \theta\}$ and define $[A]^{\leq\theta}$ and $[A]^{<\theta}$ similarly. For a sequence of sets $\langle A_i \mid i \in A\rangle$, let $\prod_{i\in I} D_i := \{f : I \to \bigcup_{i\in I} D_i \mid \forall i \in I[f(i) \in D_i]\}$.

1.3. Preliminaries. A partial ordered set $(D, \leq_D)$ is *directed* iff for every $x, y \in D$ there is $z \in D$ such that $x \leq_D z$ and $y \leq_D z$. We say that a subset X of a directed set D is *bounded* if there is some $d \in D$ such that $x \leq_D d$ for each $x \in X$. Otherwise, X is *unbounded* in D. We say that a subset X of a directed D is *cofinal* if for every $d \in D$ there exists some $x \in X$ such that $d \leq_D x$. Let $\mathrm{cf}(D)$ denote the minimal cardinality of a cofinal subset of D. If D and E are two directed sets, we say that $f : D \to E$ is a *Tukey function* if $f``X := \{f(x) \mid x \in X\}$ is unbounded in E whenever X is unbounded in D. If such a Tukey function exists we say that D is *Tukey reducible* to E, and write $D \leq_T E$. If $D \leq_T E$ and $E \not\leq_T D$, we write $D <_T E$. A function $g : E \to D$ is called a *convergent/cofinal map* from E to D if for every $d \in D$ there is an $e_d \in E$ such that for every $c \geq e_d$ we have $g(c) \geq d$. There is a convergent map $g : E \to D$ iff $D \leq_T E$. Note that for a convergent map $g : E \to D$ and a cofinal subset $Y \subseteq E$, the set $g``Y$ is cofinal in D. We say that two directed sets D and E are *cofinally/Tukey equivalent* and write $D \equiv_T E$ iff $D \leq_T E$ and $D \geq_T E$. Formally, a *cofinal type* is an equivalence class under the Tukey order, we abuse the notation and call every representative of the class a cofinal type. Notice that a directed set D

is cofinally equivalent to any cofinal subset of D. In [15], Tukey proved that $D \equiv_T E$ iff there is a directed set $(X, \leq_X)$ such that both D and E are isomorphic to a cofinal subset of X. We denote by $\mathcal{D}_\kappa$ the set of all cofinal types of directed sets of cofinality $\leq \kappa$.

Consider a sequence of directed sets $\langle D_i \mid i \in I\rangle$, we define the directed set which is the product of them $(\prod_{i\in I} D_i, \leq)$ ordered by everywhere-dominance, i.e., for two elements $d, e \in \prod_{i\in I} D_i$ we let $d \leq e$ if and only if $d(i) \leq_{D_i} e(i)$ for each $i \in I$. For $X \subseteq \prod_{i\in I} D_i$, let π_{D_i} be the projection to the i-coordinate. A simple observation [12, Proposition 2] is that if n is finite, then $D_1 \times \cdots \times D_n$ is the least upper bound of $D_1, \dots, D_n$ in the Tukey order. Similarly, we define a θ-support product $\prod^{\leq\theta}_{i\in I} D_i$; for each $i \in I$, we fix some element $0_{D_i} \in D_i$ (usually minimal). Every element $v \in \prod^{\leq\theta}_{i\in I} D_i$ is such that $|\operatorname{supp}(v)| \leq \theta$, where $\operatorname{supp}(v) := \{i \in I \mid v(i) \neq 0_{D_i}\}$. The order is coordinate wise.

§2. Characteristics of directed sets. We commence this section with the following two lemmas which will be used throughout the paper.

Lemma 2.1 (Pouzet [7]). *Suppose D is a directed set such that $\operatorname{cf}(D) = \kappa$ is infinite, then there exists a cofinal directed set $P \subseteq D$ of size κ such that every subset of size κ of P is unbounded*

Proof. Let $X \subseteq D$ be a cofinal subset of cardinality κ and let $\{x_\alpha \mid \alpha < \kappa\}$ be an enumeration of X. Let $P := \{x_\alpha \mid \alpha < \kappa \text{ and for all } \beta < \alpha[x_\alpha \not<_D x_\beta]\}$. We claim that P is cofinal. In order to prove this, fix $d \in D$. As X is cofinal in D, fix a minimal $\alpha < \kappa$ such that $d <_D x_\alpha$. If $x_\alpha \in P$, then we are done. If not, then fix some $\beta < \alpha$ minimal such that $x_\alpha <_D x_\beta$. We claim that $x_\beta \in P$, i.e., there is no $\gamma < \beta$ such that $x_\beta <_D x_\gamma$. Suppose there is some $\gamma < \beta$ such that $x_\beta <_D x_\gamma$, then $x_\alpha <_D x_\gamma$, which is a contradiction to the minimality of β. Note that $d <_D x_\beta \in P$ as sought. As P is cofinal in D, $\operatorname{cf}(D) = \kappa$, $P \subseteq X$ and $|X| = \kappa$, we get that $|P| = \kappa$.

Finally, let us show that every subset of size κ of P is unbounded. Suppose on the contrary that $X \subseteq P$ is a bounded subset of P of size κ. Fix some $x_\beta \in P$ above X and $\beta < \alpha < \kappa$ such that $x_\alpha \in X$, but this is an absurd as $x_\alpha <_D x_\beta$ and $x_\alpha \in P$. ⊣

Fact 2.2 (Kuzeljević–Todorčević [6, Lemma 2.3]). *Let $\lambda \geq \omega$ be a regular cardinal and $n < \omega$ be positive. The directed set $[\lambda^{+n}]^{\leq\lambda}$ contains a cofinal subset $\mathfrak{D}_{[\lambda^{+n}]^{\leq\lambda}}$ of size λ^{+n} with the property that every subset of $\mathfrak{D}_{[\lambda^{+n}]^{\leq\lambda}}$ of size $> \lambda$ is unbounded in $[\lambda^{+n}]^{\leq\lambda}$. In particular, $[\lambda^{+n}]^{\leq\lambda}$ belongs to $\mathcal{D}_{\lambda^{+n}}$, i.e. $\operatorname{cf}([\lambda^{+n}]^{\leq\lambda}) \leq \lambda^{+n}$.*

Recall that any directed set is Tukey equivalent to any of its cofinal subsets, hence $\mathfrak{D}_{[\lambda^{+n}]^{\leq\lambda}} \equiv_T [\lambda^{+n}]^{\leq\lambda}$.

As part of our analysis of the class $\mathcal{D}_{\aleph_n}$, we would like to find certain traits of directed sets which distinguish them from one another in the Tukey order. This was done previously, in [4, 9, 14]. We use that the language of cardinal functions of ideals.

Definition 2.3. For a set D and an ideal $\mathcal{I}$ over D, consider the following cardinal characteristics of $\mathcal{I}$:

- $\operatorname{add}(\mathcal{I}) := \min\{\kappa \mid \mathcal{A} \subseteq I,\ |\mathcal{A}| = \kappa,\ \bigcup \mathcal{A} \notin \mathcal{I}\}$;
- $\operatorname{non}(\mathcal{I}) := \min\{|X| \mid X \subseteq D,\ X \notin \mathcal{I}\}$;

https://doi.org/10.1017/jsl.2023.32 Published online by Cambridge University Press

- $\text{out}(\mathcal{I}) := \min\{\theta \leq |D|^+ \mid \mathcal{I} \cap [D]^\theta = \emptyset\}$;
- $\text{in}(\mathcal{I}, \kappa) = \{\theta \leq \kappa \mid \forall X \in [D]^\kappa\ \exists Y \in [X]^\theta \cap \mathcal{I}\}$.

Notice that $\text{add}(\mathcal{I}) \leq \text{non}(\mathcal{I}) \leq \text{out}(\mathcal{I})$.

Definition 2.4. For a directed set D, denote by $\mathcal{I}_{\text{bd}}(D)$ the ideal of bounded subsets of D.

Proposition 2.5. *Let D be a directed set. Then*:

1. $\text{non}(\mathcal{I}_{\text{bd}}(D))$ *is the minimal size of an unbounded subset of D, so every subset of size less than* $\text{non}(\mathcal{I}_{\text{bd}}(D))$ *is bounded.*
2. *If* $\theta < \text{out}(\mathcal{I}_{\text{bd}}(D))$, *then there exists in D some bounded subset of size* θ.
3. *If* $\theta \geq \text{out}(\mathcal{I}_{\text{bd}}(D))$, *then every subset X of size* θ *is unbounded in D.*
4. *If* $\theta \in \text{in}(\mathcal{I}_{\text{bd}}(D), \kappa)$, *then for every* $X \in [D]^\kappa$ *there exists some* $B \in [X]^\theta$ *bounded.*
5. *For every* $\theta < \text{add}(\mathcal{I}_{\text{bd}}(D))$ *and a family* $\mathcal{A}$ *of size* θ *of bounded subsets of D, the subset* $\bigcup \mathcal{A}$ *is also bounded in D.*

Let us consider another intuitive feature of a directed set, containing information about the cardinality of hereditary unbounded subsets, this was considered previously by Isbell [4].

Definition 2.6 (Hereditary unbounded sets). For a directed set D, set

$$\text{hu}(D) := \{\kappa \in \text{Card}(|D|^+) \mid \exists X \in [D]^\kappa[\forall Y \in [X]^\kappa \text{ is unbounded}]\}.$$

Proposition 2.7. *Let D be a directed set. Then*:

- *If* $\text{cf}(D)$ *is an infinite cardinal, then* $\text{cf}(D) \in \text{hu}(D)$.
- *If* $\text{out}(\mathcal{I}_{\text{bd}}(D)) \leq \kappa \leq |D|$, *then* $\kappa \in \text{hu}(D)$.
- *For an infinite cardinal* κ *we have that* $\text{non}(\mathcal{I}_{\text{bd}}(\kappa)) = \text{cf}(\kappa)$, $\text{out}(\mathcal{I}_{\text{bd}}(D)) = \kappa$ *and* $\text{hu}(\kappa) = \{\lambda \in \text{Card}(\kappa^+) \mid \lambda = \text{cf}(\kappa)\}$.
- *If* $\kappa = \text{cf}(D) = \text{non}(\mathcal{I}_{\text{bd}}(D))$, *then* $D \equiv_T \kappa$.
- *For two infinite cardinals* $\kappa > \theta$ *we have that* $\text{non}(\mathcal{I}_{\text{bd}}([\kappa]^{<\theta})) = \text{cf}(\theta)$.
- *For a regular cardinal* κ *and a positive* $n < \omega$, $\text{out}(\mathcal{I}_{\text{bd}}(\mathfrak{D}_{[\kappa^{+n}]^{\leq\kappa}})) > \kappa$ *and* $\text{hu}(\mathfrak{D}_{[\kappa^{+n}]^{\leq\kappa}}) = \{\kappa^{+(m+1)} \mid m < n\}$.
- *If* $\kappa = \text{cf}(D)$ *is regular,* $\theta = \text{out}(\mathcal{I}_{\text{bd}}(D)) = \text{non}(\mathcal{I}_{\text{bd}}(D))$ *and* $\theta^{+n} = \kappa$ *for some* $n < \omega$, *then* $D \equiv_T [\kappa]^{<\theta}$.

In the rest of this section we consider various scenarios in which the traits of a certain directed set give us information about its position in the poset $(\mathcal{D}_\kappa, <_T)$.

Lemma 2.8. *Suppose D is a directed set,* κ *is an infinite regular cardinal and* $X \subseteq D$ *is an unbounded subset of size* κ *such that every subset of X of size* $< \kappa$ *is bounded. Then* $\kappa \in \text{hu}(D)$.

Proof. Enumerate $X := \{x_\alpha \mid \alpha < \kappa\}$, by the assumption, for every $\alpha < \kappa$ we may fix some $z_\alpha \in D$ above the bounded initial segment $\{x_\beta \mid \beta < \alpha\}$. We show that $Z := \{z_\alpha \mid \alpha < \kappa\}$, witnesses $\kappa \in \text{hu}(D)$. First, let us show that $|Z| = \kappa$. Suppose on the contrary that $Z := \{z_\alpha \mid \alpha < \kappa\}$ is of cardinality $< \kappa$. Then for some $\alpha < \kappa$, the element z_α is above the subset X, hence X is bounded which is absurd. Now, let us prove that Z is hereditarily unbounded. We claim that every subset of Z of

https://doi.org/10.1017/jsl.2023.32 Published online by Cambridge University Press

cardinality κ is also unbounded. Suppose not, let us fix some $W \in [Z]^\kappa$ bounded by some $d \in D$, but then d is above X contradicting the fact that X is unbounded. $\dashv$

LEMMA 2.9. *Suppose D is a directed set and κ is an infinite cardinal in* $\mathrm{hu}(D)$, *then* $\kappa \leq_T D$.

PROOF. Fix $X \subseteq D$ of cardinality κ such that every subset of X of size κ is unbounded and a one-to-one function $f : \kappa \to X$, notice that f is a Tukey function from κ to D as sought. $\dashv$

COROLLARY 2.10. *Suppose D is directed set, κ is regular and $X \subseteq D$ is an unbounded subset of size κ such that every subset of X of size $< \kappa$ is bounded, then* $\kappa \leq_T D$.

The reader may check the following:

- For any two infinite cardinals λ and κ of the same cofinality, we have $\lambda \equiv_T \kappa$.
- For an infinite regular cardinal κ, we have $\kappa \equiv_T [\kappa]^{<\kappa}$.
- $\mathrm{hu}(\prod_{n<\omega}^{<\omega} \omega_{n+1}) = \{\omega_n \mid n < \omega\}$.

LEMMA 2.11. *Suppose D and E are two directed sets such that for some $\theta \in$* $\mathrm{hu}(D)$ *regular we have $\theta >$* $\mathrm{cf}(E)$, *then $D \not\leq_T E$.*

PROOF. By passing to a cofinal subset, we may assume that $|E| = \mathrm{cf}(E)$. Fix $\theta \in \mathrm{hu}(D)$ regular such that $\mathrm{cf}(E) < \theta$ and $X \in [D]^\theta$ witnessing $\theta \in \mathrm{hu}(D)$, i.e., every subset of X of size θ is unbounded. Suppose on the contrary that there exists a Tukey function $f : D \to E$. By the pigeonhole principle, there exists some $Z \in [X]^\theta$ and $e \in E$ such that $f``Z = \{e\}$. As $f$ is Tukey and the subset $Z \subseteq X$ is unbounded, $f``Z$ is unbounded in E which is absurd. $\dashv$

Notice that for every directed set D, if $\mathrm{cf}(D) > 1$, then $\mathrm{cf}(D)$ is an infinite cardinal. As a corollary from the previous lemma, $\lambda \not\leq_T \kappa$ for any two regular cardinals $\lambda > \kappa$ where λ is infinite. Furthermore, the reader can check that $\lambda \not\leq_T \kappa$, whenever $\lambda < \kappa$ are infinite regular cardinals.

LEMMA 2.12. *Suppose C and D are directed sets such that $C \leq_T D$, then* $\mathrm{cf}(C) \leq \mathrm{cf}(D)$.

