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# Arc-smooth functions on closed sets

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## ABSTRACT

By an influential theorem of Boman, a function  $f$  on an open set  $U$  in  $\mathbb{R}^d$  is smooth ( $\mathcal{C}^\infty$ ) if and only if it is *arc-smooth*, that is,  $f \circ c$  is smooth for every smooth curve  $c : \mathbb{R} \rightarrow U$ . In this paper we investigate the validity of this result on closed sets. Our main focus is on sets which are the closure of their interior, so-called *fat* sets. We obtain an analogue of Boman’s theorem on fat closed sets with Hölder boundary and on fat closed subanalytic sets with the property that every boundary point has a basis of neighborhoods each of which intersects the interior in a connected set. If  $X \subseteq \mathbb{R}^d$  is any such set and  $f : X \rightarrow \mathbb{R}$  is arc-smooth, then  $f$  extends to a smooth function defined on  $\mathbb{R}^d$ . We also get a version of the Bochnak–Siciak theorem on all closed fat subanalytic sets and all closed sets with Hölder boundary: if  $f : X \rightarrow \mathbb{R}$  is the restriction of a smooth function on  $\mathbb{R}^d$  which is real analytic along all real analytic curves in  $X$ , then  $f$  extends to a holomorphic function on a neighborhood of  $X$  in  $\mathbb{C}^d$ . Similar results hold for non-quasianalytic Denjoy–Carleman classes (of Roumieu type). We will also discuss sharpness and applications of these results.

## 1. Introduction

In this paper we study differentiability of functions defined on closed subsets of  $\mathbb{R}^d$ . One way to endow an arbitrary set  $X$  with a smooth structure is by declaring which curves  $\mathbb{R} \rightarrow X$  and which functions  $X \rightarrow \mathbb{R}$  should be *smooth*. Together with a natural compatibility condition, this leads to the notion of a *Frölicher space*; cf. [FK88, KM97]. Here we study the Frölicher space generated by the inclusion of a closed set  $X$  in  $\mathbb{R}^d$  and some of its relatives. We will not use the terminology of Frölicher spaces in the paper but the connection is made precise in Remark 1.7.

### 1.1 Boman’s theorem and its relatives

Let  $f : U \rightarrow \mathbb{R}$  be a function defined in an open subset  $U$  of  $\mathbb{R}^d$ . Then  $f$  induces a mapping  $f_* : U^{\mathbb{R}} \rightarrow \mathbb{R}^{\mathbb{R}}$ ,  $f_*(c) = f \circ c$ , whose invariance properties encode the regularity of  $f$ .

RESULT 1.1 (Boman [Bom67]). *A function  $f : U \rightarrow \mathbb{R}$  is smooth ( $\mathcal{C}^\infty$ ) if and only if  $f_*\mathcal{C}^\infty(\mathbb{R}, U) \subseteq \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ .*

Similarly, Hölder differentiability can be characterized by  $f_*$ ; we denote by  $\mathcal{C}^{k,\alpha}$ , for  $k \in \mathbb{N}$ ,  $\alpha \in (0, 1]$ , the class of  $k$ -times continuously differentiable functions whose partial derivatives of order  $k$  satisfy a local  $\alpha$ -Hölder condition.

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RESULT 1.2 [FK88, FF89, KM97]. A function  $f : U \rightarrow \mathbb{R}$  is of class  $\mathcal{C}^{k,\alpha}$  if and only if  $f_*\mathcal{C}^\infty(\mathbb{R}, U) \subseteq \mathcal{C}^{k,\alpha}(\mathbb{R}, \mathbb{R})$ .

Furthermore, there is an ultradifferentiable version of Boman’s theorem. We recall that, for a positive sequence  $M = (M_k)_{k \in \mathbb{N}}$ , the *Denjoy–Carleman class* (of Roumieu type)  $\mathcal{C}^M(U, \mathbb{R}^m)$  is the set of all functions  $f \in \mathcal{C}^\infty(U, \mathbb{R}^m)$  such that for all compact  $K \subseteq U$ ,

$$\exists C, \rho > 0 \forall k \in \mathbb{N} \forall x \in K : \|f^{(k)}(x)\|_{L_k(\mathbb{R}^d, \mathbb{R}^m)} \leq C \rho^k k! M_k. \tag{1.1}$$

The sequence  $M$  is called non-quasianalytic if  $\mathcal{C}^M$  contains non-trivial functions with compact support. If  $M$  is log-convex, then  $\mathcal{C}^M$  is stable under composition. We refer to § 2.1 for this and more on Denjoy–Carleman classes.

RESULT 1.3 [KMR09]. Assume that  $M = (M_k)$  is non-quasianalytic and log-convex. A function  $f : U \rightarrow \mathbb{R}$  is of class  $\mathcal{C}^M$  if and only if  $f_*\mathcal{C}^M(\mathbb{R}, U) \subseteq \mathcal{C}^M(\mathbb{R}, \mathbb{R})$ .

Remark 1.4. Boman actually showed that  $f$  is smooth if and only if  $f_*\mathcal{C}^M(\mathbb{R}, U) \subseteq \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ , for some arbitrary non-quasianalytic log-convex sequence  $M$ .

A glance at the proofs confirms that the curves along which the regularity in question is tested can be taken to have compact support.

A function  $f : U \rightarrow \mathbb{R}$  with the property that  $f \circ c$  is real analytic ( $\mathcal{C}^\omega$ ) for all real analytic  $c : \mathbb{R} \rightarrow U$  clearly does not need to be real analytic on  $U \subseteq \mathbb{R}^d$ , let alone continuous; see [BMP91]. But there is the following result.

RESULT 1.5 (Bochnak and Siciak [Boc70, Sic70, BS71]). A function  $f : U \rightarrow \mathbb{R}$  is real analytic if and only if  $f_*\mathcal{C}^\infty(\mathbb{R}, U) \subseteq \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$  and  $f_*\mathcal{C}^\omega(\mathbb{R}, U) \subseteq \mathcal{C}^\omega(\mathbb{R}, \mathbb{R})$ .

Actually, a smooth function  $f \in \mathcal{C}^\infty(U)$  which is real analytic on affine lines is real analytic on  $U$ .

We remark that if  $M = (M_k)$  is quasianalytic such that  $\mathcal{C}^\omega \subsetneq \mathcal{C}^M$ , then a  $\mathcal{C}^\infty$ -function  $f : U \rightarrow \mathbb{R}$  which satisfies  $f_*\mathcal{C}^M(\mathbb{R}, U) \subseteq \mathcal{C}^M(\mathbb{R}, \mathbb{R})$  need not be of class  $\mathcal{C}^M$ ; see [Jaf16].

### 1.2 Arc-smooth functions

In this paper we investigate the validity of the above results on non-open subsets  $X \subseteq \mathbb{R}^d$ . For arbitrary subsets  $X \subseteq \mathbb{R}^d$  we define

$$\begin{aligned} \mathcal{A}^\infty(X) &:= \{f : X \rightarrow \mathbb{R} : f_*\mathcal{C}^\infty(\mathbb{R}, X) \subseteq \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})\}, \\ \mathcal{A}^M(X) &:= \{f : X \rightarrow \mathbb{R} : f_*\mathcal{C}^M(\mathbb{R}, X) \subseteq \mathcal{C}^M(\mathbb{R}, \mathbb{R})\}, \\ \mathcal{A}_M^\omega(X) &:= \{f : X \rightarrow \mathbb{R} : f_*\mathcal{C}^M(\mathbb{R}, X) \subseteq \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})\}, \end{aligned}$$

where we set

$$\begin{aligned} \mathcal{C}^\infty(\mathbb{R}, X) &:= \{c \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^d) : c(\mathbb{R}) \subseteq X\}, \\ \mathcal{C}^M(\mathbb{R}, X) &:= \{c \in \mathcal{C}^M(\mathbb{R}, \mathbb{R}^d) : c(\mathbb{R}) \subseteq X\}. \end{aligned}$$

We call the elements of  $\mathcal{A}^\infty(X)$  *arc-smooth functions* and those of  $\mathcal{A}^M(X)$  *arc- $\mathcal{C}^M$  functions* on  $X$ . We will also consider

$$\mathcal{A}^\omega(X) := \{f \in \mathcal{A}^\infty(X) : f_*\mathcal{C}^\omega(\mathbb{R}, X) \subseteq \mathcal{C}^\omega(\mathbb{R}, \mathbb{R})\},$$

where

$$\mathcal{C}^\omega(\mathbb{R}, X) := \{c \in \mathcal{C}^\omega(\mathbb{R}, \mathbb{R}^d) : c(\mathbb{R}) \subseteq X\}.$$

We will not speak of *arc-analytic* functions, since such are not assumed to be smooth in the literature.

Evidently,  $\mathcal{A}^\omega(X) \subseteq \mathcal{A}^\infty(X) \subseteq \mathcal{A}_M^\infty(X) \supseteq \mathcal{A}^M(X)$ . (We will see below that there is no hope of the analogue of Result 1.2 holding on even very simple non-open sets such as the closed half-space.)

With this notation, Results 1.1, 1.3, and 1.5 amount to

$$\mathcal{A}^\infty(X) = \mathcal{C}^\infty(X), \quad \mathcal{A}^M(X) = \mathcal{C}^M(X), \quad \mathcal{A}^\omega(X) = \mathcal{C}^\omega(X), \tag{1.2}$$

if  $X \subseteq \mathbb{R}^d$  is a non-empty open set and  $M = (M_k)$  is a non-quasianalytic log-convex sequence.

*Remark 1.6.* The identities (1.2) imply that, in the definition of  $\mathcal{A}^\infty(X)$ ,  $\mathcal{A}^M(X)$ , and  $\mathcal{A}^\omega(X)$ , we could equivalently replace the families of curves  $c : \mathbb{R} \rightarrow X$  by families of plots  $p : U \rightarrow X$  (of the same regularity), where  $U$  is any open subset of  $\mathbb{R}^e$  with varying  $e$ .

*Remark 1.7.* Recall that a *Frölicher space* is a triple  $(X, \mathcal{C}_X, \mathcal{F}_X)$  consisting of a set  $X$ , a subset  $\mathcal{C}_X \subseteq X^{\mathbb{R}}$  and a subset  $\mathcal{F}_X \subseteq \mathbb{R}^X$  such that:

- (i)  $f : X \rightarrow \mathbb{R}$  belongs to  $\mathcal{F}_X$  if and only if  $f \circ c \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$  for all  $c \in \mathcal{C}_X$ ;
- (ii)  $c : \mathbb{R} \rightarrow X$  belongs to  $\mathcal{C}_X$  if and only if  $f \circ c \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$  for all  $f \in \mathcal{F}_X$ .

Any subset  $\mathcal{F} \subseteq \mathbb{R}^X$  generates a unique Frölicher space  $(X, \mathcal{C}_X, \mathcal{F}_X)$  by setting

$$\begin{aligned} \mathcal{C}_X &:= \{c : \mathbb{R} \rightarrow X : f \circ c \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}) \text{ for all } f \in \mathcal{F}\}, \\ \mathcal{F}_X &:= \{f : X \rightarrow \mathbb{R} : f \circ c \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}) \text{ for all } c \in \mathcal{C}_X\}. \end{aligned}$$

In this paper we are investigating the Frölicher spaces generated by the inclusion map  $\iota_X : X \rightarrow \mathbb{R}^d$  of subsets  $X$  of  $\mathbb{R}^d$ , that is,  $(X, \mathcal{C}^\infty(\mathbb{R}, X), \mathcal{A}^\infty(X))$ . For suitable sets  $X$  we try to identify the corresponding set of functions  $\mathcal{F}_X = \mathcal{A}^\infty(X)$ . More on Frölicher spaces can be found in [FK88, KM97].

### 1.3 Admissible sets

Let  $X \subseteq \mathbb{R}^d$  be an arbitrary subset. A function  $f : X \rightarrow \mathbb{R}$  is said to be *smooth* if for each  $x \in X$  there exist a neighborhood  $U$  in  $\mathbb{R}^d$  and a smooth function  $F : U \rightarrow \mathbb{R}$  such that  $F|_{U \cap X} = f|_{U \cap X}$ . If  $X$  is open, then this notion of smoothness coincides with the usual one. We denote by  $\mathcal{C}^\infty(X)$  the set of all smooth functions on  $X$ .

DEFINITION 1.8. A subset  $X \subseteq \mathbb{R}^d$  is called  *$\mathcal{A}^\infty$ -admissible* if  $\mathcal{A}^\infty(X) = \mathcal{C}^\infty(X)$ , that is, the arc-smooth functions on  $X$  are precisely the smooth functions.

Boman’s theorem states that open subsets  $X \subseteq \mathbb{R}^d$  are  $\mathcal{A}^\infty$ -admissible. We will look for non-open  $\mathcal{A}^\infty$ -admissible sets. It follows from a result of Kriegl [Kri97] that closed convex subsets  $X \subseteq \mathbb{R}^d$  with non-empty interior are  $\mathcal{A}^\infty$ -admissible. It is natural to consider closed sets with dense interior.

DEFINITION 1.9. A non-empty closed subset  $X$  of  $\mathbb{R}^d$  is called *fat* if  $X = \overline{\text{int}(X)}$ .

If  $X \subseteq \mathbb{R}^d$  is fat, then there are other natural possibilities for defining ‘smooth’ functions on  $X$  which we compare in the following lemma.