PROOF. Suppose $|D| = \mathrm{cf}(D)$ and let $f : C \to D$ be a Tukey function. As f is Tukey, for every $d \in D$ the set $\{x \in C \mid f(x) = d\}$ is bounded in C by some $c_d \in C$. Note that for every $x \in C$, we have $x \leq_C c_{f(x)}$, hence the set $\{c_d \mid d \in D\}$ is cofinal in C. So $\mathrm{cf}(C) \leq |D| = \mathrm{cf}(D)$ as sought. $\dashv$

LEMMA 2.13. *Let κ and θ be two cardinals such that $\theta < \kappa =$* $\mathrm{cf}(\kappa)$.
Suppose D is a directed set such that $\mathrm{cf}(D) \leq \kappa$ *and* $\mathrm{non}(\mathcal{I}_{\mathrm{bd}}(D)) \geq \theta$, *then $D \leq_T [\kappa]^{<\theta}$. Furthermore, if $\theta \in$* $\mathrm{in}(\mathcal{I}_{\mathrm{bd}}(D), \kappa)$, *then $D <_T [\kappa]^{<\theta}$.*

PROOF. First, we show that there exists a Tukey function $f : D \to [\kappa]^{<\theta}$. Let us fix a cofinal subset $X \subseteq D$ of cardinality $\leq \kappa$ such that every subset of X of cardinality $< \theta$ is bounded. As $|X| \leq \kappa$ we may fix an injection $f : X \to [\kappa]^1$, we will show f is a Tukey function. Let $Y \subseteq X$ be a subset unbounded in D, this implies $|Y| \geq \theta$. As f is an injection, the set $\bigcup f``Y$ is of cardinality $\geq \theta$. Note that every subset of $[\kappa]^{<\theta}$ whose union is of cardinality $\geq \theta$ is unbounded in $[\kappa]^{<\theta}$, hence $f``Y$ is an unbounded subset in $[\kappa]^{<\theta}$ as sought.

https://doi.org/10.1017/jsl.2023.32 Published online by Cambridge University Press

Assume $\theta \in \mathrm{in}(\mathcal{I}_{\mathrm{bd}}(D), \kappa)$, we are left to show that $[\kappa]^{<\theta} \not\leq_T D$. Suppose on the contrary that $g : [\kappa]^{<\theta} \to D$ is a Tukey function. We split to two cases:

► Suppose $|g``[\kappa]^1| < \kappa$. As $\kappa$ is regular, by the pigeonhole principle there exists a set $X \subseteq [\kappa]^1$ of cardinality $\kappa$, and $d \in D$ such that $g(x) = d$ for each $x \in X$. Notice $g``X$ is a bounded subset of D. As $X \subseteq [\kappa]^1$ is of cardinality κ and $\kappa > \theta$, it is unbounded in $[\kappa]^{<\theta}$. Since g is a Tukey function, we get that $g``X$ is unbounded which is absurd.

► Suppose $|g``[\kappa]^1| = \kappa$. Let $X := g``[\kappa]^1$, by our assumption on D, there exists a bounded subset $B \in [X]^\theta$. Since B is of size θ, we get that $(g^{-1}[B]) \cap [\kappa]^1$ is of cardinality $\geq \theta$, hence unbounded in $[\kappa]^{<\theta}$, which is absurd to the assumption g is Tukey. $\dashv$

Remark 2.14. For every two directed sets, D and E, if $\mathrm{non}(\mathcal{I}_{\mathrm{bd}}(D)) < \mathrm{non}(\mathcal{I}_{\mathrm{bd}}(E))$, then $D \not\leq_T E$. For example, $\theta \not\leq_T [\kappa]^{\leq\theta}$.

Lemma 2.15. *Let κ be a regular infinite cardinal. Suppose D and E are two directed sets such that $|D| \geq \kappa$ and $\mathrm{out}(\mathcal{I}_{\mathrm{bd}}(D)) \in \mathrm{in}(\mathcal{I}_{\mathrm{bd}}(E), \kappa)$, then $D \not\leq_T E$.*

Proof. Let $\theta := \mathrm{out}(\mathcal{I}_{\mathrm{bd}}(D))$. By the definition of $\mathrm{in}(\mathcal{I}_{\mathrm{bd}}(E), \kappa)$, as $\theta \in \mathrm{in}(\mathcal{I}_{\mathrm{bd}}(E), \kappa)$, we know that $\theta \leq \kappa$. Notice that every subset of D of size $\geq \theta$ is unbounded in D and every subset of size κ of E contains a bounded subset in E of size θ.

Suppose on the contrary that there exists a Tukey function $f : D \to E$. We split to two cases:

► Suppose $|f``D| < \kappa$, then by the pigeonhole principle there exists some $X \in [D]^\kappa$ and $e \in E$ such that $f``X = \{e\}$. As $|X| = \kappa \geq \theta$, we know that X is unbounded in D, but $f``X$ is bounded in E which is absurd as f is a Tukey function.

► Suppose $|f``D| \geq \kappa$, by the assumption there exists a subset $Y \in [f``D]^\theta$ which is bounded in E. Notice that $X := f^{-1}Y$ is a subset of D of size $\geq \theta$, hence unbounded in D. So X is an unbounded subset of D such that $f``X = Y$ is bounded in E, contradicting the fact that f is a Tukey function. $\dashv$

Lemma 2.16. *Suppose κ is a regular uncountable cardinal, C and $\langle D_m \mid m < n\rangle$ are directed sets such that $|C| < \kappa \leq \mathrm{cf}(D_m)$ and $\mathrm{non}(\mathcal{I}_{\mathrm{bd}}(D_m)) > \theta$ for every $m < n$. Then $\theta \in \mathrm{in}(\mathcal{I}_{\mathrm{bd}}(C \times \prod_{m<n} D_m), \kappa)$.*

Proof. Suppose $X \subseteq C \times \prod_{m<n} D_m$ is of size κ, we show that X contains a bounded subset of size θ. As $|C| < \kappa$, by the pigeonhole principle we can fix some $Y \in [X]^\kappa$ and $c \in C$ such that $\pi_C``Y = \{c\}$. Suppose on the contrary that some subset $Z \subseteq Y$ of size $\theta$ is unbounded, it must be that for some $m < n$ the set $\pi_{D_m}``Z$ is unbounded in D_m, but this is absurd as $\mathrm{non}(\mathcal{I}_{\mathrm{bd}}(D_m)) > \theta$ and $|\pi_{D_m}``Z| \leq \theta$. $\dashv$

Lemma 2.17. *Suppose C, D and E are directed sets such that:*

- *for every partition $D = \bigcup_{\gamma<\kappa} D_\gamma$, there exists an ordinal $\gamma < \kappa$, and an unbounded $X \subseteq D_\gamma$ of size κ;*
- $|C| \leq \kappa$;
- $\mathrm{non}(\mathcal{I}_{\mathrm{bd}}(E)) > \kappa$.

Then $D \not\leq_T C \times E$.

https://doi.org/10.1017/jsl.2023.32 Published online by Cambridge University Press

PROOF. Suppose on the contrary, that there exists a Tukey function $h : D \to C \times E$. For $c \in C$, let $D_c := \{x \in D \mid \exists e \in E[h(x) = (c, e)]\}$. Since h is a function, $D := \bigcup_{c \in C} D_c$ is a partition to $\leq \kappa$ many sets. By the assumption, there exists $c \in C$ and an unbounded subset $X \subseteq D_c$ of cardinality κ. Enumerate $X = \{x_\xi \mid \xi < \kappa\}$ and let $e_\xi \in E$ be such that $h(x_\xi) = (c, e_\xi)$, for each $\xi < \kappa$. As $\mathrm{non}(\mathcal{I}_{\mathrm{bd}}(E)) > \kappa$, there exists some upper bound $e \in E$ to the set $\{e_\xi \mid \xi < \kappa\}$. Since X is unbounded and h is Tukey, $h"X = \{(c, e_\xi) \mid \xi < \kappa\}$ must be unbounded, which is absurd as (c, e) is bounding it. $\dashv$

Note that the lemma is also true when the partition of D is of size less than κ.

§3. The Catalan structure. The sequence of Catalan numbers $\langle c_n \mid n < \omega \rangle = \langle 1, 1, 2, 5, 14, 42, \dots \rangle$ is an ubiquitous sequence of integers with many characterizations, for a comprehensive review of the subject, we refer the reader to Stanley's book [11]. One of the many representations of c_n, is the number of good n-paths (Dyck paths), where a *good n-path* is a monotonic lattice path along the edges of a grid with $n \times n$ square cells, which do not pass above the diagonal. A *monotonic path* is one which starts in the lower left corner, finishes in the upper right corner, and consists entirely of edges pointing rightwards or upwards. An equivalent representation of a good n-path, which we will consider from now on, is a vector $\vec{p}$ of the columns' heights of the path (ignoring the first trivial column), i.e., a vector $\vec{p} = \langle p_0, \dots, p_{n-2} \rangle$ of length $n - 1$ of $\leq$-increasing numbers satisfying $0 \leq p_k \leq k + 1$, for every $0 \leq k \leq n - 2$. We consider the poset $(\mathcal{K}_n, \lhd)$ where $\mathcal{K}_n$ is the set of all good n-paths and the relation $\lhd$ is defined such that $\vec{a} \lhd \vec{b}$ if and only if the two paths are distinct and for every k with $0 \leq k \leq n - 2$ we have $b_k \leq a_k$, in other words, the path $\vec{b}$ is below the path $\vec{a}$ (allowing overlaps). Notice that for two distinct good n-paths $\vec{a}$ and $\vec{b}$, either $\vec{a} \not\lhd \vec{b}$ or $\vec{b} \not\lhd \vec{a}$. A good n-path $\vec{b}$ is an immediate successor of a good n-path $\vec{a}$ if $\vec{a} \lhd \vec{b}$ and $\vec{a} - \vec{b}$ is a vector with value 0 at all coordinates except one of them which gets the value 1.

Suppose $\vec{a}$ and $\vec{b}$ are two good n-paths where $\vec{b}$ is an immediate successor of $\vec{a}$. Let $i \leq n - 2$ be the unique coordinate on which $\vec{a}$ and $\vec{b}$ are different and a_i be the value of $\vec{a}$ on this coordinate, i.e., $a_i = b_i + 1$. We say that the pair $(\vec{a}, \vec{b})$ is on the k-diagonal if and only if $i + 1 - a_i = k$ and $\vec{b}$ is an immediate successor of $\vec{a}$ (Figure 1).

In this section we show the connection between the Catalan numbers and cofinal types. Let us fix $n < \omega$. Recall that for every $k < \omega$, we set $V_k := \{1, \omega_k, [\omega_k]^{<\omega_m} \mid 0 \leq m < k\}$, $\mathcal{F}_n := \bigcup_{k \leq n} V_k$ and let $\mathcal{S}_n$ be the set of all finite products of elements in $\mathcal{F}_n$. Our goal is to construct a coding which gives rise to an order-isomorphism between $(\mathcal{S}_n/\equiv_T, <_T)$ and $(\mathcal{K}_{n+2}, \lhd)$.

To do that, we first consider a "canonical form" of directed sets in $\mathcal{S}_n$. By Lemma 2.13 the following hold:

(a) For all $0 \leq l < m < k < \omega$ we have $1 <_T \omega_k <_T [\omega_k]^{<\omega_m} <_T [\omega_k]^{<\omega_l}$.
(b) For all $0 \leq l \leq t < m \leq k < \omega$ with $(l, k) \neq (t, m)$ we have $[\omega_m]^{<\omega_t} <_T [\omega_k]^{<\omega_l}$ and $\omega_m <_T [\omega_k]^{<\omega_l}$.

Notice that (a) implies $(V_k, <_T)$ is linearly ordered. A basic fact is that for two directed sets C and D such that $C \leq_T D$, we have $C \times D \equiv_T D$. Hence, for every

https://doi.org/10.1017/jsl.2023.32 Published online by Cambridge University Press

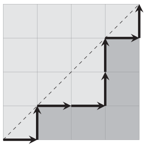

FIGURE 1. The good 4-path $\langle 1, 1, 3\rangle$.

$D \in \mathcal{S}_n$ we can find a sequence of elements $\langle D^k \mid k \leq n\rangle$, where $D^k \in V_k$ for every $k \leq n$, such that $D \equiv_T \prod_{k \leq n} D^k$. As we are analyzing the class $\mathcal{D}_{\aleph_n}$ under the Tukey relation $<_T$, two directed sets which are of the same $\equiv_T$-equivalence class are indistinguishable, so from now on we consider only elements of this form in $\mathcal{S}_n$.

We define a function $\mathfrak{F} : \mathcal{S}_n \to \mathcal{S}_n$ as follows: Fix $D \in \mathcal{S}_n$ where $D = \prod_{k \leq n} D^k$. Next, we construct a sequence $\langle D_k \mid k \leq n\rangle$ by reverse recursion on $k \leq n$. At the top case, set $D_n := D^n$. Next, for $0 \leq k < n$. If by (b), we get that $D^k <_T D^m$ for some $k < m \leq n$, then set $D_k := 1$. Else, let $D_k := D^k$. Finally, let $\mathfrak{F}(D) := \prod_{k \leq n} D_k$. Notice that we constructed $\mathfrak{F}(D)$ such that $\mathfrak{F}(D) \equiv_T D$. We define $\mathcal{T}_n := \mathrm{Im}(\mathfrak{F})$.

The coding. We encode each product $D \in \mathcal{T}_n$ by an $(n+2)$-good path $\vec{v}_D := \langle v_0, \dots, v_n\rangle$. Recall that $D := \prod_{k \leq n} D_k$, where $D_k \in V_k$ for every $k \leq n$. We define by reverse recursion on $0 \leq k \leq n$, the elements of the vector $\vec{v}_D$ such that $v_k \leq k+1$ as follows: Suppose one of the elements of $\langle [\omega_k]^{<\omega}, \dots, [\omega_k]^{<\omega_{k-1}}, \omega_k\rangle$ is equal to D_k, then let v_k be its coordinate (starting from 0). Suppose this is not the case, then if $k = n$, we let $v_k := n+1$ else $v_k := \min\{v_{k+1}, k+1\}$.

Notice that by (b), if $0 \leq i < j \leq n$, then $v_i \leq v_j$. Hence, every element $D \in \mathcal{T}_n$ is encoded as a good $(n+2)$-path.

To see that the coding is one-to-one, suppose $C, D \in \mathcal{T}_n$ are distinct. Let $k := \max\{i \leq n \mid C_i \neq D_i\}$. We split to two cases:

► Suppose both C_k and D_k are not equal to 1, then clearly the column height of $\vec{v}_C$ and $\vec{v}_D$ are different at coordinate $k+1$.

► Suppose one of them is equal to 1, say C_k, then $D_k \neq 1$. Let $\vec{v}_C := \langle v_0^C, \dots v_n^C\rangle$ and $\vec{v}_D := \langle v_0^D, \dots v_n^D\rangle$. Suppose $k = n$, then clearly $v_n^D < v_n^C$. Suppose $k < n$, then $v_i^D = v_i^C$ for $k < i \leq n$. By the coding, $v_k^D < k+1$ and by (b) $v_k^D < v_{k+1}^D = v_{k+1}^C$, but $v_k^C := \min\{k+1, v_{k+1}^C\}$. Hence $v_k^D < v_k^C$ as sought.

To see that the coding is onto, let us fix a good $(n+2)$-path $\vec{v} := \langle v_0, \dots, v_n\rangle$. We construct $\langle D_k \mid k \leq n\rangle$ by reverse recursion on $k \leq n$. At the top case, set D_n to be the v_n element of the vector $\langle [\omega_n]^{<\omega}, \dots, [\omega_n]^{<\omega_{n-1}}, \omega_n, 1\rangle$. For $k < n$, if $v_k = v_{k+1}$, let $D_k := 1$. Else, let D_k be the kth element of the vector $\langle [\omega_k]^{<\omega}, \dots, [\omega_k]^{<\omega_{k-1}}, \omega_k, 1\rangle$. Let $D = \prod_{k \leq n} D_k$, notice that as $\vec{v}$ represents a good $(n+2)$-path we have $D = \mathfrak{F}(D)$, hence $D \in \mathcal{T}_n$. Furthermore, $\vec{v}_D = \vec{v}$, hence the coding is onto as sought. As a Corollary we get that $|\mathcal{T}_n| = c_{n+2}$.