LEMMA 1.10. Let  $X \subseteq \mathbb{R}^d$  be a fat closed set. Consider the following conditions:

- (1) there exists  $F \in \mathcal{C}^\infty(\mathbb{R}^d)$  such that  $F|_X = f$ ;
- (2)  $f \in \mathcal{C}^\infty(X)$ ;
- (3)  $f|_{\text{int}(X)} \in \mathcal{C}^\infty(\text{int}(X))$  and the Fréchet derivatives  $(f|_{\text{int}(X)})^{(n)}$  of all orders have continuous extensions  $f^{(n)} : X \rightarrow L_n(\mathbb{R}^d, \mathbb{R})$ ;
- (4)  $f|_{\text{int}(X)} \in \mathcal{C}^\infty(\text{int}(X))$  and the directional derivatives  $d_v^n f|_{\text{int}(X)}$  for all  $v \in \mathbb{R}^d$  and all  $n \in \mathbb{N}$  have continuous extensions to  $X$ ;
- (5)  $f|_{\text{int}(X)} \in \mathcal{C}^\infty(\text{int}(X))$  and the partial derivatives  $\partial^\alpha f|_{\text{int}(X)}$  for all  $\alpha \in \mathbb{N}^d$  have continuous extensions to  $X$ .

Then (1)  $\Rightarrow$  (2)  $\Rightarrow$  (3)  $\Leftrightarrow$  (4)  $\Leftrightarrow$  (5). All five conditions are equivalent if  $X$  has the following regularity property.

- (6) For all  $x \in X$  there exist  $m \in \mathbb{N}_{>0}$ ,  $C > 0$ , and a compact neighborhood  $K$  of  $x$  in  $X$  such that any two points  $y_1, y_2 \in K$  can be joined by a rectifiable path  $\gamma$  which lies in  $\text{int}(X)$ , except perhaps for finitely many points, and has length

$$\ell(\gamma) \leq C|y_1 - y_2|^{1/m}.$$

*Proof.* (1)  $\Rightarrow$  (2)  $\Rightarrow$  (3) are obvious.

(3)  $\Leftrightarrow$  (4)  $\Leftrightarrow$  (5) This follows from the fact that at points  $x \in \text{int}(X)$  Fréchet, directional, and partial derivatives can be converted into one another in a linear way; cf. [KM97, Lemma 7.13].

(5)  $\Rightarrow$  (1) By the regularity property (6),  $f$  defines a Whitney jet on  $X$ ; see [Bie80, Proposition 2.16]. So Whitney’s extension theorem implies (1).  $\square$

In general the implication (5)  $\Rightarrow$  (1) is false; see Example 10.9.

Another natural condition for  $\mathcal{A}^\infty$ -admissibility is the following; see Example 10.5.

DEFINITION 1.11. A closed subset  $X \subseteq \mathbb{R}^d$  is called *simple* if each  $x \in X$  has a basis of neighborhoods  $\mathcal{U}$  such that  $U \cap \text{int}(X)$  is connected for all  $U \in \mathcal{U}$ .

A function  $f : X \rightarrow \mathbb{R}$  is said to be *real analytic* if for each  $x \in X$  there exist a neighborhood  $U$  of  $x$  in  $\mathbb{C}^d$  and a holomorphic function  $F : U \rightarrow \mathbb{C}$  such that  $F|_{U \cap X} = f|_{U \cap X}$ . We denote by  $\mathcal{C}^\omega(X)$  the set of all real analytic functions on  $X$ .

If  $M = (M_k)$  is a positive sequence, we set

$$\mathcal{C}^M(X) := \{f \in \mathcal{C}^\infty(X) : (1.1) \text{ holds for all compact } K \subseteq X\}.$$

Note that we do not require that a function  $f \in \mathcal{C}^M(X)$  is locally a restriction of a  $\mathcal{C}^M$ -function on  $\mathbb{R}^d$ . We shall discuss in § 10.1 when a function in  $\mathcal{C}^M(X)$  extends to a  $\mathcal{C}^M$ -function on  $\mathbb{R}^d$ .

DEFINITION 1.12. A subset  $X \subseteq \mathbb{R}^d$  is called  $\mathcal{A}^\omega$ -admissible (respectively,  $\mathcal{A}^M$ -admissible) if  $\mathcal{A}^\omega(X) = \mathcal{C}^\omega(X)$  (respectively,  $\mathcal{A}^M(X) = \mathcal{C}^M(X)$ ).

By the Bochnak–Siciak Theorem 1.5 and Result 1.3, all open subsets  $X \subseteq \mathbb{R}^d$  are  $\mathcal{A}^\omega$ -admissible and  $\mathcal{A}^M$ -admissible, for each log-convex non-quasianalytic  $M$ .

### 1.4 Main results

Our results can be arranged in groups with respect to two criteria: *regularity of the functions* (smooth, real analytic, ultradifferentiable) and *regularity of the domains* (Hölder sets, fat subanalytic sets).

By a *Hölder set* we mean the *closure* of an open set which has the uniform cusp property of index  $\alpha$  for some  $0 < \alpha \leq 1$ . If  $\alpha = 1$  we speak of a *Lipschitz set*. The collection of all Hölder sets in  $\mathbb{R}^d$  is denoted by  $\mathcal{H}(\mathbb{R}^d)$ . (We use the term *Hölder set* instead of *domain*, since the latter is usually reserved for open sets.) For precise definitions we refer to § 3.

*The smooth case.*

THEOREM 1.13. *Every  $X \in \mathcal{H}(\mathbb{R}^d)$  is  $\mathcal{A}^\infty$ -admissible. We even have*

$$\mathcal{A}_M^\infty(X) = \mathcal{A}^\infty(X) = \mathcal{C}^\infty(X), \quad (1.3)$$

for any non-quasianalytic log-convex positive sequence  $M = (M_k)$ .

Theorem 1.13 is proved in § 4.

THEOREM 1.14. *Every simple fat closed subanalytic set  $X \subseteq \mathbb{R}^d$  is  $\mathcal{A}^\infty$ -admissible.*

This is proved in § 5. The proof is based on the L-regular decomposition of subanalytic sets and the fact that fat closed subanalytic sets are uniformly polynomially cuspidal. It uses the result for Hölder sets (Theorem 1.13).

Remark 1.15. Hölder sets  $X \in \mathcal{H}(\mathbb{R}^d)$  and fat closed subanalytic subsets  $X \subseteq \mathbb{R}^d$  satisfy Lemma 1.10(6) and hence all items (1)–(5) in Lemma 1.10 are equivalent; cf. Proposition 3.8 and Theorem 5.6.

Notice that the assumption that  $X$  is simple is necessary; see Example 10.5. Hölder sets are always simple; see Proposition 3.9.

*The real analytic case.*

THEOREM 1.16. *Let  $X \subseteq \mathbb{R}^d$  be a fat closed subanalytic set. Let  $f \in \mathcal{C}^\infty(X)$  be real analytic on real analytic curves in  $X$ . Then  $f$  extends to a holomorphic function defined on an open neighborhood of  $X$  in  $\mathbb{C}^d$ .*

The proof of Theorem 1.16 (in § 6) is based on the uniformization theorem of subanalytic sets and a result of Eakin and Harris [EH77] (proved earlier by Gabriélov [Gab73]). The following consequence will also be proved in § 6.

COROLLARY 1.17. *Let  $X \subseteq \mathbb{R}^d$  be a closed set such that for all  $z \in \partial X$  there is a closed fat subanalytic set  $X_z$  such that  $z \in X_z \subseteq X$ . Let  $f \in \mathcal{C}^\infty(X)$  be real analytic on real analytic curves in  $X$ . Then  $f$  extends to a holomorphic function defined on an open neighborhood of  $X$  in  $\mathbb{C}^d$ .*

Note that all Hölder sets satisfy the assumption in Corollary 1.17. Interestingly, for these results we need not assume that  $X$  is simple (note that we already suppose that  $f \in \mathcal{C}^\infty(X)$ ). Together with Theorems 1.13 and 1.14, we obtain the following corollary.

COROLLARY 1.18. *Every  $X \in \mathcal{H}(\mathbb{R}^d)$  and every simple fat closed subanalytic  $X \subseteq \mathbb{R}^d$  is  $\mathcal{A}^\omega$ -admissible.*

*The ultradifferentiable case.* Let  $M = (M_k)$  be a non-quasianalytic log-convex positive sequence. For positive integers  $a$  let  $M^{(a)}$  denote the sequence defined by  $M_k^{(a)} := M_{ak}$ .

**THEOREM 1.19.** *Let  $M = (M_k)$  be a non-quasianalytic log-convex positive sequence. Every Lipschitz set  $X \subseteq \mathbb{R}^d$  satisfies  $\mathcal{C}^M(X) \subseteq \mathcal{A}^M(X) \subseteq \mathcal{C}^{M^{(2)}}(X)$ .*

A similar statement can be expected for Hölder sets (with the loss of regularity also depending on the Hölder index). We will not pursue this in this paper. Instead, combining our results with a result of [CC99, BBC18], we show in Theorem 8.4 that for fat closed subanalytic sets the loss of regularity can be controlled in a precise way.

In an earlier version of the paper we claimed that every Lipschitz set  $X \subseteq \mathbb{R}^d$  is  $\mathcal{A}^M$ -admissible. That is doubtful, but we do not have a counterexample.

### 1.5 Permanence of admissibility

The main results all concern subsets  $X \subseteq \mathbb{R}^d$  with maximal dimension  $d$ . The following permanence properties yield further examples of admissible sets both of maximal dimension and of codimension at least 1.

**PROPOSITION 1.20.** *Let  $X \subseteq \mathbb{R}^d$  be  $\mathcal{A}^\infty$ -admissible. If  $U$  is an open neighborhood of  $X$  in  $\mathbb{R}^d$  and  $\varphi : U \rightarrow \mathbb{R}^e$  is a smooth embedding, then  $\varphi(X) \subseteq \mathbb{R}^e$  is  $\mathcal{A}^\infty$ -admissible.*

*Proof.* Let  $Y := \varphi(X)$ . If  $f \in \mathcal{A}^\infty(Y)$ , then  $g := f \circ \varphi \in \mathcal{A}^\infty(X)$ . Since  $M := \varphi(U)$  is an embedded submanifold of  $\mathbb{R}^e$ , it suffices to show that for each  $y \in Y$  there exist a neighborhood  $V$  in  $M$  and a smooth function  $F : V \rightarrow \mathbb{R}$  such that  $F|_{V \cap Y} = f|_{V \cap Y}$ .

Since  $X$  is  $\mathcal{A}^\infty$ -admissible, for each  $x \in X$  there exist a neighborhood  $W$  in  $\mathbb{R}^d$  and a smooth function  $G : W \rightarrow \mathbb{R}$  such that  $G|_{W \cap X} = g|_{W \cap X}$ . Taking  $U \cap W$  instead of  $W$ , we may assume that  $W \subseteq U$ . Then  $F := G \circ \varphi^{-1}|_{\varphi(W)}$  is smooth on  $V := \varphi(W)$  and satisfies  $F|_{V \cap Y} = f|_{V \cap Y}$ .  $\square$

The same proof yields the following proposition.

**PROPOSITION 1.21.** *Let  $X \subseteq \mathbb{R}^d$  be  $\mathcal{A}^\omega$ -admissible. If  $U$  is an open neighborhood of  $X$  in  $\mathbb{R}^d$  and  $\varphi : U \rightarrow \mathbb{R}^e$  is a real analytic embedding, then  $\varphi(X) \subseteq \mathbb{R}^e$  is  $\mathcal{A}^\omega$ -admissible.*

In the ultradifferentiable case we have the following. Note that if  $M = (M_k)$  is log-convex, then  $\mathcal{C}^M$  is stable under composition and the  $\mathcal{C}^M$  inverse function theorem holds. If  $N \subseteq \mathbb{R}^e$  is an embedded submanifold of class  $\mathcal{C}^M$  (i.e. the chart change maps are of class  $\mathcal{C}^M$ ), then we define  $\mathcal{C}^M(N)$  to be the set of  $f \in \mathcal{C}^\infty(N)$  which are of class  $\mathcal{C}^M$  in every local coordinate chart. If  $Y \subseteq N$ , then let  $\mathcal{C}^M(Y)$  be the set of  $\mathcal{C}^\infty$ -functions on  $Y$  such that the defining estimates hold for all compact subsets in  $Y$  in all local coordinate charts. The proof of Proposition 1.20 implies the following.

**PROPOSITION 1.22.** *Let  $M = (M_k)$  be non-quasianalytic and log-convex, and let  $N = (N_k)$  be a sequence with  $M \leq N$ . Assume that  $X \subseteq \mathbb{R}^d$  satisfies  $\mathcal{C}^M(X) \subseteq \mathcal{A}^M(X) \subseteq \mathcal{C}^N(X)$ . If  $U$  is an open neighborhood of  $X$  in  $\mathbb{R}^d$  and  $\varphi : U \rightarrow \mathbb{R}^e$  is a  $\mathcal{C}^M$ -embedding, then  $Y := \varphi(X) \subseteq \mathbb{R}^e$  satisfies  $\mathcal{C}^M(Y) \subseteq \mathcal{A}^M(Y) \subseteq \mathcal{C}^N(Y)$ .*

### 1.6 Sharpness of the results

We discuss in § 10.2 counterexamples which show that none of the conditions in the main results can in general be omitted without suitable replacement.

In particular, Example 10.4, which is based on a division theorem of [JP90], shows that the  $\infty$ -flat cusp

$$X := \{(x, y) \in \mathbb{R}^2 : x \geq 0, 0 \leq y \leq \exp(-1/x)\}$$

is not  $\mathcal{A}^\infty$ -admissible; in this case  $\mathcal{A}^\infty(X)$  is strictly larger than  $\mathcal{C}^\infty(X)$ . Note, however, that for  $Y := \mathbb{R}^2 \setminus \text{int}(X)$  we have  $f \in \mathcal{A}^\infty(Y)$  if and only if  $f$  satisfies Lemma 1.10(3), but  $\mathcal{A}^\infty(Y) \neq \mathcal{C}^\infty(Y)$ ; see Example 10.9.