In Figure 2 we present all good 4-paths and the corresponding types in $\mathcal{T}_2$ they encode.

LEMMA 3.1. *Suppose $C, D \in \mathcal{T}_n$ and $\vec{v}_D \lhd \vec{v}_C$, then $D \leq_T C$.*

https://doi.org/10.1017/jsl.2023.32 Published online by Cambridge University Press

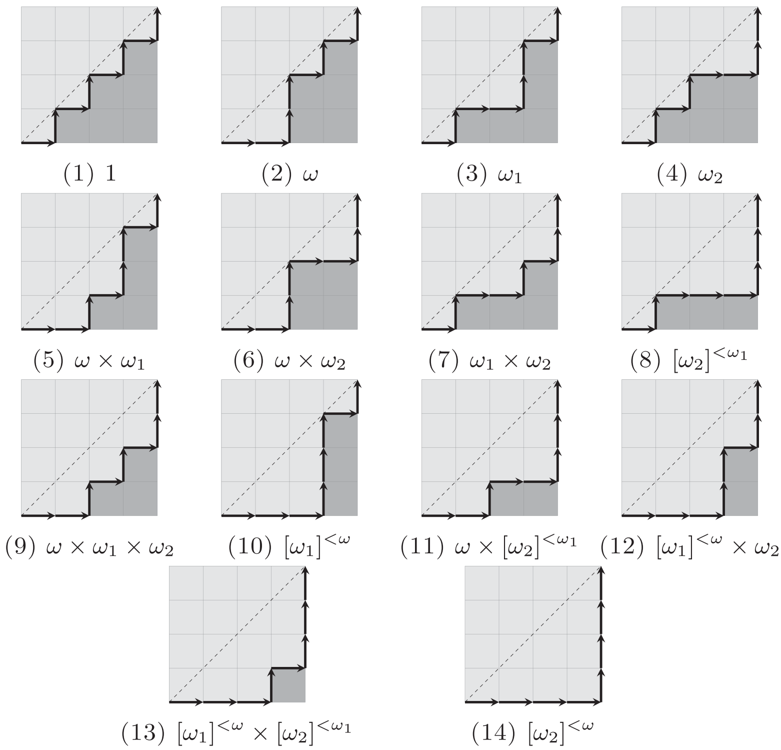

FIGURE 2. All good 4-paths and the corresponding types in $\mathcal{T}_2$ they encode.

PROOF. Let $D = \prod_{k \leq n} D_k$ and $C = \prod_{k \leq n} C_k$. Note that if $D_k \leq_T C$ for every $k \leq n$, then $D \leq_T C$ as sought. Fix $k \leq n$, if $D_k = 1$, then clearly $D_k \leq C$. Suppose $D_k \neq 1$, we split to two cases:

▶ Suppose $C_k \neq 1$. As $v_k^C < v_k^D$ and by (a) we have $D_k \leq_T C_k \leq_T C$ as sought.

▶ Suppose $C_k = 1$, let $m := \max\{i \leq n \mid k < i,\ v_i^C = v_k^C\}$. As $v_i^C \leq i + 1$, by the coding m is well-defined and $v_m^C = v_k^C \leq k < m$. Notice that $C_m = [\omega_m]^{<\omega_p}$ where $p = v_m^C$ and $D_k \equiv_T [\omega_k]^{<\omega_p}$. So by (b), $D_k \leq_T C_m \leq_T C$ as sought. ⊣

LEMMA 3.2. *Suppose $C, D \in \mathcal{T}_n$ and $\vec{v}_D \ntriangleleft \vec{v}_C$, then $D \nleq_T C$.*

PROOF. Let $D = \prod_{k \leq n} D_k$, $C = \prod_{k \leq n} C_k$, $\vec{v}_C := \langle v_0^C, \dots, v_n^C \rangle$ and $\vec{v}_D := \langle v_0^D, \dots, v_n^D \rangle$ As $\vec{v}_D \ntriangleleft \vec{v}_C$, we can define $i = \min\{k \leq n \mid v_k^C > v_k^D\}$.

Let $p := v_i^D$ and $r = \max\{k \leq n \mid v_i^D = v_k^D\}$, notice that $p \leq i$. We define a directed set F such that $F \leq_T D$.

▶ Suppose $p = i$ and let $F = \omega_i$. If $r = i$, then clearly $F = D_i$ and $F \leq_T D$ as sought. Else, by the coding $D_r = [\omega_r]^{<\omega_p}$. By Lemma 2.13, we have $F \leq_T D$ as sought.

▶ Suppose $p < i$ and let $F = \mathfrak{D}_{[\omega_i]^{<\omega_p}}$. By the coding $D_r = [\omega_r]^{<\omega_p}$ and by Clause (b), we have $F \leq_T D$ as sought.

https://doi.org/10.1017/jsl.2023.32 Published online by Cambridge University Press

Notice that $\mathrm{out}(\mathcal{I}_{\mathrm{bd}}(F)) = \omega_p$ and $\mathrm{cf}(F) = \omega_i$. As $F \leq_T D$, it is enough to verify that $F \not\leq_T C$.

As $\vec{v}_C$ is a good $(n+2)$-path, we know that $v^C_k > p$ for every $k \geq i$. Consider $A := \{i \leq k \leq n \mid C_k \neq 1\}$. We split to two cases:

▶ Suppose $A = \emptyset$. Then $\mathrm{cf}(\prod_{k\leq n} C_k) < \omega_i$. As $\mathrm{cf}(F) = \omega_i$, by Lemma 2.12 we have that $F \not\leq_T \prod_{k\leq n} C_k$ as sought.

▶ Suppose $A \neq \emptyset$. Let $E := \prod_{k<i} C_k \times \prod_{k\in A} C_k$ Notice that $\mathrm{cf}(\prod_{k<i} C_k) < \omega_i$, $\prod_{i\leq k\leq n} C_k \equiv_T \prod_{k\in A} C_k$ and $C \equiv_T E$. Furthermore, for each $k \in A$, we have $\mathrm{non}(\mathcal{I}_{bd}(C_k)) > \omega_p$. By Lemma 2.16, we have $\omega_p \in \mathrm{in}(\mathcal{I}_{\mathrm{bd}}(E), \omega_i)$. Recall $\mathrm{out}(\mathcal{I}_{\mathrm{bd}}(F)) = \omega_p$. By Lemma 2.15, we get that $F \not\leq_T E$, hence $F \not\leq_T C$ as sought. ⊣

THEOREM 3.3. *The posets $(\mathcal{T}_n, <_T)$ and $(\mathcal{K}_{n+2}, \lhd)$ are isomorphic.*

PROOF. Define f from $(\mathcal{T}_n, <_T)$ to $(\mathcal{K}_{n+2}, \lhd)$, where for $C \in \mathcal{T}_n$, we let $f(C) := \vec{v}_C$. By Lemmas 3.1 and 3.2, this is indeed an isomorphism of posets.

Furthermore, we claim that $\mathcal{T}_n$ contains one unique representative from each equivalence class of $(\mathcal{S}_n, \equiv_T)$. Recall that the function $\mathfrak{F}$ is preserving Tukey equivalence classes. Consider two distinct $C, D \in \mathcal{T}_n$. As the coding is a bijection, $\vec{v}_C$ and $\vec{v}_D$ are different. Notice that either $\vec{v}_C \ntrianglelefteq \vec{v}_D$ or $\vec{v}_D \ntrianglelefteq \vec{v}_C$, hence by Lemma 3.2, $C \not\equiv_T D$ as sought. ⊣

Consider the poset $(\mathcal{T}_n, <_T)$, clearly 1 is a minimal element and by Lemma 2.13, $[\omega_n]^{<\omega}$ is a maximal element. By the previous theorem, the set of immediate successors of an element D in the poset $(\mathcal{T}_n, <_T)$, is the set of all directed sets $C \in \mathcal{T}_n$ such that $\vec{v}_C$ is an $\lhd$-immediate successor of $\vec{v}_D$.

LEMMA 3.4. *Suppose $G, H \in \mathcal{T}_n$, H is an immediate successor of G in the poset $(\mathcal{T}_n, <_T)$ and $(\vec{v}_G, \vec{v}_H)$ are on the l-diagonal. Then there are C, E, M, N directed sets such that*:

- *$G \equiv_T C \times M \times E$ and $H \equiv_T C \times N \times E$;*
- *for some $k \leq n$, $\mathrm{cf}(N) = \omega_k$, $|C| < \omega_k$ and either $E \equiv_T 1$ or $\mathrm{non}(\mathcal{I}_{\mathrm{bd}}(E)) > \omega_{k-l}$.*

Furthermore,

- *If $l = 0$, then $M = 1$ and $N = \omega_k$.*
- *If $l = 1$, then $k > 1$ and $M = \omega_k$ and $N = [\omega_k]^{<\omega_{k-1}}$.*
- *If $l > 1$, then $k > l$ and $M = [\omega_k]^{<\omega_{k-l+1}}$ and $N = [\omega_k]^{<\omega_{k-l}}$.*

PROOF. As H is an immediate successor of G in the poset $(\mathcal{T}_n, <_T)$, we know that $\vec{v}_H$ is an immediate successor of $\vec{v}_H$ in $(\mathcal{K}_{n+2}, \lhd)$. Let k be the unique $k \leq n$ such that $v^k_G = v^k_H + 1$.

Let $\vec{v}_G := \langle v^0_G, \dots, v^n_G\rangle$ be a good $(n+2)$-path coded by G. We construct $\langle M_i \mid i \leq n\rangle$ by letting M_i be the ith element of the vector $\langle [\omega_i]^{<\omega}, \dots, [\omega_i]^{<\omega_{i-1}}, \omega_i, 1\rangle$ for every $i \leq n$. Notice that $G \equiv_T \prod_{i\leq n} M_i$. Similarly, we may construct $\langle N_i \mid i \leq n\rangle$ such that $H \equiv_T \prod_{i\leq n} N_i$. Clearly, $M_i = N_i$ for every $i \neq k$.

Let $C := \prod_{i<k} M_i$ and $E := \prod_{i>k} M_i$. Notice that $|C| = \mathrm{cf}(C) < \omega_k$ and either $E \equiv_T 1$ or $\mathrm{non}(\mathcal{I}_{\mathrm{bd}}(E)) > \omega_{k-l}$. Moreover, $G \equiv_T C \times M_k \times E$ and $H \equiv_T C \times N_k \times E$. We split to cases:

https://doi.org/10.1017/jsl.2023.32 Published online by Cambridge University Press

- If $l = 0$, then $v_H^k = k + 1$, hence $M_k = 1$ and $N_k = \omega_k$.
- If $l = 1$, then $v_H^k = k$, hence $M_k = \omega_k$ and $N_k = [\omega_k]^{<\omega_{k-1}}$.
- If $l > 1$, then $v_H^k = k - l + 1$, hence $M_k = [\omega_k]^{<\omega_{k-l+1}}$ and $N_k = [\omega_k]^{<\omega_{k-l}}$.

⊣

THEOREM 3.5. *Suppose $G, H \in \mathcal{T}_n$, H is an immediate successor of G in the poset $(\mathcal{T}_n, <_T)$ and $(\vec{v}_G, \vec{v}_H)$ are on the l-diagonal.*

- *If $l = 0$, then there is no directed set $D \in \mathcal{D}_{\aleph_n}$ such that $G <_T D <_T H$.*
- *If $l > 0$, then consistently there exist a directed set $D \in \mathcal{D}_{\aleph_n}$ such that $G <_T D <_T H$.*

PROOF. Let C, E, M, N be as in the previous lemma, so $G \equiv_T C \times M \times E$, $H \equiv_T C \times N \times E$ and for some $k \leq n$, $|C| < \omega_k$ and either $E \equiv_T 1$ or $\mathrm{non}(\mathcal{I}_{\mathrm{bd}}(E)) > \omega_{k-l}$. We split to three cases:

- Suppose $l = 0$, then $G \equiv_T C \times E$ and $H \equiv_T C \times \omega_k \times E$, by Theorem 4.1 there is no directed set D such that $G <_T D <_T H$.
- Suppose $l = 1$, then $k \geq 1$ and $N = [\omega_k]^{<\omega_{k-1}}$ and $M = \omega_k$.
  - Suppose $k = 1$, then under the assumption $\mathfrak{b} = \omega_1$, by Theorem 5.9 there exists a directed set D such that $G <_T D <_T H$.
  - Suppose $k > 1$, then under the assumption $2^{\aleph_{k-2}} = \aleph_{k-1}$ and $2^{\aleph_{k-1}} = \aleph_k$, by Corollary 5.1 there exists a directed set D such that $G <_T D <_T H$.
- Suppose $l > 1$, then $k \geq 2$. Let $\theta = \omega_{k-l}$ and $\lambda = \omega_{k-1}$. Notice $N = [\omega_k]^{\leq\theta}$ and $M = [\omega_k]^{<\theta}$. In Corollary 5.11, we shall show that under the assumption $\lambda^\theta < \lambda^+$ and $\clubsuit_J^{\omega_{k-1}}(S, 1)$ for some stationary set $S \subseteq E_\theta^{\omega_k}$, there exists a directed set D such that $G <_T D <_T H$. ⊣

§4. Empty intervals in $D_{\aleph_n}$. Consider two successive directed sets in the poset $(\mathcal{T}_n, <_T)$, we can ask whether there exists some other directed set in between in the Tukey order. The following theorem give us a scenario in which there is a no such directed set.

THEOREM 4.1. *Let κ be a regular cardinal. Suppose C and E are two directed sets such that $\mathrm{cf}(C) < \kappa$ and either $E \equiv_T 1$ or $\kappa \in \mathrm{in}(\mathcal{I}_{\mathrm{bd}}(E), \kappa)$ and $\kappa \leq \mathrm{cf}(E)$. Then there is no directed set D such that $C \times E <_T D <_T C \times \kappa \times E$.*

PROOF. By the upcoming Lemmas 4.2 and 4.3. ⊣

LEMMA 4.2. *Let κ be a regular cardinal. Suppose C is a directed set such that $\mathrm{cf}(C) < \kappa$, then there is no directed set D such that $C <_T D <_T C \times \kappa$.*

PROOF. Suppose D is a directed set such that $C <_T D <_T C \times \kappa$. Let us assume D is a directed set of size $\mathrm{cf}(D)$ such that every subset of D of size $\mathrm{cf}(D)$ is unbounded in D. By Lemma 2.12 we get that $\mathrm{cf}(C) \leq \mathrm{cf}(D) \leq \kappa$. We split to two cases:

► Suppose $\mathrm{cf}(C) \leq \mathrm{cf}(D) < \kappa$. Let $g : D \to C \times \kappa$ be a Tukey function. As $|D| = \mathrm{cf}(D) < \kappa$ and κ is regular there exists some $\alpha < \kappa$ such that $g"D \subseteq C \times \alpha$. We claim that $\pi_C \circ g$ is a Tukey function from D to C, hence $D \leq_T C$ which is absurd. Suppose $X \subseteq D$ is unbounded in D, as g is a Tukey function, we know that $g"X$ is unbounded in $C \times \kappa$. But as $(\pi_\kappa \circ g)"X$ is bounded by α, we get that $(\pi_C \circ g)"X$ is unbounded in C as sought.

https://doi.org/10.1017/jsl.2023.32 Published online by Cambridge University Press

▶ Suppose $\mathrm{cf}(D) = \kappa$, notice that $\kappa \in \mathrm{hu}(D)$ is regular so by Lemma 2.9 we get that $\kappa \leq_T D$. We also know that $C \leq_T D$, thus $\kappa \times C \leq_T D$ which is absurd. ⊣

Note that $\mathrm{non}(\mathcal{I}_{\mathrm{bd}}(E)) > \kappa$ implies that $\kappa \in \mathrm{in}(\mathcal{I}_{\mathrm{bd}}(E), \kappa)$.