Interestingly, the analogue for finite differentiability (i.e. Result 1.2) fails even on convex fat closed sets such as the half-space; see Example 10.7, which is a consequence of Glaeser's inequality.

### 1.7 Applications

As a corollary of the real analytic result (i.e. Theorem 1.16) we obtain that smooth solutions of real analytic equations on Hölder sets or closed fat subanalytic sets must be real analytic; see Theorem 9.1. Furthermore, we obtain sufficient conditions for the existence of real analytic solutions  $g$  of the equation  $f = g \circ \varphi \in \mathcal{C}^\omega(M)$ , where  $\varphi : M \rightarrow \mathbb{R}^d$  is a real analytic map defined on a real analytic manifold  $M$ ; see Corollary 9.3.

The usefulness of the smooth result is illustrated by some consequences for the division of smooth functions (see Theorem 9.5) and for pseudo-immersions (see Theorem 9.6).

### 1.8 Structure of the paper

We recall facts on weight sequences and Denjoy–Carleman classes in § 2, and we revisit and adapt the  $\mathcal{C}^M$  curve lemma which is an essential tool for proving some results of the paper. In § 3 we introduce Hölder sets and collect some of their properties. The proofs of Theorems 1.13, 1.14, 1.16 and 1.19 are given in §§ 4, 5, 6, and 7, respectively. In § 8 we discuss the ultradifferentiable case on subanalytic sets. Applications are given in § 9. The final § 10 contains complements, examples, and counterexamples.

Some of the results of this paper were announced in [Rai17].

## 2. A $\mathcal{C}^M$ -curve lemma

This section is only of relevance for the ultradifferentiable results in the paper.

### 2.1 Weight sequences and Denjoy–Carleman classes

Let  $M = (M_k)_{k \in \mathbb{N}}$  be a positive sequence of reals. Let  $U \subseteq \mathbb{R}^d$  be open and let  $\mathcal{C}^M(U, \mathbb{R}^m)$  be the corresponding Denjoy–Carleman class (of Roumieu type) as defined in § 1.1.

If  $N = (N_k)$  is another positive sequence such that  $(M_k/N_k)^{1/k}$  is bounded, then  $\mathcal{C}^M(U) \subseteq \mathcal{C}^N(U)$ . The converse holds if  $k!M_k$  is logarithmically convex (*log-convex* for short). It follows that the class  $\mathcal{C}^M(U)$  is preserved by replacing  $M = (M_k)_k$  by  $(C^k M_k)_k$  for some positive constant  $C$ .

We shall assume that the sequence  $M$  is log-convex (which entails log-convexity of  $k!M_k$ ). We may assume that  $M_0 = 1$  and that  $M$  is increasing. Indeed, the sequence  $N_k := C^k M_k / M_0$  for some constant  $C \geq M_0 / M_1$ , is log-convex, increasing, satisfies  $N_0 = 1$ , and  $\mathcal{C}^M(U) = \mathcal{C}^N(U)$ . This motivates the following definition.

**DEFINITION 2.1.** An increasing log-convex sequence  $M = (M_k)$  with  $M_0 = 1$  is called a *weight sequence*.



The regularity properties of a weight sequence  $M = (M_k)$  entail stability properties of the class  $\mathcal{C}^M$ ; cf. [RS16]. Of particular interest in this paper is the fact that, for a weight sequence  $M$ , the composite of  $\mathcal{C}^M$  mappings is  $\mathcal{C}^M$ . By the celebrated Denjoy–Carleman theorem, the condition

$$\sum_k \frac{M_k}{(k+1)M_{k+1}} < \infty \tag{2.1}$$

holds if and only if  $\mathcal{C}^M$  is non-quasianalytic, that is, the Borel mapping which sends germs at some point  $a$  of smooth functions to their infinite Taylor expansion at  $a$  is not injective on  $\mathcal{C}^M$ -germs. Then there exist non-trivial  $\mathcal{C}^M$ -functions with compact support. Note that (2.1) is equivalent to

$$\sum_k (k! M_k)^{-1/k} < \infty. \tag{2.2}$$

DEFINITION 2.2. Let  $M = (M_k)$  be a weight sequence. We say that  $M$  is *non-quasianalytic* if it satisfies (2.1); otherwise it is said to be *quasianalytic*. A weight sequence  $M$  is called *strongly non-quasianalytic* if

$$\exists C > 0 \forall k \in \mathbb{N} : \sum_{j \geq k} \frac{M_{j-1}}{jM_j} \leq C \frac{M_{k-1}}{M_k}. \tag{2.3}$$

It is said to be of *moderate growth* if

$$\exists C > 0 \forall j, k \in \mathbb{N} : M_{j+k} \leq C^{j+k} M_j M_k. \tag{2.4}$$

A weight sequence is called *strongly regular* if it is strongly non-quasianalytic and of moderate growth.

Example 2.3. The Gevrey sequences  $G_k^s = k!^s$ ,  $s > 0$ , which give rise to the Gevrey classes  $\mathcal{C}^{G^s}$  are strongly regular weight sequences. They appear naturally in the theory of (partial) differential equations. For  $s = 0$  we recover the real analytic functions  $\mathcal{C}^{G^0} = \mathcal{C}^\omega$  which obviously form a quasianalytic class.

Note that  $\mathcal{C}^\omega(U) \subseteq \mathcal{C}^M(U) \subseteq \mathcal{C}^\infty(U)$  for every weight sequence  $M$ . In fact, the Denjoy–Carleman classes form an a scale of spaces intermediate between the real analytic and the smooth functions.

### 2.2 The $\mathcal{C}^M$ curve lemma revisited

We generalize the  $\mathcal{C}^M$  curve lemma (see [KMR09, § 3.6] and [KMR11, § 2.5]) which was inspired by [Bom67, Lemma 2].

LEMMA 2.4. *There are sequences  $t_k \rightarrow t_\infty$  and  $s_k > 0$  in  $\mathbb{R}$  with the following property. For any non-quasianalytic weight sequence  $M = (M_k)$  and each  $a \in \mathbb{N}_{\geq 2}$  there is a real positive sequence  $\lambda_k \rightarrow 0$  satisfying*

$$\lambda_k \left( \frac{M_{ak}}{M_k} \right)^{1/(ak+1)} \rightarrow 0 \quad \text{as } k \rightarrow \infty \tag{2.5}$$

such that the following holds. Let  $E$  be a Banach space. Let  $c_k \in C^\infty(\mathbb{R}, E)$  be a sequence such that

$$\{\lambda_k^{-1} c_k^{(\ell)}(t) : t \in I, \ell, k \in \mathbb{N}\} \tag{2.6}$$

is bounded in  $E$ , for every bounded interval  $I \subseteq \mathbb{R}$ . Then there exists a  $\mathcal{C}^M$ -curve  $c : \mathbb{R} \rightarrow E$  with compact support and  $c(t_k + t) = c_k(t)$  for  $|t| \leq s_k$ .

*Proof.* There exists a non-quasianalytic weight sequence  $L = (L_k)$  such that  $(M_k/L_k)^{1/k} \rightarrow \infty$  (this follows, for example, from [Kom79, Lemma 6]). Choose a  $\mathcal{C}^L$ -function  $\varphi : \mathbb{R} \rightarrow [0, 1]$  which is 0 on  $\{t : |t| \geq 1/2\}$  and 1 on  $\{t : |t| \leq 1/3\}$ .

Let  $T \in (0, 1]$  and  $R > 0$ . Assume that  $\gamma \in C^\infty(\mathbb{R}, E)$  is such that

$$\|\gamma^{(\ell)}(t)\| \leq R \quad \text{for all } |t| \leq 1/2, \ell \in \mathbb{N}.$$

Then there exist  $C, \rho \geq 1$  such that for the curve  $c(t) := \varphi(t/T)\gamma(t)$  we have

$$\begin{aligned} \|c^{(\ell)}(t)\| &= \left\| \sum_{j=0}^{\ell} \binom{\ell}{j} T^{-j} \varphi^{(j)}\left(\frac{t}{T}\right) \gamma^{(\ell-j)}(t) \right\| \\ &\leq R \sum_{j=0}^{\ell} \binom{\ell}{j} T^{-j} C \rho^j j! L_j \leq CR \left(1 + \frac{\rho}{T}\right)^\ell \ell! L_\ell \leq CR \left(\frac{2\rho}{T}\right)^\ell \ell! L_\ell. \end{aligned} \tag{2.7}$$

Choose a sequence

$$T_j \in (0, 1] \text{ with } \sum_j T_j < \infty, \text{ and let } t_k := 2 \sum_{j < k} T_j + T_k. \tag{2.8}$$

Now choose  $\lambda_j$  such that the following conditions are fulfilled:

$$0 < \frac{\lambda_j}{T_j^k} \leq \frac{M_k}{L_k} \quad \text{for all } j, k, \tag{2.9}$$

$$\frac{\lambda_j}{T_j^k} \rightarrow 0 \quad \text{as } j \rightarrow \infty \text{ for all } k. \tag{2.10}$$

It suffices to take  $\lambda_j \leq \inf_k T_j^{k+1} M_k/L_k$ . Clearly, we may in addition require that  $\lambda_j$  tends to zero fast enough so that (2.5) holds.

By (2.6), there is  $R > 0$  such that

$$\|c_k^{(\ell)}(t)\| \leq R \lambda_k \quad \text{for all } |t| \leq 1/2, \ell, k \in \mathbb{N}.$$

Define

$$c(t) := \sum_j \varphi\left(\frac{t-t_j}{T_j}\right) c_j(t-t_j).$$

The summands have disjoint supports (the support of the  $j$ th summand is contained in  $[t_j - T_j/2, t_j + T_j/2]$ ). Thus  $c$  is  $\mathcal{C}^\infty$  on  $\mathbb{R} \setminus \{t_\infty\}$ . By (2.7),

$$\|c^{(\ell)}(t)\| \leq CR \lambda_j \left(\frac{2\rho}{T_j}\right)^\ell \ell! L_\ell \quad \text{for } |t-t_j| \leq \frac{T_j}{2}.$$

Consequently, by (2.9),

$$\|c^{(\ell)}(t)\| \leq CR(2\rho)^\ell \ell! M_\ell \quad \text{for } t \neq t_\infty.$$

It follows that  $c : \mathbb{R} \rightarrow E$  has compact support and is  $\mathcal{C}^M$ ; cf. [KM97, Lemma 2.9] and [KMR09, Lemma 3.7]. □

*Remark 2.5.* A similar statement holds for convenient vector spaces  $E$ . The proof can be easily adapted to this case; cf. [KMR09] or [KMR11].

The next lemma is a variant of [KM97, Lemma 2.8]. Recall that, given some sequence  $\mu_k \rightarrow \infty$ , a sequence  $x_k$  in  $E$  is called  $\mu$ -convergent to  $x$  if  $\mu_k(x_k - x)$  is bounded.

LEMMA 2.6. For any non-quasianalytic weight sequence  $M = (M_k)$  there is a positive sequence  $\lambda_k \rightarrow 0$  such that the following holds. Let  $E$  be a Banach space. Let  $x_n \rightarrow x$  be  $1/\lambda_k$ -convergent in  $E$ . Then the infinite polygon through the  $x_n$  and  $x$  can be parameterized as a  $C^M$ -curve  $c : \mathbb{R} \rightarrow E$  such that  $c(1/n) = x_n$  and  $c(0) = x$ .

Proof. Let  $L = (L_k)$  be a non-quasianalytic weight sequence with  $(M_k/L_k)^{1/k} \rightarrow \infty$ . Set  $T_j := 1/(j(j+1))$  and choose  $\lambda_j$  such that the conditions (2.9) and (2.10) are satisfied. Let  $\varphi : \mathbb{R} \rightarrow [0, 1]$  be a  $C^L$ -function which vanishes on  $(-\infty, 0]$  and is 1 on  $[1, \infty)$ . Let  $t_n := 1/n$  and define

$$c(t) := \begin{cases} x & \text{if } t \leq 0, \\ x_{n+1} + \varphi\left(\frac{t - t_{n+1}}{t_n - t_{n+1}}\right)(x_n - x_{n+1}) & \text{if } t_{n+1} \leq t \leq t_n, \\ x_1 & \text{if } t \geq 1. \end{cases}$$

Clearly,  $c$  is  $C^\infty$  on  $\mathbb{R} \setminus \{0\}$ . For  $t_{n+1} \leq t \leq t_n$  we have

$$\begin{aligned} c^{(k)}(t) &= \varphi^{(k)}\left(\frac{t - t_{n+1}}{t_n - t_{n+1}}\right)(n(n+1))^k(x_n - x_{n+1}) \\ &= \varphi^{(k)}\left(\frac{t - t_{n+1}}{t_n - t_{n+1}}\right) \cdot \frac{\lambda_n}{T_n^k} \cdot \frac{x_n - x_{n+1}}{\lambda_n}. \end{aligned}$$

Condition (2.10) guarantees that  $c^{(k)}(t) \rightarrow 0$  as  $t \rightarrow 0$  for all  $k$ , and hence  $c$  is  $C^\infty$  on  $\mathbb{R}$ . That  $c$  is of class  $C^M$  follows from (2.9). □

### 3. Hölder sets

#### 3.1 Uniform cusp property and Hölder sets

We denote by  $B(x, \epsilon) := \{y \in \mathbb{R}^d : |x - y| < \epsilon\}$  the open ball with center  $x$  and radius  $\epsilon$  in  $\mathbb{R}^d$ .