Lemma 4.3. *Let κ be a regular cardinal. Suppose C and E are two directed sets such that $\mathrm{cf}(C) < \kappa \leq \mathrm{cf}(E)$ and $\kappa \in \mathrm{in}(\mathcal{I}_{\mathrm{bd}}(E), \kappa)$. Then there is no directed set D such that $C \times E <_T D <_T C \times \kappa \times E$.*

Proof. Suppose D is a directed set such that $C \times E \leq_T D \leq_T C \times \kappa \times E$, we will show that either $D \equiv_T C \times E$ or $D \equiv_T C \times \kappa \times E$. We may assume that every subset of D of size $\mathrm{cf}(D)$ is unbounded and $|C| = \mathrm{cf}(C)$. By Lemma 2.12, we have that $\mathrm{cf}(E) = \mathrm{cf}(D)$.

Suppose first there exists some unbounded subset $X \in [D]^\kappa$ such that every subset $Y \in [X]^{<\kappa}$ is bounded. By Corollary 2.10, this implies that $\kappa \leq_T D$. But as $C \times E \leq_T D$ and $D \leq_T C \times \kappa \times E$, this implies that $C \times \kappa \times E \equiv_T D$ as sought.

Hereafter, suppose for every unbounded subset $X \in [D]^\kappa$ there exists some subset $Y \in [X]^{<\kappa}$ unbounded. Let $g : D \to C \times \kappa \times E$ be a Tukey function. Define $h := \pi_{C\times E} \circ g$. Now, there are two main cases to consider:

- ▶ Suppose every unbounded subset $X \subseteq D$ of size $\lambda > \kappa$ which contain no unbounded subset of smaller cardinality is such that h"X is unbounded in $C \times E$.

  We show that h is Tukey, it is enough to verify that for every cardinal $\omega \leq \mu \leq \kappa$ and every unbounded subset $X \subseteq D$ of size μ which contain no unbounded subset of smaller cardinality is such that h"X is unbounded in $C \times E$.

  As g is Tukey, the set g"X is unbounded in $C \times \kappa \times E$. Notice that if the set $\pi_{C\times E} \circ g$"X is unbounded, then we are done. Assume that $\pi_{C\times E} \circ g$"X is bounded, then $\pi_\kappa \circ g$"X is unbounded.

  ▶▶ Suppose $|X| < \kappa$. As $|g"X| < \kappa$, we have that $\pi_\kappa \circ g$"X is bounded, which is absurd.

  ▶▶ Suppose $|X| = \kappa$, by the case assumption there exists some $Y \in [X]^{<\kappa}$ unbounded in D. But this is absurd as the assumption on X was that X contains no subset of size smaller than $|X|$ which is unbounded.

  ▶▶ Suppose $|X| > \kappa$, by the case assumption, h"X is unbounded in $C \times E$ as sought.
- ▶ Suppose for some unbounded subset $X \subseteq D$ of size $\lambda > \kappa$ which contains no unbounded subset of smaller cardinality is such that h"X is bounded in $C \times E$. As g is Tukey, $\pi_\kappa \circ g$"X is unbounded.

  Let $X_\alpha := X \cap g^{-1}(C \times \{\alpha\} \times E)$ and $U_\alpha := \bigcup_{\beta\leq\alpha} X_\beta$ for every $\alpha < \kappa$. As g is Tukey and g"U_α is bounded, we get that U_α is also bounded by some $y_\alpha \in D$. Let $Y := \{y_\alpha \mid \alpha < \kappa\}$. We claim that Y is of cardinality κ. If it wasn't, then by the pigeonhole principle as κ is regular there would be some $\alpha < \kappa$ such that y_α bounds the set X in D and that is absurd. Similarly, as X is unbounded, the set Y and also every subset of it of size κ must be unbounded.

  Next, we aim to get $Z \in [Y]^\kappa$ such that $\pi_{C\times E} \circ g$"Z is bounded by some $(c, e) \in C \times E$. This can be done as follows: As $|C| < \kappa$ and κ is regular, by the pigeonhole principle, there exists some $Z_0 \in [Y]^\kappa$ and $c \in C$ such that g"$Z_0 \subseteq$

https://doi.org/10.1017/jsl.2023.32 Published online by Cambridge University Press

$\{c\} \times \kappa \times E$. Similarly, if $|\pi_E \circ g\text{"}Z_0| < \kappa$, by the pigeonhole principle, there exists some $Z \in [Z_0]^\kappa$ and $e \in E$ such that $g\text{"}Z \subseteq \{c\} \times \kappa \times \{e\}$. Else, if $|\pi_E\text{"}Z_0| = \kappa$, then as $\kappa \in \mathrm{in}(\mathcal{I}_{\mathrm{bd}}(E), \kappa)$ for some $B \in [\pi_E \circ g\text{"}Z_0]^\kappa$ and $e \in E$, B is bounded in E by e. Fix some $Z \in [Z_0]^\kappa$ such that $g\text{"}Z \subseteq \{c\} \times \kappa \times B$.

Note that Z is a subset of Y of size κ, hence, unbounded in D. By the assumption, there exists some subset $W \in [Z]^{<\kappa}$ unbounded in D. Note that as κ is regular, for some $\alpha < \kappa$, $\pi_\kappa \circ g\text{"}W \subseteq \alpha$. As g is Tukey, the subset $g\text{"}W \subseteq \{c\} \times \kappa \times E$ is unbounded in $C \times \kappa \times E$, but this is absurd as $g\text{"}W$ is bounded by (c, α, e). ⊣

§5. Non-empty intervals. In this section we consider three types of intervals in the poset $(\mathcal{T}_n, <_T)$ and show each one can consistently have a directed set inside.

5.1. Directed set between $\theta^+ \times \theta^{++}$ and $[\theta^{++}]^{\leq\theta}$. In [6, Theorem 1.1], the authors constructed a directed set between $\omega_1 \times \omega_2$ and $[\omega_2]^{\leq\omega}$ under the assumption $2^{\aleph_0} = \aleph_1$, $2^{\aleph_1} = \aleph_2$ and the existence of an $\aleph_2$-Souslin tree. In this subsection we generalize this result while waiving the assumption concerning the Souslin tree. The main corollary of this subsection is:

Corollary 5.1. *Assume θ is an infinite cardinal such that $2^\theta = \theta^+$, $2^{\theta^+} = \theta^{++}$. Suppose C and E are directed sets such that* $\mathrm{cf}(C) \leq \theta^+$ *and either* $\mathrm{non}(\mathcal{I}_{\mathrm{bd}}(E)) > \theta^+$ *or $E \equiv_T 1$. Then there exists a directed set D such that $C \times \theta^+ \times \theta^{++} \times E <_T C \times D \times E <_T C \times [\theta^{++}]^{\leq\theta} \times E$.*

The result follows immediately from Theorems 5.3 and 5.4. First, we prove the following required lemma.

Lemma 5.2. *Suppose θ is a infinite cardinal and D, J, E are three directed sets such that*:

- $\mathrm{cf}(D) = \mathrm{cf}(J) = \theta^{++}$;
- $\theta^+ \in \mathrm{in}(\mathcal{I}_{\mathrm{bd}}(D), \theta^{++})$ *and* $\mathrm{out}(\mathcal{I}_{\mathrm{bd}}(J)) \leq \theta^+$;
- $\mathrm{non}(\mathcal{I}_{\mathrm{bd}}(E)) > \theta^+$ *or* $E \equiv_T 1$;
- $D \times E \leq_T J \times E$.

Then $J \times E \not\leq_T D \times E$. In particular, $D \times E <_T J \times E$.

Proof. Notice that D is a directed set such that every subset of size θ^{++} contains a bounded subset of size θ^+. Let us fix a cofinal subset $A \subseteq J$ of size θ^{++} such that every subset of A of size $> \theta$ is unbounded in J.

Suppose on the contrary that $J \times E \leq_T D \times E$. As $D \times E \leq_T J \times E$ we get that $D \times E \equiv_T J \times E$, hence there exists some directed set X such that both $D \times E$ and $J \times E$ are cofinal subsets of X.

We may assume that D has an enumeration $D := \{d_\alpha \mid \alpha < \theta^{++}\}$ such that for every $\beta < \alpha < \theta^{++}$ we have $d_\alpha \not< d_\beta$. Fix some $e \in E$. Now, for each $a \in A$ take a unique $x_a \in D$ and some $e_a \in E$ such that $(a, e) \leq_X (x_a, e_a)$. To do that, enumerate $A = \{a_\alpha \mid \alpha < \theta^{++}\}$. Suppose we have constructed already the increasing sequence $\langle v_\beta \mid \beta < \alpha\rangle$ of elements in θ^{++}. Pick some $\xi < \theta^{++}$ above $\{v_\beta \mid \beta < \alpha\}$. As $D \times E$ is a directed set we may fix some $(x_{a_\alpha}, e_a) := (d_{v_\alpha}, e_a) \in D \times E$ above (a_α, e) and (d_ξ, e).

https://doi.org/10.1017/jsl.2023.32 Published online by Cambridge University Press

Set $T = \{x_a \mid a \in A\}$, since $A \times E$ is cofinal in X, the set $T \times E$ is also cofinal in X and $D \times E$. As $|T| = \theta^{++}$ we get that there exists some subset $B \in [T]^{\theta^+}$ bounded in D. Let $c \in D$ be such that $b \leq c$ for each $b \in B$. Consider the set $K = \{a \in A \mid x_a \in B\}$. Since either $\mathrm{non}(\mathcal{I}_{\mathrm{bd}}(E)) > \theta^+$ or $E \equiv_T 1$, as $\{e_a \mid a \in K\}$ is of size $\leq \theta^+$, it is bounded in E by some $\tilde{e} \in E$. So $P := \{(x_a, e_a) \mid a \in K\}$ is bounded in X. Since B is of size $> \theta$, the set K is also of size $> \theta$. Thus, by the assumption on A, the set $K \times \{e\}$ is unbounded in $J \times E$, but also in X because $J \times E$ is a cofinal subset of X. Then, for each $a \in K$ we have $(a, e) \leq_X (x_a, e_a) \leq_X (c, \tilde{e})$, contradicting the unboundedness of $K \times \{e\}$ in $J \times E$. ⊣

THEOREM 5.3. *Suppose θ is an infinite cardinal and C, D, E are directed sets such that*:

(1) $\mathrm{cf}(D) = \theta^{++}$.
(2) *For every partition $D = \bigcup_{\gamma<\theta^+} D_\gamma$, there is an ordinal $\gamma < \theta^+$, and an unbounded $K \subseteq D_\gamma$ of size θ^+.*
(3) $\theta^+ \in \mathrm{in}(\mathcal{I}_{\mathrm{bd}}(D), \theta^{++})$ *and* $\mathrm{non}(\mathcal{I}_{\mathrm{bd}}(D)) = \theta^+$.
(4) $\mathrm{non}(\mathcal{I}_{\mathrm{bd}}(E)) > \theta^+$ *or* $E \equiv_T 1$.
(5) *C is a directed set such that* $\mathrm{cf}(C) \leq \theta^+$.

Then $C \times \theta^+ \times \theta^{++} \times E <_T C \times D \times E <_T C \times [\theta^{++}]^{\leq\theta} \times E$.

PROOF. As $\mathrm{cf}(D) = \theta^{++}$, we may assume that every subset of D of size θ^{++} is unbounded. ⊣

CLAIM 5.3.1. $\theta^+ \times \theta^{++} \leq_T D$.

PROOF. As $\mathrm{cf}(D) = \theta^{++}$, we get by Lemma 2.9 that $\theta^{++} \leq_T D$. Let K be an unbounded subset of D of size θ^+, as every subset of size θ is bounded, by Corollary 2.10 we get that $\theta^+ \leq_T D$. Hence, $\theta^+ \times \theta^{++} \leq_T D$ as sought. ⊣

CLAIM 5.3.2. $D \leq_T [\theta^{++}]^{\leq\theta}$.

PROOF. As $\mathrm{cf}(D) = \theta^{++}$ and $\mathrm{non}(\mathcal{I}_{\mathrm{bd}}(D)) = \theta^+$, by Lemma 2.13, $D \leq_T [\theta^{++}]^{\leq\theta}$ as sought. ⊣

Notice this implies that $C \times \theta^+ \times \theta^{++} \times E \leq_T C \times D \times E \leq_T C \times [\theta^{++}]^{\leq\theta} \times E$.

By Lemma 2.17, as $|C \times \theta^+| \leq \theta^+$, $\mathrm{non}(\mathcal{I}_{\mathrm{bd}}(\theta^{++} \times E)) > \theta^+$ and Clause (2) we get that $D \not\leq_T C \times \theta^+ \times \theta^{++} \times E$.

CLAIM 5.3.3. $C \times [\theta^{++}]^{\leq\theta} \times E \not\leq_T C \times D \times E$.

PROOF. Recall that $\mathfrak{D}_{[\theta^{++}]^{\leq\theta}} \equiv_T [\theta^{++}]^{\leq\theta}$. Notice that following:

- $\mathrm{cf}(C \times D) = \mathrm{cf}(C \times \mathfrak{D}_{[\theta^{++}]^{\leq\theta}}) = \theta^{++}$.
- By Clause (3) we have that $\theta^+ \in \mathrm{in}(\mathcal{I}_{\mathrm{bd}}(C \times D), \theta^{++})$ and $\mathrm{out}(\mathcal{I}_{\mathrm{bd}}(C \times \mathfrak{D}_{[\theta^{++}]^{\leq\theta}})) \leq \theta^+$.
- $\mathrm{non}(\mathcal{I}_{\mathrm{bd}}(E)) > \theta^+$ or $E = 1$.
- $C \times D \times E \leq_T C \times \mathfrak{D}_{[\theta^{++}]^{\leq\theta}} \times E$.

So by Lemma 5.2 we are done. ⊣

We are left with proving the following theorem, in which we define a directed set D_c using a coloring c.

https://doi.org/10.1017/jsl.2023.32 Published online by Cambridge University Press

THEOREM 5.4. *Suppose θ is an infinite cardinal such that $2^\theta = \theta^+$ and $2^{\theta^+} = \theta^{++}$. Then there exists a directed set D such that*:

(1) $\mathrm{cf}(D) = \theta^{++}$.
(2) *For every partition $D = \bigcup_{\gamma<\theta^+} D_\gamma$, there is an ordinal $\gamma < \theta^+$, and an unbounded $K \subseteq D_\gamma$ of size θ^+.*
(3) $\theta^+ \in \mathrm{in}(\mathcal{I}_{\mathrm{bd}}(D), \theta^{++})$ *and* $\mathrm{non}(\mathcal{I}_{\mathrm{bd}}(D)) = \theta^+$.

The rest of this subsection is dedicated to proving Theorem 5.4. The arithmetic hypothesis will only play a role later on. Let θ be an infinite cardinal. For two sets of ordinals A and B, we denote $A \circledast B := \{(\alpha, \beta) \in A \times B \mid \alpha < \beta\}$. Recall that by [3, Corollary 7.3], $\mathsf{onto}(\mathcal{S}, J^{\mathrm{bd}}[\theta^{++}], \theta^+)$ holds for $\mathcal{S} := [\theta^{++}]^{\theta^{++}}$. This means that we may fix a coloring $c : [\theta^{++}]^2 \to \theta^+$ such that for every $S \in \mathcal{S}$ and unbounded $B \subseteq \theta^{++}$, there exists $\delta \in S$ such that $c``(\{\delta\} \circledast B) = \theta^+$.