DEFINITION 3.1 (Truncated open cusp). Let us consider  $\mathbb{R}^d = \mathbb{R}^{d-1} \times \mathbb{R}$  with the Euclidean coordinates  $x = (x_1, \dots, x_d) = (x', x_d)$ . For  $0 < \alpha \leq 1$  and  $r, h > 0$ , consider the truncated open cusp

$$\Gamma_d^\alpha(r, h) := \{(x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R} : |x'| < r, h(|x'|/r)^\alpha < x_d < h\}.$$

For  $\alpha = 1$  this is a truncated open cone.

DEFINITION 3.2 (Uniform cusp property). Let  $U \subseteq \mathbb{R}^d$  be an open set and let  $\alpha \in (0, 1]$ . We say that  $U$  has the uniform cusp property of index  $\alpha$  if for every  $x \in \partial U$  there exist  $\epsilon > 0$ , a truncated open cusp  $\Gamma = \Gamma_d^\alpha(r, h)$ , and an orthogonal linear map  $A \in O(d)$  such that for all  $y \in \bar{U} \cap B(x, \epsilon)$  we have  $y + A\Gamma \subseteq U$ .

DEFINITION 3.3 (Hölder set). By an  $\alpha$ -set we mean a closed fat set  $X \subseteq \mathbb{R}^d$  such that  $\text{int}(X)$  has the uniform cusp property of index  $\alpha$ . We say that  $X \subseteq \mathbb{R}^d$  is a Hölder set if it is an  $\alpha$ -set for some  $\alpha \in (0, 1]$ .

We denote by  $\mathcal{H}^\alpha(\mathbb{R}^d)$  the collection of all  $\alpha$ -sets in  $\mathbb{R}^d$  and by

$$\mathcal{H}(\mathbb{R}^d) := \bigcup_{0 < \alpha \leq 1} \mathcal{H}^\alpha(\mathbb{R}^d)$$

the collection of all Hölder sets in  $\mathbb{R}^d$ . Note that  $\mathcal{H}^\alpha(\mathbb{R}^d) \subseteq \mathcal{H}^\beta(\mathbb{R}^d)$  if  $\alpha \geq \beta$ .

*Remark 3.4.* A bounded open subset  $U \subseteq \mathbb{R}^d$  has the uniform cusp property of index  $\alpha$  if and only if  $U$  has Hölder boundary of index  $\alpha$  with uniformly bounded Hölder constant; see [DZ11, Theorem 6.9, p. 116] and [Gri85, Theorem 1.2.2.2]. That means the following. At each boundary point  $p$  there is an orthogonal system of coordinates  $(x', x_d)$  and an  $\alpha$ -Hölder function  $a = a(x')$  such that in a neighborhood of  $p$  the boundary of  $U$  is given by  $\{x_d = a(x')\}$  and the set  $U$  is of the form  $\{x_d > a(x')\}$ . There is a uniform bound for the Hölder constant of  $a$  which is independent of the boundary point  $p$ .

The boundary of an  $\alpha$ -set with  $\alpha < 1$  can be quite irregular. It may have Hausdorff dimension strictly larger than  $d - 1$  and hence its Hausdorff measure  $\mathcal{H}^{d-1}$  may be locally infinite. See [DZ11, Theorem 6.10, p. 116].

*Example 3.5.* (1) The set  $X = \{(x, y) \in \mathbb{R}^2 : x \geq 0, |y| \leq x^{1/\alpha}\}$  is an  $\alpha$ -set.

(2) The set  $X = \{(x, y) \in \mathbb{R}^2 : x \geq 0, x^2 \leq y \leq 2x^2\}$  is not a Hölder set, but  $X$  is the image of the Hölder set  $\{(x, y) \in \mathbb{R}^2 : x \geq 0, |y| \leq x^2/2\}$  under the diffeomorphism  $(x, y) \mapsto (x, y + 3x^2/2)$  of  $\mathbb{R}^2$ .

(3) The set  $X = \{(x, y) \in \mathbb{R}^2 : x \geq 0, x^{3/2} \leq y \leq 2x^{3/2}\}$  is not a Hölder set and there is no smooth diffeomorphism of  $\mathbb{R}^2$  which maps  $X$  to a Hölder set.

(4) Let  $C \subseteq [0, 1]$  be the Cantor set and let  $f : [0, 1] \rightarrow \mathbb{R}$  be defined by  $f(x) := \text{dist}(x, C)^\alpha$ . Then the set  $X = \{(x, y) \in \mathbb{R}^2 : -1 \leq x \leq 2, f(x) \leq y \leq 2 \text{ if } x \in [0, 1], 0 \leq y \leq 2 \text{ if } x \notin [0, 1]\}$  is an  $\alpha$ -set.

### 3.2 $c^\infty$ -topology on Hölder sets

The  $c^\infty$ -topology on a locally convex space  $E$  is the final topology with respect to all smooth curves  $c : \mathbb{R} \rightarrow E$ . The  $c^\infty$ -topology on  $\mathbb{R}^d$  coincides with the usual topology; cf. [KM97, Theorem 4.11]. The  $c^\infty$ -topology on a subset  $X \subseteq E$  is the final topology with respect to all smooth curves  $c : \mathbb{R} \rightarrow E$  satisfying  $c(\mathbb{R}) \subseteq X$ .

**PROPOSITION 3.6.** *Let  $X \in \mathcal{H}(\mathbb{R}^d)$ . Then the  $c^\infty$ -topology of  $X$  coincides with the trace topology from  $\mathbb{R}^d$ .*

*Proof.* Let  $A \subseteq X$  be  $c^\infty$ -closed in  $X$ . Let  $\bar{A}$  be the closure of  $A$  in  $\mathbb{R}^d$ . We have to show that  $\bar{A} \cap X = \bar{A} \subseteq A$ . The converse implication is obvious.

Let  $x \in \bar{A}$ . Then there is a sequence  $x_n \in A$  which tends to  $x$ . It suffices to find a smooth curve  $c \in C^\infty(\mathbb{R}, X)$  passing through a subsequence of  $x_n$  and through  $x$ . Since  $A$  is  $c^\infty$ -closed in  $X$ , this shows  $x \in A$ .

Since  $X$  is an  $\alpha$ -set, for some  $0 < \alpha \leq 1$ , we may assume that there exist a neighborhood  $U$  of  $x$  in  $X$  and a cusp  $\Gamma = \Gamma_d^\alpha(r, h)$  such that for all  $y \in U$  we have  $y + \Gamma \subseteq \text{int}(X)$ . By rescaling, we may assume that  $r = h = 1$ .

Consider  $C(y, r) := y + \Gamma_d^\alpha(r, r^\alpha)$  for  $0 < r \leq 1$ . It is easy to see that there is a universal constant  $c > 0$  such that  $C(y_1, r_1) \cap C(y_2, r_2) \neq \emptyset$  provided that  $|y_1 - y_2| \leq c \min\{r_1, r_2\}$ .