We fix some $S \in \mathcal{S}$. For our purpose, it will suffice to assume that S is nothing but the whole of θ^{++}. Let

$$D_c := \{X \in [\theta^{++}]^{\leq\theta^+} \mid \forall\delta \in S[\{c(\delta, \beta) \mid \beta \in X \setminus (\delta + 1)\} \in \mathrm{NS}_{\theta^+}]\}.$$

Consider D_c ordered by inclusion, and notice that D_c is a directed set since NS_{θ^+} is an ideal.

PROPOSITION 5.5. *The following hold*:

- $[\theta^{++}]^{\leq\theta} \subseteq D_c \subseteq [\theta^{++}]^{\leq\theta^+}$.
- $\mathrm{non}(\mathcal{I}_{\mathrm{bd}}(D_c)) \geq \mathrm{add}(\mathcal{I}_{\mathrm{bd}}(D_c)) \geq \theta^+$, *i.e., every family of bounded subsets of D_c of size $< \theta^+$ is bounded.*
- *If $2^{\theta^+} = \theta^{++}$, then $|D_c| = \theta^{++}$, and hence $D_c \in \mathcal{D}_{\theta^{++}}$.*

LEMMA 5.6. *For every partition $D_c = \bigcup_{\gamma<\theta^+} D_\gamma$, there is an ordinal $\gamma < \theta^+$, and an unbounded $E \subseteq D_\gamma$ of size θ^+.*

PROOF. As $[\theta^{++}]^1$ is a subset of D_c, the family $\{D_\gamma \mid \gamma < \theta^+\}$ is a partition of the set $[\theta^{++}]^1$ to at most θ^+ many sets. As $\theta^+ < \theta^{++} = \mathrm{cf}(\theta^{++})$, by the pigeonhole principle we get that for some $\gamma < \theta^+$ and $b \in [\theta^{++}]^{\theta^{++}}$, we have $[b]^1 \subseteq D_\gamma$. Notice that by the assumption on the coloring c, there exists some $\delta \in S$ and $\delta < b' \in [b]^{\theta^+}$ such that $c``(\delta \circledast b') = \theta^+$. Clearly the set $E := [b']^1$ is a subset of D_γ of size θ^+ which is unbounded in D_c. ⊣

LEMMA 5.7. *Suppose $2^\theta = \theta^+$, then $\theta^+ \in \mathrm{in}(\mathcal{I}_{\mathrm{bd}}(D_c), \theta^{++})$.*

PROOF. We follow the proof of [6, Lemma 5.4].

Let D' be a subset of D_c of size θ^{++} we will show it contains a bounded subset of size θ^+, let us enumerate it as $\{T_\gamma \mid \gamma < \theta^{++}\}$. Let, for each $X \in D_c$ and $\gamma \in S$, N^X_γ denote the non-stationary set $\{c(\gamma, \beta) \mid \beta \in X \setminus (\gamma + 1)\}$, and let G^X_γ denote a club in θ^+ disjoint from N^X_γ.

As $2^\theta = \theta^+$ we may fix a sufficiently large regular cardinal χ, and an elementary submodel $M \prec H_\chi$ of cardinality θ^+ containing all the relevant objects and such that $M^\theta \subseteq M$. Denote $\delta = M \cap \theta^{++}$, notice $\delta \in E^{\theta^{++}}_{\theta^+}$. Fix an increasing sequence $\langle\gamma_\xi \mid \xi < \theta^+\rangle$ in δ such that $\sup\{\gamma_\xi \mid \xi < \theta^+\} = \delta$. Enumerate $\delta \cap S = \{s_\xi \mid \xi < \theta^+\}$. In order to simplify notation, let G^γ_ξ denote the set $G^{T_\gamma}_{s_\xi}$ for each $\gamma < \theta^{++}$ and $\xi < \theta^+$.

https://doi.org/10.1017/jsl.2023.32 Published online by Cambridge University Press

We construct by recursion on $\xi < \theta^+$ three sequences $\langle \delta_\xi \mid \xi < \theta^+\rangle$, $\langle \Gamma_\xi \mid \xi < \theta^+\rangle$ and $\langle \eta_\xi \mid \xi < \theta^+\rangle$ with the following properties:

(1) $\langle \delta_\xi \mid \xi < \theta^+\rangle$ is an increasing sequence converging to δ.
(2) $\langle \Gamma_\xi \mid \xi < \theta^+\rangle$ is a decreasing $\subseteq$-chain of stationary subsets of θ^{++} each one containing δ and definable in M.
(3) $\langle \eta_\xi \mid \xi < \theta^+\rangle$ is an increasing sequence of ordinals below θ^+.
(4) $G^\delta_\zeta \cap \eta_\mu = G^{\delta_\mu}_\zeta \cap \eta_\mu$ for $\zeta \leq \mu < \theta^+$.

► Base case: Let η_0 be the first limit point of G^δ_0. Notice that $G^\delta_0 \cap \eta_0$ is an infinite set of size $\leq \theta$ below δ, hence it is inside of M. Let

$$\Gamma_0 := \{\gamma < \theta^{++} \mid G^\delta_0 \cap \eta_0 = G^\gamma_0 \cap \eta_0\}.$$

Since $\delta \in \Gamma_0$, the set Γ_0 is stationary in θ^{++}. Let $\delta_0 := \min(\Gamma_0)$.

► Suppose $\xi_0 < \theta^+$, and that δ_ξ, Γ_ξ and η_ξ have been constructed for each $\xi < \xi_0$. Let η_{ξ_0} be the first limit point of $G^\delta_{\xi_0} \setminus \sup\{\eta_\xi \mid \xi < \xi_0\}$. Consider the set

$$\Gamma_{\xi_0} = \{\gamma \in \bigcap_{\xi<\xi_0} \Gamma_\xi \mid \forall \xi \leq \xi_0 [G^\delta_\xi \cap \eta_{\xi_0} = G^\gamma_\xi \cap \eta_{\xi_0}]\}.$$

Since Γ_{ξ_0} belongs to M, and since $\delta \in \Gamma_{\xi_0}$, it must be that Γ_{ξ_0} is stationary in θ^{++}. Since Γ_{ξ_0} is cofinal in θ^{++} and belongs to M, the set $\delta \cap \Gamma_{\xi_0}$ is cofinal in δ. Define δ_{ξ_0} be the minimal ordinal in $\delta \cap \Gamma_{\xi_0}$ greater than both $\sup\{\delta_\xi \mid \xi < \xi_0\}$ and γ_{ξ_0}. It is clear from the construction that conditions $(1-4)$ are satisfied. ⊣

The following claim gives us the wanted result.

Claim 5.7.1. *The set $\{T_{\delta_\xi} \mid \xi < \theta^+\}$ is a subset of D' of size θ^+ which is bounded in D_c.*

Proof. As the order on D_c is $\subseteq$, it suffices to prove that the union $T = \bigcup_{\xi<\theta^+} T_{\delta_\xi} \in D_c$. Since, for each $\xi < \theta^+$, both δ_ξ and $\langle T_\gamma \mid \gamma < \theta^{++}\rangle$ belong to M, it must be that $T_{\delta_\xi} \in M$. Since $\theta^+ \in M$ and $M \models |T_{\delta_\xi}| \leq \theta^+$, we have $T_{\delta_\xi} \subseteq M$. Thus $T \subseteq M$ and furthermore $T \subseteq \delta$. This means that, in order to prove that $T \in D_c$, it is enough to prove that for each $t \in S \cap \delta$, the set $\{c(t,\beta) \mid \beta \in T \setminus (t+1)\}$ is non-stationary in θ^+. Fix some $t \in S \cap \delta$. Let $\zeta < \theta^+$ be such that $s_\zeta = t$. Define

$$G := G^\delta_\zeta \cap (\bigcap_{\xi \leq \zeta} G^{\delta_\xi}_\zeta) \cap (\triangle_{\xi<\theta^+} G^{\delta_\xi}_\zeta).$$

Since the intersection of $<\theta^+$-many clubs in θ^+ is a club, and since diagonal intersection of θ^+ many clubs is a club, we know that G is a club in θ^+.

We will prove that $G \cap \{c(t,\beta) \mid \beta \in T \setminus (t+1)\} = \emptyset$. Suppose $\alpha < \theta^+$ is such that $\alpha \in G \cap \{c(t,\beta) \mid \beta \in T \setminus (t+1)\}$. This means that $\alpha \in G$ and that for some $\mu < \theta^+$ and $\beta \in T_{\delta_\mu} \setminus (t+1)$ we have $\alpha = c(t,\beta)$. So $\alpha \in N^{T_{\delta_\mu}}_t$. Note that this implies that $\alpha \notin G^{\delta_\mu}_\zeta$. Let us split to three cases:

► Suppose $\mu \leq \zeta$, then since $\alpha \in \bigcap_{\xi\leq\zeta} G^{\delta_\xi}_\zeta$, we have that $\alpha \in G^{\delta_\mu}_\zeta$ which is clearly contradicting $\alpha \notin G^{\delta_\mu}_\zeta$.

https://doi.org/10.1017/jsl.2023.32 Published online by Cambridge University Press

▶ Suppose $\mu > \zeta$ and $\alpha < \eta_\mu$. Then by (4), we have that $G^\delta_\zeta \cap \eta_\mu = G^{\delta_\mu}_\zeta \cap \eta_\mu$. As $\alpha \notin G^{\delta_\mu}_\zeta$ and $\alpha < \eta_\mu$, it must be that $\alpha \notin G^\delta_\zeta$. Recall that $\alpha \in G$, but this is absurd as $G \subseteq G^\delta_\zeta$ and $\alpha \notin G^\delta_\zeta$.

▶ Suppose $\mu > \zeta$ and $\alpha \geq \eta_\mu \geq \mu$. As $\alpha \in G$, we have that $\alpha \in \triangle_{\xi<\theta^+} G^{\delta_\xi}_\zeta$. As $\alpha > \mu$, we get that $\alpha \in G^{\delta_\mu}_\zeta$ which is clearly contradicting $\alpha \notin G^{\delta_\mu}_\zeta$. ⊣

5.2. Directed set between $\omega \times \omega_1$ and $[\omega_1]^{<\omega}$. As mentioned in [8], by the results of Todorčević [13], it follows that under the assumption $\mathfrak{b} = \omega_1$ there exists a directed set of size ω_1 between the directed sets $\omega \times \omega_1$ and $[\omega_1]^{<\omega}$. In this subsection we spell out the details of this construction.

For two functions $f, g \in {}^\omega\omega$, we define the order $<^*$ by $f <^* g$ iff the set $\{n < \omega \mid g(n) \geq f(n)\}$ is finite. Furthermore, by $f \lhd g$ we means that there exists $m < \omega$ such that for all $n < m$ we have $f(n) \leq g(n)$ and $f(k) < g(k)$ whenever $m \leq k < \omega$. Assuming $f \leq^* g$, we let $\Delta(f, g) := \min\{m < \omega \mid \forall n \geq m[f(n) \leq g(n)]\}$.

The following fact is a special case of [13, Theorem 1.1] in the case $n = 0$, for complete details we give the proof as suggested by the referee.

Fact 5.8 (Todorčević [13, Theorem 1.1]). *Suppose A is an uncountable sequence of ${}^\omega\omega$ of increasing functions which are $<^*$-increasing and $\leq^*$-unbounded, then there are $f, g \in A$ such that $f \lhd g$.*

Proof. Let $A := \{g_\alpha \mid \alpha < \omega_1\}$ be an uncountable sequence of increasing functions of ${}^\omega\omega$ which are $<^*$-increasing and $\leq^*$-unbounded.

Let us fix a countable elementary sub-model $M \prec (H_{\omega_2}, \in)$ with $A \in M$. Let $\delta := \omega_1 \cap M$, $B := \omega_1 \setminus (\delta + 1)$ and write $B_n := \{\beta \in B \mid \Delta(g_\delta, g_\beta) = n\}$. As $B = \bigcup_{n<\omega} B_n$, let us fix some $n < \omega$ such that B_n is uncountable. As $\{g_\alpha \mid \alpha \in B_n\}$ is unbounded, we get that the set $K := \{m < \omega \mid \sup\{g_\beta(m) \mid \beta \in B_n\} = \omega\}$ is non-empty, so consider the minimal element, $m := \min(K)$. For $t \in {}^m\omega$, denote $B^t_n := \{\beta \in B_n \mid t \subseteq g_\beta\}$. By minimality of m, the set $\{t \in {}^m\omega \mid B^t_n \neq \emptyset\}$ is finite, so we can easily find some $t \in {}^m\omega$ such that $\sup\{g_\beta(m) \mid \beta \in B^t_n\} = \omega$.

Note that the set $\{\beta < \omega_1 \mid t \subseteq g_\beta\}$ is a non-empty set that is definable from A and t, hence it is in M. Let us fix some $\alpha \in M \cap \omega_1$ such that $t \subseteq g_\alpha$. Put $k := \Delta(g_\alpha, g_\delta)$, and then pick $\beta \in B^t_n$ such that $g_\beta(m) > g_\alpha(k + n)$. Of course, $\alpha < \delta < \beta$. We claim that $g_\alpha \lhd g_\beta$ as sought.

Let us divide to three cases:

- If $i < m$, then $g_\alpha(i) = t(i) = g_\beta(i)$.
- If $m \leq i \leq k + n$, then $g_\alpha(i) \leq g_\alpha(k + n) < g_\beta(m) \leq g_\beta(i)$ recall that every function in A is increasing.
- If $k + n < i < \omega$, then $\Delta(g_\alpha, g_\delta) = k < i$ and $g_\alpha(i) \leq g_\delta(i)$, as well as $\Delta(g_\delta, g_\beta) = n < i$ and $g_\delta(i) \leq g_\beta(i)$. Altogether, $g_\alpha(i) \leq g_\beta(i)$. ⊣

Theorem 5.9. *Assume $\mathfrak{b} = \omega_1$. Suppose E is a directed set such that* $\mathrm{non}(\mathcal{I}_{\mathrm{bd}}(E)) > \omega$ *or $E \equiv_T 1$. Then there exists a directed set D such that*

$$\omega \times \omega_1 \times E <_T D \times E <_T [\omega_1]^{<\omega} \times E.$$

https://doi.org/10.1017/jsl.2023.32 Published online by Cambridge University Press

PROOF. Let $\mathcal{F} := \langle f_\alpha \mid \alpha < \omega_1 \rangle \subseteq {}^\omega\omega$ witness $\mathfrak{b} = \omega_1$. Recall $\mathcal{F}$ is a $<^*$-increasing and unbounded sequence, i.e., for every $g \in {}^\omega\omega$, there exists some $\alpha < \omega_1$ such that $f_\beta \not\leq^* g$, whenever $\alpha < \beta < \omega_1$.

For a finite set of functions $F \subseteq {}^\omega\omega$, we define a function $h := \max(F)$ which is $\lhd$-above every function in F by letting $h(n) := \max\{f(n) \mid f \in F\}$. We consider the directed set $D := \{\max(F) \mid F \subseteq \mathcal{F},\ |F| < \aleph_0\}$, ordered by the relation $\lhd$, clearly D is a directed set. ⊣

CLAIM 5.9.1. *Every uncountable subset $X \subseteq D$ contains a countable $B \subset X$ which is unbounded in D.*

PROOF. Let X be an uncountable subset of D. As $\mathcal{F}$ is a $<^*$-increasing and unbounded, also X contains an uncountable $<^*$-unbounded subset $Y \subseteq X$. As no function $g : \omega \to \omega$ is $<^*$-bounding the set Y, we can find an infinite countable subset $B \subseteq Y$ and $n < \omega$ such that $\{f(n) \mid f \in B\}$ is infinite. Clearly B is $\lhd$-unbounded in D as sought. ⊣

CLAIM 5.9.2. $\omega \in \mathrm{in}(\mathcal{I}_{\mathrm{bd}}(D), \omega_1)$.

PROOF. We show that every uncountable subset of D contains a countable infinite bounded subset. Let $A \subseteq D$ be an uncountable set, we may refine A and assume that it is $<^*$-increasing and unbounded. We enumerate $A := \{g_\alpha \mid \alpha < \omega_1\}$ and define a coloring $c : [\omega_1]^2 \to 2$, letting for $\alpha < \beta < \omega_1$ the color $c(\alpha, \beta) = 1$ iff $g_\alpha \lhd g_\beta$. Recall that Erdös and Rado showed that $\omega_1 \to (\omega_1, \omega + 1)^2$, so either there is an uncountable homogeneous set of color 0 or there exists an homogeneous set of color 1 of order-type $\omega + 1$. Notice that Fact 5.8 contradicts the first alternative, so the second one must hold. Let $X \subseteq \omega_1$ be a set such that $\mathrm{otp}(X) = \omega + 1$ and $c``[X]^2 = \{1\}$, notice that $\{g_\alpha \mid \alpha \in X\}$ is an infinite countable subset of A which is $\lhd$-bounded by the function $g_{\max(X)} \in A$ as sought.here ⊣

Note that $\mathrm{cf}(D) = \omega_1$, hence $D \times E \leq_T [\omega_1]^{<\omega} \times E$.

CLAIM 5.9.3. $\omega \times \omega_1 \times E \leq_T D \times E$.

PROOF. As every subset of D of size ω_1 is unbounded, we get by Lemma 2.9 that $\omega_1 \leq_T D$. As D is a directed set, every finite subset of D is bounded. By Claim 5.9.1, D contains an infinite countable unbounded subset, so by Corollary 2.10 we have $\omega \leq_T D$. Finally, $\omega \times \omega_1 \leq_T D$ as sought. ⊣

CLAIM 5.9.4. $D \not\leq_T \omega \times E$.

PROOF. Recall that either $\mathrm{non}(\mathcal{I}_{\mathrm{bd}}(E)) > \omega$ or $E \equiv_T 1$. Note that if $E \equiv_T 1$, then as $\mathrm{cf}(D) = \omega_1 > \mathrm{cf}(\omega)$, we have by Lemma 2.12 that $D \not\leq_T \omega \times E$ as sought. Note that for every partition $D = \bigcup\{D_n \mid n < \omega\}$ of D, there exists some $n < \omega$ such that D_n is uncountable, and by Claim 5.9.1, there exists some $X \subseteq D_n$ infinite and unbounded in D. As $\mathrm{non}(\mathcal{I}_{\mathrm{bd}}(E)) > \omega$, by Lemma 2.17 we have $D \not\leq_T \omega \times E$ as sought. ⊣

CLAIM 5.9.5. $[\omega_1]^{<\omega} \not\leq_T D \times E$.

PROOF. By Claim 5.9.2, every uncountable subset of D contains an infinite countable bounded subset and every countable subset of E is bounded, we get that $\omega \in \mathrm{in}(\mathcal{I}_{\mathrm{bd}}(D \times E), \omega_1)$. As $\mathrm{out}(\mathcal{I}_{\mathrm{bd}}([\omega_1]^{<\omega})) = \omega$ by Lemma 2.15 we get that $[\omega_1]^{<\omega} \not\leq_T D \times E$ as sought. ⊣

5.3. Directed set between $[\lambda]^{<\theta} \times [\lambda^+]^{\leq\theta}$ and $[\lambda^+]^{<\theta}$. In [6, Theorem 1.2], the authors constructed a directed set between $[\omega_1]^{<\omega} \times [\omega_2]^{\leq\omega}$ and $[\omega_2]^{<\omega}$ under the assumption $2^{\aleph_0} = \aleph_1$, $2^{\aleph_1} = \aleph_2$ and the existence of a non-reflecting stationary subset of $E^{\omega_2}_{\omega}$. In this subsection we generalize this result while waiving the assumption concerning the non-reflecting stationary set.

We commence by recalling some classic guessing principles and introducing a weak one, named $\clubsuit^{\mu}_{J}(S,1)$, which will be useful for our construction.

DEFINITION 5.10. For a stationary subset $S \subseteq \kappa$:

(1) $\diamondsuit(S)$ asserts the existence of a sequence $\langle C_\alpha \mid \alpha \in S\rangle$ such that:
- for all $\alpha \in S$, $C_\alpha \subseteq \alpha$;
- for every $B \subseteq \kappa$, the set $\{\alpha \in S \mid B \cap \alpha = C_\alpha\}$ is stationary.

(2) $\clubsuit(S)$ asserts the existence of a sequence $\langle C_\alpha \mid \alpha \in S\rangle$ such that:
- for all $\alpha \in S \cap \mathrm{acc}(\kappa)$, C_α is a cofinal subset of α of order type $\mathrm{cf}(\alpha)$;
- for every cofinal subset $B \subseteq \kappa$, the set $\{\alpha \in S \mid C_\alpha \subseteq B\}$ is stationary.

(3) $\clubsuit^{\mu}_{J}(S,1)$ asserts the existence of a sequence $\langle C_\alpha \mid \alpha \in S\rangle$ such that:
- for all $\alpha \in S \cap \mathrm{acc}(\kappa)$, C_α is a cofinal subset of α of order type $\mathrm{cf}(\alpha)$;
- for every partition $\langle A_\beta \mid \beta < \mu\rangle$ of κ there exists some $\beta < \mu$ such that the set $\{\alpha \in S \mid \sup(C_\alpha \cap A_\beta) = \alpha\}$ is stationary.

Recall that by a Theorem of Shelah [10], for every uncountable cardinal λ which satisfy $2^\lambda = \lambda^+$ and every stationary $S \subseteq E^{\lambda^+}_{\neq\mathrm{cf}(\lambda)}$, $\diamondsuit(S)$ holds. It is clear that $\diamondsuit(S) \Rightarrow \clubsuit(S) \Rightarrow \clubsuit^{\lambda}_{J}(S,1)$. The main corollary of this subsection is:

COROLLARY 5.11. *Let $\theta < \lambda$ be two regular cardinals. Assume $\lambda^\theta < \lambda^+$ and $\clubsuit^{\lambda}_{J}(S,1)$ holds for some stationary $S \subseteq E^{\lambda^+}_{\theta}$. Suppose C and E are two directed sets such that $\mathrm{cf}(C) < \lambda^+$ and $\mathrm{non}(\mathcal{I}_{\mathrm{bd}}(E)) > \theta$ or $E \equiv_T 1$. Then there exists a directed set $D_{\mathcal{C}}$ such that*:

$$C \times [\lambda]^{<\theta} \times [\lambda^+]^{\leq\theta} \times E <_T C \times [\lambda]^{<\theta} \times D_{\mathcal{C}} \times E <_T C \times [\lambda^+]^{<\theta} \times E.$$

In the rest of this subsection we prove this result.

Suppose $\mathcal{C} := \langle C_\alpha \mid \alpha \in S\rangle$ is a C-sequence for some stationary set $S \subseteq E^{\lambda^+}_{\theta}$, i.e., C_α is a cofinal subset of α of order-type θ, whenever $\alpha \in S$. We define the directed set $D_{\mathcal{C}} := \{Y \in [\lambda^+]^{\leq\theta} \mid \forall\alpha \in S[|Y \cap C_\alpha| < \theta]\}$ ordered by $\subseteq$. Notice that $\mathrm{non}(\mathcal{I}_{\mathrm{bd}}(D_{\mathcal{C}})) = \theta$ and $[\lambda^+]^{<\theta} \subseteq D_{\mathcal{C}}$.

Recall that by Hausdorff's formula $(\lambda^+)^\theta = \max\{\lambda^+, \lambda^\theta\}$, so if $\lambda^\theta < \lambda^+$, then $(\lambda^+)^\theta = \lambda^+$. So we may assume $|D_{\mathcal{C}}| = \lambda^+$.

CLAIM 5.11.1. *Suppose $|D_{\mathcal{C}}| = \lambda^+$, then $[\lambda^+]^{\leq\theta} \leq_T D_{\mathcal{C}}$.*

PROOF. Fix a bijection $\phi : D_{\mathcal{C}} \to \lambda^+$. Denote $X := \{x \cup \{\phi(x)\} \mid x \in D_{\mathcal{C}}\}$, clearly X is cofinal subset of $D_{\mathcal{C}}$. Let us fix some injective function $g : [\lambda^+]^{\leq\theta} \to X$. We claim that g is a Tukey function, which witness that $[\lambda^+]^{\leq\theta} \leq_T D_{\mathcal{C}}$. Fix some $B \subseteq [\lambda^+]^{\leq\theta}$ unbounded in $[\lambda^+]^{\leq\theta}$, note that $|B| > \theta$. As g is injective, we get that $g``B$ is a set of size $> \theta$. Notice that there exists $Z \in [\lambda^+]^{\theta^+}$ such that $Z \subseteq \bigcup g``B$. Assume that $g``B$ is bounded by $d \in D_{\mathcal{C}}$ in $D_{\mathcal{C}}$. As $D_{\mathcal{C}}$ is ordered by $\subseteq$, we get that $Z \subseteq d$, so $|d| \geq \theta^+$. But this is a absurd as every set in $D_{\mathcal{C}}$ is of size $\leq \theta$. ⊣

https://doi.org/10.1017/jsl.2023.32 Published online by Cambridge University Press

Notice that by Lemma 2.13 and Claim 5.11.1, as $(\lambda^+)^\theta = \lambda^+$ we have $[\lambda]^{<\theta} \times [\lambda^+]^{\leq\theta} \leq_T [\lambda]^{<\theta} \times D_{\mathcal{C}} \leq_T [\lambda^+]^{<\theta}$. Hence, $C \times [\lambda]^{<\theta} \times [\lambda^+]^{\leq\theta} \times E \leq_T C \times [\lambda]^{<\theta} \times D_{\mathcal{C}} \times E \leq_T C \times [\lambda^+]^{<\theta} \times E$.

CLAIM 5.11.2. *Suppose $\mathcal{C}$ is a $\clubsuit^\lambda_J(S,1)$-sequence and:*

(i) *C is a directed set such that $|C| < \lambda^+$;*
(ii) *E is a directed set such that $\mathrm{non}(\mathcal{I}_{\mathrm{bd}}(E)) > \theta$ and $\mathrm{cf}(E) \geq \lambda^+$.*

Then $C \times D_{\mathcal{C}} \not\leq_T C \times E$.

PROOF. Suppose that $f : C \times D_{\mathcal{C}} \to C \times E$ is a Tukey function. Fix some $o \in C$ and for each $\xi < \lambda^+$, denote $(c_\xi, x_\xi) := f(o, \{\xi\})$. Consider the set $\{(c_\xi, x_\xi) \mid \xi < \lambda^+\}$. For every $c \in C$, we define $A_c := \{\xi < \lambda^+ \mid c_\xi = c\}$, clearly $\langle A_c \mid c \in C\rangle$ is a partition of λ^+ to less than λ^+ many sets.

As $\mathcal{C}$ is a $\clubsuit^\lambda_J(S,1)$-sequence, there exists some $c \in C$ and $\alpha \in S$ such that $|C_\alpha \cap A_c| = \theta$. Let us fix some $B \in [C_\alpha \cap A_c]^\theta$. Notice that the set $G := \{(o, \{\xi\}) \mid \xi \in B\}$ is unbounded in $C \times D_{\mathcal{C}}$, hence as f is Tukey, $f``G$ is unbounded in $C \times E$. The subset $\{x_\xi \mid \xi \in B\}$ of $E$ is of size $\theta$, hence bounded by some $e$. Note that $f``G = \{(c, x_\xi) \mid \xi \in B\}$ is bounded by (c, e) in $C \times E$ which is absurd. ⊣

By the previous claim, as $\lambda^\theta < \lambda^+$, we get that $C \times D_{\mathcal{C}} \times [\lambda]^{<\theta} \times E \not\leq_T C \times [\lambda]^{<\theta} \times [\lambda^+]^{\leq\theta} \times E$. The following claim gives a negative answer to the question of whether there is a C-sequence $\mathcal{C}$ such that $D_{\mathcal{C}} \equiv_T [\lambda^+]^{<\theta}$.

In the following claim we use the fact that the sets in the sequence $\mathcal{C}$ are of a bounded cofinality.

CLAIM 5.11.3. *Assume $\lambda^\theta < \lambda^+$. Suppose $S \subseteq E^{\lambda^+}_\theta$ is a stationary set and $\mathcal{C} := \langle C_\alpha \mid \alpha \in S\rangle$ is a C-sequence, then $D_{\mathcal{C}} \not\geq_T [\lambda^+]^{<\theta}$.*

PROOF. Let $S \subseteq E^{\lambda^+}_\theta$ and $\mathcal{C} := \langle C_\alpha \mid \alpha \in S\rangle$ be a C-sequence. Suppose we have $[\lambda^+]^{<\theta} \leq_T D_{\mathcal{C}}$, let $f : [\lambda^+]^{<\theta} \to D_{\mathcal{C}}$ be a Tukey function and $Y := f``[\lambda^+]^1$. Let us split to two cases:

► Suppose $|Y| < \lambda^+$. By the pigeonhole principle, we can find a subset $Q \subseteq [\lambda]^1$ of size θ such that $f``Q = \{x\}$ for some $x \in D_{\mathcal{C}}$. As $f$ is Tukey and $Q$ is unbounded in $[\lambda^+]^{<\theta}$, the set $f``Q$ is unbounded which is absurd.

► Suppose $|Y| = \lambda^+$. As f is Tukey, every subset of Y of size θ is unbounded which is absurd to the following claim. ⊣

SUBCLAIM 5.11.3.1. *There is no subset $Y \subseteq D_{\mathcal{C}}$ of size λ^+ such that every subset of Y of size θ is unbounded.*

PROOF. Assume towards a contradiction that Y is such a set. As $\lambda^\theta < \lambda^+$, we may refine Y and assume that $Y = \{y_\alpha \mid \alpha < \lambda^+\}$ is a Δ-system with a root R separated by a club $C \subseteq \lambda^+$, i.e., such that for every $\alpha < \beta < \lambda^+$, $y_\alpha \setminus R < \eta < y_\beta \setminus R$ for some $\eta \in C$.

We define an increasing sequence of ordinals $\langle \beta_\nu \mid \nu \leq \theta^2\rangle$ where for each $\nu \leq \theta^2$ we let $\beta_\nu := \sup\{y_\xi \mid \xi < \nu\}$. As C is a club, we get that $\beta_{\theta\cdot\nu} \in C$ for each $\nu < \theta$.

We aim to construct a subset $X = \{x_j \mid j < \theta\}$ of Y, we split to two cases: Suppose $\beta_{\theta^2} \in S$. Recall that $\mathrm{otp}(C_{\beta_{\theta^2}}) = \theta$ and $\sup(C_{\beta_{\theta^2}}) = \beta_{\theta^2}$, so for every $j < \theta$ we have that the interval $[\beta_{\theta\cdot j}, \beta_{\theta\cdot(j+1)})$ contains $< \theta$ many elements of the ladder $C_{\beta_{\theta^2}}$, let us

https://doi.org/10.1017/jsl.2023.32 Published online by Cambridge University Press

fix some $x_j \in Y$ such that $x_j \setminus R \subset [\beta_{\theta\cdot j}, \beta_{\theta\cdot(j+1)})$ and $x_j \setminus R$ is disjoint from $C_{\beta_{\theta^2}}$. If $\beta_{\theta^2} \notin S$, define $X := \{x_j \mid j < \theta\}$ where $x_j := y_{\theta\cdot j}$.

Let us show that $X = \{x_j \mid j < \theta\}$ is a bounded subset of Y, which is a contradiction to the assumption. It is enough to show that for every $\alpha \in S$, we have that $|(\bigcup X) \cap C_\alpha| < \theta$. Let $\alpha \in S$.

► Suppose $\alpha > \beta_{\theta^2}$, as C_α is a cofinal subset of α of order-type θ and $\bigcup X$ is bounded by β_{θ^2} it is clear that $|(\bigcup X) \cap C_\alpha| < \theta$.

► Suppose $\alpha < \beta_{\theta^2}$. As C_α if cofinal in α and of order-type θ, there exists some $j < \theta$ such that for all $j < \rho < \theta$, we have $(x_\rho \setminus R) \cap C_\alpha = \emptyset$. As $x_\rho \in D_{\mathcal{C}_R}$ for every $\rho < \theta$ and θ is regular, we get that $|(\bigcup X) \cap C_\alpha| < \theta$ as sought.

► Suppose $\alpha = \beta_{\theta^2}$. Notice this implies that we are in the first case of the construction of the set X. Recall that the Δ-system $\{x_j \mid j < \theta\}$ is such that $(x_j \setminus R) \cap C_\alpha = \emptyset$, hence $(\bigcup X) \cap C_\alpha = R \cap C_\alpha$. Recall that as $x_0 \in D_{\mathcal{C}}$, we get that $R \cap C_\alpha$ is of size $< \theta$, hence also $(\bigcup X) \cap C_\alpha$ is as sought. ⊣

CLAIM 5.11.4. *Assume $\lambda^\theta < \lambda^+$. Suppose C and E are two directed sets such that $|C| < \lambda^+$ and either* $\mathrm{non}(\mathcal{I}_{\mathrm{bd}}(E)) > \theta$ *or $E \equiv_T 1$. Then for every C-sequence $\mathcal{C}$ on a stationary $S \subseteq E^{\lambda^+}_\theta$, $C \times [\lambda^+]^{<\theta} \times E \not\leq_T C \times D_{\mathcal{C}} \times E$.*

PROOF. Let $\mathcal{C} := \langle C_\alpha \mid \alpha \in S\rangle$ be a C-sequence where $S \subseteq E^{\lambda^+}_\theta$. Suppose on the contrary that $C \times [\lambda^+]^{<\theta} \times E \leq_T C \times D_{\mathcal{C}} \times E$. Hence, $[\lambda^+]^{<\theta} \leq_T C \times D_{\mathcal{C}} \times E$, let us fix a Tukey function $f : [\lambda^+]^{<\theta} \to C \times D_{\mathcal{C}} \times E$ witnessing that. Consider $X = [\lambda^+]^1$.

By the pigeonhole principle, there exists some $c \in C$ and some set $Z \subseteq X$ of size λ^+ such that $f``Z \subseteq \{c\} \times D_{\mathcal{C}} \times E$. Let $Y := \pi_{D_{\mathcal{C}}}(f``Z)$. Let us split to two cases:

► Suppose $|Y| < \lambda^+$. By the pigeonhole principle, we can find a subset $Q \subseteq Z$ of size θ such that $f``Q = \{c\} \times \{x\} \times E$ for some $x \in D_{\mathcal{C}}$. As $f$ is Tukey and $Q$ is unbounded, we must have that $f``Q$ is unbounded, but this is absurd as $\mathrm{non}(\mathcal{I}_{\mathrm{bd}}(E)) > \theta$.

► Suppose $|Y| = \lambda^+$. As f is Tukey and either $\mathrm{non}(\mathcal{I}_{\mathrm{bd}}(E)) > \theta$ or $E = 1$, every subset of Y of size θ is unbounded which is impossible by Claim 5.11.3.1. ⊣

5.4. Structure of $D_{\mathcal{C}}$. In [12, Lemmas 1, 2], Todorčević defined for every κ regular and $S \subseteq \kappa$ the directed set $D(S) := \{C \subseteq [S]^{\leq\omega} \mid \forall\alpha < \omega_1[\sup(C \cap \alpha) \in C]\}$ ordered by inclusion; and studied the structure of such directed sets. In this section we follow this line of study but for directed sets of the form $D_{\mathcal{C}}$, constructing a large $<_T$-antichain and chain of directed sets using θ-support product.

5.4.1. Antichain

THEOREM 5.12. *Suppose $2^\lambda = \lambda^+$, $\lambda^\theta < \lambda^+$, then there exists a family $\mathcal{F}$ of size 2^{λ^+} of directed sets of the form $D_{\mathcal{C}}$ such that every two of them are Tukey incomparable.*

PROOF. As $2^\lambda = \lambda^+$ holds, by Shelah's Theorem we get that $\diamondsuit(S)$ holds for every $S \subseteq E^{\lambda^+}_\theta$ stationary subset. Let us fix some stationary subset $S \subseteq E^{\lambda^+}_\theta$ and a partition of S into λ^+-many stationary subsets $\langle S_\alpha \mid \alpha < \lambda^+\rangle$. For each S_α we fix a $\clubsuit(S_\alpha)$ sequence $\langle C_\beta \mid \beta \in S_\alpha\rangle$.

Let us fix a family $\mathcal{F}$ of size 2^{λ^+} of subsets of S such that for every two $R, T \in \mathcal{F}$ there exists some S_α such that $R \setminus T \supseteq S_\alpha$. For each $T \in \mathcal{F}$ let us define

https://doi.org/10.1017/jsl.2023.32 Published online by Cambridge University Press

a C-sequence $\mathcal{C}_T := \langle C_\alpha \mid \alpha \in T\rangle$. Clearly the following lemma shows the family $\{D_{\mathcal{C}_T} \mid T \in \mathcal{F}\}$ is as sought. ⊣

Claim 5.12.1. *Suppose $\mathcal{C}_T := \langle C_\beta \mid \beta \in T\rangle$ and $\mathcal{C}_R := \langle C_\beta \mid \beta \in R\rangle$ are two C-sequences such that $T, R \subseteq E^{\lambda^+}_\theta$ are stationary subsets. Then if $\langle C_\beta \mid \beta \in T \setminus R\rangle$ is a ♣-sequence, then $D_{\mathcal{C}_T} \not\leq_T D_{\mathcal{C}_R}$.*

Proof. Suppose $f : D_{\mathcal{C}_T} \to D_{\mathcal{C}_R}$ is a Tukey function. Fix a subset $W \subseteq [\lambda^+]^1 \subseteq D_{\mathcal{C}_T}$ of size λ^+, we split to two cases:

► Suppose $f``W \subseteq [\alpha]^\theta$ for some $\alpha < \lambda^+$. As $\lambda^\theta < \lambda^+$, by the pigeonhole principle we can find a subset $X \subseteq W$ of size $\lambda^+$ such that $f``X = \{z\}$ for some $z \in D_{\mathcal{C}_R}$. As $\langle C_\beta \mid \beta \in T \setminus R\rangle$ is a ♣-sequence and $\bigcup X \in [\lambda^+]^{\lambda^+}$, there exists some $\beta \in T \setminus R$ such that $C_\beta \subseteq \bigcup X$. So X is an unbounded subset of $\mathcal{C}_T$ such that $f``X$ is bounded in $\mathcal{C}_R$ which is absurd.

► As $|f``W| = \lambda^+$, using $\lambda^\theta < \lambda^+$ we may fix a subset $Y = \{y_\beta \mid \beta < \lambda^+\} \subseteq f``W$ which forms a Δ-system with a root R_1. In other words, for $\alpha < \beta < \lambda^+$ we have $y_\alpha \setminus R_1 < y_\beta \setminus R_1$ and $y_\alpha \cap y_\beta = R_1$. For each $\alpha < \lambda^+$, we fix $x_\alpha \in W$ such that $f(x_\alpha) = y_\alpha$. Finally, without loss of generality we may use the Δ-system lemma again and refine our set Y to get that there exists a club $E \subseteq \lambda^+$ such that, for all $\alpha < \beta < \lambda^+$ we have:

- $x_\alpha \cap x_\beta = \emptyset$;
- $y_\alpha \cap y_\beta = R_1$;
- there exists some $\gamma \in E$ such that $x_\alpha < \gamma < x_\beta$ and $y_\alpha \setminus R_1 < \gamma < y_\beta \setminus R_1$;
- $f(x_\alpha) = y_\alpha$.

Furthermore, we may assume that between any two elements of $\xi < \eta$ in E there exists a unique $\alpha < \lambda^+$ such that $\xi < x_\alpha \cup (y_\alpha \setminus R_1) < \eta$.

As $\langle C_\beta \mid \beta \in T \setminus R\rangle$ is a ♣-sequence, there exists some $\beta \in (T \setminus R) \cap \mathrm{acc}(E)$ such that $C_\beta \subseteq \bigcup\{x_\alpha \mid \alpha < \lambda^+\}$. Construct by recursion an increasing sequence $\langle \beta_\nu \mid \nu < \theta\rangle \subseteq C_\beta$ and a sequence $\langle z_\nu \mid \nu < \theta\rangle \subseteq \{x_\alpha \mid \alpha < \lambda^+\}$ such that $\beta_\nu \in z_\nu < \beta$.

Clearly, $\{z_\nu \mid \nu < \theta\}$ is unbounded in $D_{\mathcal{C}_T}$, so the following claim proves f is not a Tukey function. ⊣

Subclaim 5.12.1.1. *The subset $\{f(z_\nu) \mid \nu < \theta\}$ is bounded in $D_{\mathcal{C}_R}$.*

Proof. Let $Y := \bigcup f(z_\nu)$ and $\mathcal{C}_R := \langle C_\beta \mid \beta \in R\rangle$, we will show that for every $\alpha \in R$, we have $|Y \cap C_\alpha| < \theta$. By the refinement we did previously it is clear that $\{f(z_\nu) \setminus R_1 \mid \nu < \theta\}$ is a pairwise disjoint sequence, where for each $\nu < \theta$ we have some element $\gamma_\nu \in E$ such that $f(z_\nu) \setminus R_1 < \gamma_\nu < f(z_{\nu+1}) \setminus R_1 < \beta$. Let $\alpha \in R$.

► Suppose $\alpha > \beta$. As C_α is cofinal in α and of order-type θ, then $|Y \cap C_\alpha| < \theta$.

► Suppose $\alpha < \beta$. As C_α is cofinal in α and of order-type θ, there exists some $\nu < \theta$ such that for all $\nu < \rho < \theta$, we have $(f(z_\rho) \setminus R_1) \cap C_\alpha = \emptyset$. As $f(z_\rho) \in D_{\mathcal{C}_R}$ for every $\rho < \theta$ and θ is regular, we get that $|Y \cap C_\alpha| < \theta$ as sought.

As $\beta \notin R$ there are no more cases to consider. ⊣

Corollary 5.13. *Suppose $2^\lambda = \lambda^+$, $\lambda^\theta < \lambda^+$ and $S \subseteq E^{\lambda^+}_\theta$ is a stationary subset. Then there exists a family $\mathcal{F}$ of directed sets of the form $D_{\mathcal{C}} \times [\lambda]^{<\theta}$ of size 2^{λ^+} such that every two of them are Tukey incomparable.*

PROOF. Clearly by the same arguments of Theorem 5.12 the following lemma is suffices to get the wanted result. ⊣

CLAIM 5.13.1. *Suppose $\mathcal{C}_T := \langle C_\beta \mid \beta \in T\rangle$ and $\mathcal{C}_R := \langle C_\beta \mid \beta \in R\rangle$ are two C-sequences such that $T, R \subseteq E^{\lambda^+}_\theta$ are stationary subsets such that $T \setminus R$ is stationary. Then if $\langle C_\beta \mid \beta \in T \setminus R\rangle$ is a ♣-sequence, then $D_{\mathcal{C}_T} \times [\lambda]^{<\theta} \not\leq_T D_{\mathcal{C}_R} \times [\lambda]^{<\theta}$.*

PROOF. Suppose $f : D_{\mathcal{C}_T} \times [\lambda]^{<\theta} \to D_{\mathcal{C}_R} \times [\lambda]^{<\theta}$ is a Tukey function. Consider $Q = f``([\lambda^+]^1 \times \{\emptyset\})$, let us split to two cases:

► If $|Q| < \lambda^+$, then by the pigeonhole principle, there exists $x \in D_{\mathcal{C}_R}$, $F \in [\lambda]^{<\theta}$ and a set $W \subseteq [\lambda^+]^1$ of size λ^+ such that $f``(W \times \{\emptyset\}) = \{(x, F)\}$. As $\langle C_\beta \mid \beta \in T \setminus R\rangle$ is a ♣-sequence and $\bigcup W \in [\lambda^+]^{\lambda^+}$, we may fix some $\beta \in T \setminus R$ such that $C_\beta \subseteq \bigcup W$. Hence $W \times \{\emptyset\}$ is unbounded in $D_{\mathcal{C}_T} \times [\lambda]^{<\theta}$ but $f``(W \times \{\emptyset\})$ is bounded in $D_{\mathcal{C}_R} \times [\lambda]^{<\theta}$ which is absurd as f is Tukey.

► If $|Q| = \lambda^+$, then by the pigeonhole principle there exists some $F \in [\lambda]^{<\theta}$ and a set $W \subseteq [\lambda^+]^1$ of size λ^+ such that, $f``(W \times \{\emptyset\}) \subseteq D_{\mathcal{C}_R} \times \{F\}$. Let $Y := \pi_0(f``(W \times \{\emptyset\}))$. Next, we may continue with the same proof as in Lemma 5.12.1. ⊣

5.4.2. *Chain*

THEOREM 5.14. *Suppose $2^\lambda = \lambda^+$, $\lambda^\theta < \lambda^+$. Then there exists a family $\mathcal{F} = \{D_{\mathcal{C}_\xi} \mid \xi < \lambda^+\}$ of Tukey incomparable directed sets of the form $D_\mathcal{C}$ such that $\langle \prod^{\leq\theta}_{\zeta<\xi} D_{\mathcal{C}_\zeta} \mid \xi < \lambda^+\rangle$ is a $<_T$-increasing chain.*

PROOF. As in Theorem 5.12, we fix a partition $\langle S_\zeta \mid \zeta < \lambda^+\rangle$ of $E^{\lambda^+}_\theta$ to stationary subsets such that there exists a ♣(S_ζ)-sequence $\mathcal{C}_\zeta$ for $\zeta < \lambda^+$. Note that for every $A \in [\lambda^+]^{<\lambda^+}$, we have $|\prod^{\leq\theta}_{\zeta\in A} D_{\mathcal{C}_\zeta}| = \lambda^+$. Note that for every $A, B \in [\lambda^+]^{<\lambda^+}$ such that $A \subset B$, we have $\prod^{\leq\theta}_{\zeta\in A} D_{\mathcal{C}_\zeta} \leq_T \prod^{\leq\theta}_{\zeta\in B} D_{\mathcal{C}_\zeta}$. The following claim gives us the wanted result. ⊣

CLAIM 5.14.1. *Suppose $A \in [\lambda^+]^{<\lambda^+}$ and $\xi \in \lambda^+ \setminus A$, then $D_{\mathcal{C}_\xi} \not\leq_T \prod^{\leq\theta}_{\zeta\in A} D_{\mathcal{C}_\zeta}$. In particular, $\prod^{\leq\theta}_{\zeta\in A} D_{\mathcal{C}_\zeta} <_T \prod^{\leq\theta}_{\zeta\in A} D_{\mathcal{C}_\zeta} \times D_{\mathcal{C}_\xi}$.*

PROOF. Let $D := D_{\mathcal{C}_\xi}$ and $E := \prod^{\leq\theta}_{\zeta\in A} D_{\mathcal{C}_\zeta}$. Note that as $2^\lambda = \lambda^+$, then $(\lambda^+)^\lambda = \lambda^+$, so $|E| = \lambda^+$. Suppose $f : D \to E$ is a Tukey function. Consider $Q = f``[\lambda^+]^1$, let us split to cases:

► Suppose $|Q| < \lambda^+$, then by pigeonhole principle, there exists $e \in E$ and a subset $X \subseteq D$ of size λ^+ such that $f``X = \{e\}$. As $\langle C_\beta \mid \beta \in S_\xi\rangle$ is a ♣-sequence, there exists some $\beta \in S_\xi$ such that $C_\beta \subseteq \bigcup X$. So $X$ is an unbounded subset of $D$ such that $f``X$ is bounded in E which is absurd.

► Suppose $|Q| = \lambda^+$. Let us enumerate $Q := \{q_\alpha \mid \alpha < \lambda^+\}$. Recall that for every $\zeta \in A$, $D_{\mathcal{C}_\zeta} \subseteq [\lambda^+]^{\leq\theta}$. Let $z_\alpha := \bigcup\{q_\alpha(\zeta) \times \{\zeta\} \mid \zeta \in A,\ q_\alpha(\zeta) \neq 0_{D_{\mathcal{C}_\zeta}}\}$, notice that $z_\alpha \in [\lambda^+ \times A]^{\leq\theta}$. We fix a bijection $\phi : \lambda^+ \times A \to \lambda^+$.

As $\{\phi``z_\alpha \mid \alpha < \lambda^+\}$ is a subset of $[\lambda^+]^{\leq\theta}$ of size $\lambda^+$ and $\lambda^\theta < \lambda^+$, by the $\Delta$-system lemma, we may refine our sequence $Q$ and re-index such that $\{\phi``z_\alpha \mid \alpha < \lambda^+\}$ will be a Δ-system with root R'.

For each $\alpha < \lambda^+$ and $\zeta \in A$, let $y_{\alpha,\zeta} := \{\beta < \lambda^+ \mid \beta \in q_\alpha(\zeta)\}$. We claim that for each $\zeta \in A$, the set $\{y_{\alpha,\zeta} \mid \zeta \in A\}$ is a Δ-system with root $R_\zeta := \{\beta < \lambda^+ \mid$

https://doi.org/10.1017/jsl.2023.32 Published online by Cambridge University Press

$(\beta,\zeta) \in \phi^{-1}[R']\}$. Let us show that whenever $\alpha < \beta < \lambda^+$, we have $y_{\alpha,\zeta} \cap y_{\beta,\zeta} = R_\zeta$. Notice that $\delta \in y_{\alpha,\zeta} \cap y_{\beta,\zeta} \iff \delta \in q_\alpha(\zeta) \cap q_\beta(\zeta) \iff (\delta,\zeta) \in z_\alpha \cap z_\beta \iff \phi(\delta,\zeta) \in \phi\text{"}(z_\alpha \cap z_\beta) = \phi\text{"}z_\alpha \cap \phi\text{"}z_\beta = R' \iff (\delta,\zeta) \in \phi^{-1}R' \iff \delta \in R_\zeta$. For each $\alpha < \lambda^+$, we fix $x_\alpha \in [\lambda^+]^1$ such that $f(x_\alpha) = q_\alpha$.

We use the Δ-system lemma again and refine our sequence such that there exists a club $C \subseteq \lambda^+$ and for all $\alpha < \beta < \lambda^+$ we have:

(1) for every $\zeta \in A$, we have $y_{\alpha,\zeta} \cap y_{\beta,\zeta} = R_\zeta$;
(2) $x_\alpha \cap x_\beta = \emptyset$;
(3) there exists some $\gamma \in C$ such that $x_\alpha \cup (\bigcup_{\zeta\in A}(y_{\alpha,\zeta} \setminus R_\zeta)) < \gamma < x_\beta \cup (\bigcup_{\zeta\in A}(y_{\beta,\zeta} \setminus R_\zeta))$.

Furthermore, we may assume that between any two elements of $\gamma < \delta$ in C there exists some $\alpha < \lambda^+$ such that $\gamma < x_\alpha \cup (\bigcup_{\zeta\in A}(y_{\alpha,\zeta} \setminus R_\zeta)) < \delta$. We continue in the spirit of Claim 5.11.3.1.

As $\langle C_\beta \mid \beta \in S_\xi\rangle$ is a $\clubsuit$-sequence, there exists some $\beta \in S_\xi \cap \mathrm{acc}(C)$ such that $C_\beta \subseteq \bigcup\{x_\alpha \mid \alpha < \lambda^+\}$. Construct by recursion an increasing sequence $\langle \beta_\nu \mid \nu < \theta\rangle \subseteq C_\beta$ and a sequence $\langle w_\nu \mid \nu < \theta\rangle \subseteq \{x_\alpha \mid \alpha < \lambda^+\}$ such that $\beta_\nu \in w_\nu < \beta$.

Clearly, $\{w_\nu \mid \nu < \theta\}$ is unbounded in $D_{\mathcal{C}_\xi}$, so the following Claim proves f is not a Tukey function. $\dashv$

SUBCLAIM 5.14.1.1. *The subset $\{f(w_\nu) \mid \nu < \theta\}$ is bounded in E.*

PROOF. For each $\zeta \in A$, let $W_\zeta := \bigcup_{\nu<\theta} f(w_\nu)(\zeta)$. We will show that $W_\zeta \in D_{\mathcal{C}_\zeta}$, as $|\{\zeta \in A \mid W_\zeta \neq \emptyset\}| \leq \theta$ this will imply that $\prod^{\leq\theta}_{\zeta\in A} W_\zeta$ is well defined and an element of E. Clearly $f(w_\nu) \leq_E \prod^{\leq\theta}_{\zeta\in A} W_\zeta$ for every $\nu < \theta$, so the set $\{f(w_\nu) \mid \nu < \theta\}$ is bounded in E as sought.

Let $\mathcal{C}_\zeta := \langle C_\alpha \mid \alpha \in S_\zeta\rangle$, we will show that for every $\alpha \in S_\zeta$, we have $|W_\zeta \cap C_\alpha| < \theta$. By the refinement we did previously it is clear that $\{f(w_\nu)(\zeta) \setminus R_\zeta \mid \nu < \theta\}$ is a pairwise disjoint sequence, where for each $\nu < \theta$ we have some element $\gamma_\nu \in C$ such that $f(w_\nu)(\zeta) \setminus R_\zeta < \gamma_\nu < f(w_{\nu+1})(\zeta) \setminus R_\zeta$. Furthermore, $f(w_\nu)(\zeta) \subseteq \beta$ for every $\nu < \theta$. Let $\alpha \in S_\zeta$.

► Suppose $\alpha > \beta$. As C_α if cofinal in α and of order-type θ, then $|W_\zeta \cap C_\alpha| < \theta$.

► Suppose $\alpha < \beta$. As C_α if cofinal in α and of order-type θ, there exists some $\nu < \theta$ such that for all $\nu < \rho < \theta$, we have $(f(w_\rho)(\zeta) \setminus R_\zeta) \cap C_\alpha = \emptyset$. As $f(w_\rho)(\zeta) \in D_{\mathcal{C}_\zeta}$ for every $\rho < \theta$, we get that $|W_\zeta \cap C_\alpha| < \theta$ as sought.

As $\beta \notin S_\zeta$ there are no more cases to consider. $\dashv$

§6. Concluding remarks. A natural continuation of this line of research is analysing the class $\mathcal{D}_\kappa$ for cardinals $\kappa \geq \aleph_\omega$. As a preliminary finding we notice that the poset $(\mathcal{P}(\omega), \subset)$ can be embedded by a function $\mathfrak{F}$ into the class $\mathcal{D}_{\aleph_\omega}$ under the Tukey order. Furthermore, for every two successive elements A, B in the poset $(\mathcal{P}(\omega), \subset)$, i.e., $A \subset B$ and $|B \setminus A| = 1$, there is no directed set D such that $\mathfrak{F}(A) <_T D <_T \mathfrak{F}(B)$. The embedding is defined via $\mathfrak{F}(A) := \prod^{<\omega}_{n\in A} \omega_{n+1}$, and the furthermore part can be proved by Lemma 4.3. As a corollary, we get that in ZFC the cardinality of $\mathcal{D}_{\aleph_\omega}$ is at least $2^{\aleph_0}$.

https://doi.org/10.1017/jsl.2023.32 Published online by Cambridge University Press

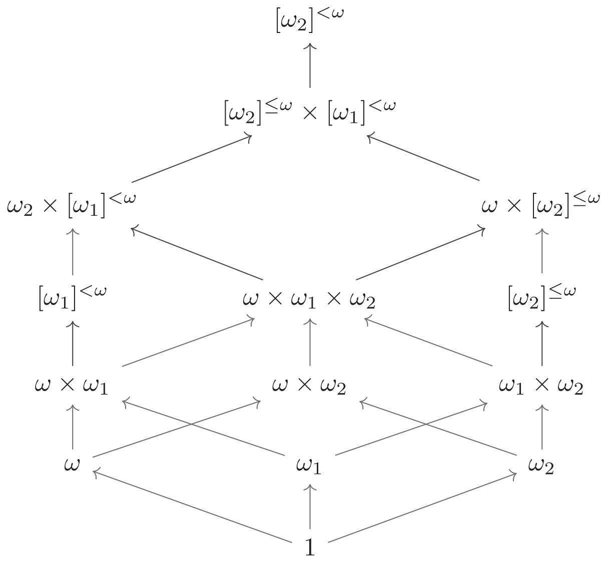

FIGURE 3. Tukey ordering of $(\mathcal{T}_2, <_T)$.

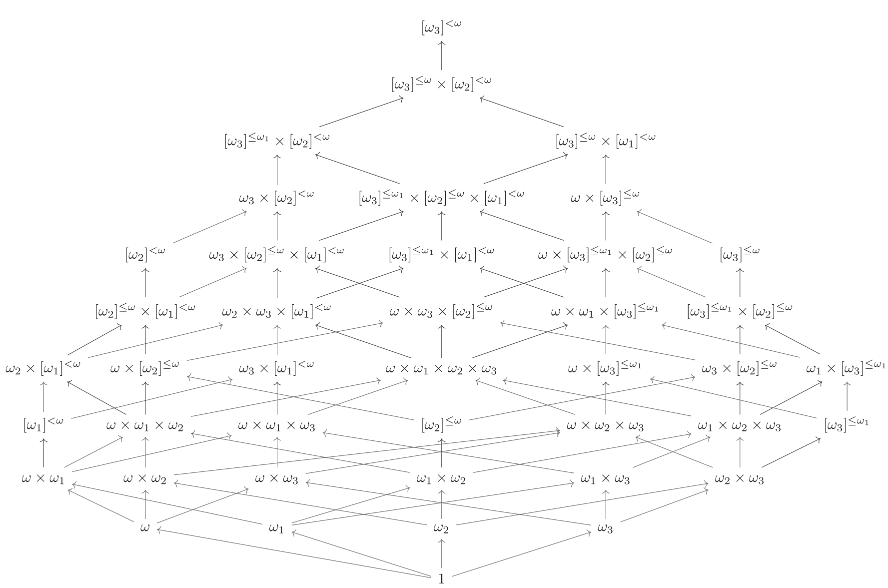

FIGURE 4. Tukey ordering of $(\mathcal{T}_3, <_T)$.

A. Appendix: Tukey ordering of simple elements of the class $\mathcal{D}_{\aleph_2}$ and $\mathcal{D}_{\aleph_3}$ We present each of the posets $(\mathcal{T}_2, <_T)$ and$(\mathcal{T}_3, <_T)$ in a diagram. In both diagrams below, for any two directed sets A and B, an arrow$A \to B$, represents the fact that $A <_T B$. If the arrow is dashed, then under GCH there exists a directed set in

https://doi.org/10.1017/jsl.2023.32 Published online by Cambridge University Press

between. If the arrow is not dashed, then there is no directed set in between A and B. Every two directed sets A and B such that there is no directed path (in the obvious sense) from A to B, are such that $A \not\leq_T B$. Note that this implies that any two directed sets on the same horizontal level are incompatible in the Tukey order (Figures 3 and 4).

Acknowledgments. This paper presents several results from the author's Ph.D. research at Bar-Ilan University under the supervision of Assaf Rinot to whom he wishes to express his deep appreciation. Our thanks go to Tanmay Inamdar for many illuminating discussions. We thank the referee for their effort and for writing a detailed thoughtful report that improved this paper.

Funding. The author is supported by the European Research Council (grant agreement ERC-2018-StG 802756).

REFERENCES

[1] G. Birkhoff, *Moore–Smith convergence in general topology*. ***Annals of Mathematics. Second Series***, vol. 38 (1937), no. 1, pp. 39–56.

[2] M. M. Day, *Oriented systems*. ***Duke Mathematical Journal***, vol. 11 (1944), pp. 201–229.

[3] T. Inamdar and A. Rinot, *Was Ulam right? I: Basic theory and subnormal ideals*. ***Topology and its Applications***, vol. 323 (2022), p. 108287.

[4] J. R. Isbell, *The category of cofinal types II*. ***Transactions of the American Mathematical Society***, vol. 116 (1965), pp. 394–416.

[5] ———, *Seven cofinal types*. ***Journal of the London Mathematical Society. Second Series***, vol. 4 (1972), pp. 651–654.

[6] B. Kuzeljević and S. Todorčević, *Cofinal types on ω_2*. ***Mathematical Logic Quarterly***, vol. 69 (2023), no. 1, pp. 99–103.

[7] M. Pouzet, *Parties cofinales des ordres partiels ne contenant pas d'antichaines infinies, 1980*. ***Journal of the London Mathematical Society***, to appear.

[8] D. Raghavan and S. Todorčević, *Combinatorial dichotomies and cardinal invariants*. ***Mathematical Research Letters***, vol. 21 (2014), no. 2, pp. 379–401.

[9] J. Schmidt, *Konfinalität*. ***Zeitschrift für Mathematische Logik und Grundlagen der Mathematik***, vol. 1 (1955), pp. 271–303.

[10] S. Shelah, *Diamonds*. ***Proceedings of the American Mathematical Society***, vol. 138 (2010), no. 6, pp. 2151–2161.

[11] R. P. Stanley, ***Catalan Numbers***, Cambridge University Press, New York, 2015.

[12] S. Todorčević, *Directed sets and cofinal types*. ***Transactions of the American Mathematical Society***, vol. 290 (1985), no. 2, pp. 711–723.

[13] ———, ***Partition Problems in Topology***, Contemporary Mathematics, vol. 84, American Mathematical Society, Providence, RI, 1989.

[14] ———, *A classification of transitive relations on ω_1*. ***Proceedings of the London Mathematical Society. Third Series***, vol. 73 (1996), no. 3, pp. 501–533.

[15] J. W. Tukey, ***Convergence and Uniformity in Topology***, Annals of Mathematics Studies, vol. 2, Princeton University Press, Princeton, NJ, 1940.

DEPARTMENT OF MATHEMATICS
BAR-ILAN UNIVERSITY
RAMAT GAN 5290002, ISRAEL
E-mail: royshalev2@gmail.com
URL: https://roy-shalev.github.io/