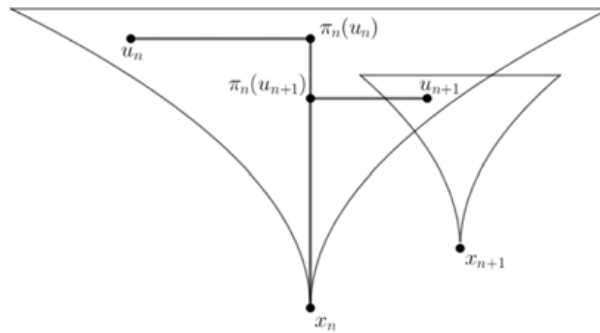


FIGURE 1. The polygon  $P_n$ .

Choose a decreasing sequence  $\mu_n$  which tends to 0 faster than any polynomial. By passing to a subsequence of  $x_n$  (again denoted by  $x_n$ ), we may assume that  $|x - x_n| \leq c\mu_{n+1}/2$  for all  $n$ . Then, for all  $n$ ,

$$|x_n - x_{n+1}| \leq |x - x_n| + |x - x_{n+1}| \leq c\mu_{n+1}.$$

Setting  $C_n := C(x_n, \mu_n)$ , this guarantees the existence of a sequence  $u_n$  such that  $u_{n+1} \in C_n \cap C_{n+1}$  for all  $n$ . By construction,  $x_n$  and  $u_n$  tend to  $x$  faster than any polynomial.

For  $u \in C_n$  define  $\pi_n(u) := x_n + u_d e_d$  (where  $\{e_i\}$  is the standard basis in  $\mathbb{R}^d$ ). Consider the polygon  $P_n$  through the points  $u_n, \pi_n(u_n), x_n, \pi_n(u_{n+1}), u_{n+1}$  (see Figure 1). It is contained in  $C_n$ . The infinite polygon consisting of the concatenation of all  $P_n$  satisfies the assumptions of [KM97, Lemma 2.8] and can hence be parameterized by a smooth curve  $c$  which is contained in  $X$  and satisfies  $c(0) = x$ .  $\square$

*Remark 3.7.* It is not difficult to modify the proof in order to obtain the following result. Let  $X \in \mathcal{H}(\mathbb{R}^d)$  and let  $M = (M_k)$  be a non-quasianalytic weight sequence. Then the final topology on  $X$  with respect to all  $C^M$ -curves  $c : \mathbb{R} \rightarrow \mathbb{R}^d$  with  $c(\mathbb{R}) \subseteq X$  coincides with the trace topology from  $\mathbb{R}^d$ . It suffices to take  $\mu_n := \lambda_n^{1/\alpha}$  for the sequence  $\lambda_n$  provided by Lemma 2.6.

### 3.3 Further properties of Hölder sets

The following proposition is well known. We include a proof for the convenience of the reader.

**PROPOSITION 3.8.** *Let  $X \in \mathcal{H}^\alpha(\mathbb{R}^d)$ . Then for each  $x \in X$  there exist a compact neighborhood  $K$  of  $x$  in  $X$  and a constant  $D > 0$  such that any two points  $y_1, y_2 \in K$  can be joined by a polygon  $\gamma$  contained in  $K$  with  $\partial X \cap \gamma \subseteq \{y_1, y_2\}$  of length*

$$\ell(\gamma) \leq D|y_1 - y_2|^\alpha.$$

*Proof.* Clearly each  $x \in \text{int}(X)$  has this property. Let  $x \in \partial X$ . We may assume that in a compact neighborhood  $K$  of  $x$  the set  $X$  is the epigraph  $\{x_d \geq f(x')\}$  of a  $\alpha$ -Hölder function  $f$  with respect to an orthogonal system of coordinates  $(x', x_d) = (x_1, \dots, x_d)$ . For two points  $y_1, y_2 \in K$  consider the segments  $S := [y_1, y_2]$  and  $S' := [y'_1, y'_2]$ . If  $(y_1, y_2) \subseteq K \cap \text{int}(X)$  there is nothing

to prove. Otherwise let  $z' \in S'$  be such that  $f(z') = \max_{y' \in S'} f(y')$  and let  $z = (z', z_d)$  with  $z_d := f(z') + \epsilon|y_1 - y_2|$  for some small  $\epsilon > 0$  such that  $z \in K \cap \text{int}(X)$ . It is possible to choose  $\epsilon$  such that it only depends on  $K$ , not on  $y_1, y_2$ . We have  $(y_i)_d \leq f(z')$  and thus  $|(y_i)_d - f(z')| \leq |f(y'_i) - f(z')|$  for at least one  $i \in \{1, 2\}$ , say for  $i = 1$ . If  $(y_2)_d \leq f(z')$ , then the polygon with vertices  $y_1, (y'_1, z_d), (y'_2, z_d), y_2$  is contained in  $K$ , meets  $\partial X$  at most at one of the points  $y_i$ , and has length

$$\begin{aligned} & |(y_1)_d - z_d| + |(y_2)_d - z_d| + |y'_1 - y'_2| \\ & \leq |f(y'_1) - f(z')| + |f(y'_2) - f(z')| + 2\epsilon|y_1 - y_2| + |y'_1 - y'_2| \\ & \leq C|y'_1 - z'|^\alpha + C|y'_2 - z'|^\alpha + (1 + 2\epsilon)|y_1 - y_2| \\ & \leq D|y_1 - y_2|^\alpha, \end{aligned}$$

for constants only depending on  $K$ . If  $(y_2)_d > f(z')$ , then the segment joining  $z$  and  $y_2$  is contained in  $K \cap \text{int}(X)$ , and thus the polygon with vertices  $y_1, (y'_1, z_d), z, y_2$  is contained in  $K$ , meets  $\partial X$  at most at one of the points  $y_i$ , and has length

$$\begin{aligned} & |(y_1)_d - z_d| + |y_2 - z| + |y'_1 - z'| \\ & \leq |f(y'_1) - f(z')| + (1 + \epsilon)|y_1 - y_2| + |y'_1 - y'_2| \\ & \leq D|y_1 - y_2|^\alpha. \end{aligned}$$

This finishes the proof. □

PROPOSITION 3.9. *Every  $X \in \mathcal{H}(\mathbb{R}^d)$  is simple in the sense of Definition 1.11.*

*Proof.* The proof of Proposition 3.8 implies that there is a basis of neighborhoods  $\mathcal{U}$  of each  $x \in X$  such that  $\text{int}(X) \cap U$  is path-connected for each  $U \in \mathcal{U}$ . □

#### 4. Arc-smooth functions on Hölder sets

The aim of this section is to prove Theorem 1.13: *Every  $X \in \mathcal{H}(\mathbb{R}^d)$  is  $\mathcal{A}^\infty$ -admissible. We even have*

$$\mathcal{A}_M^\infty(X) = \mathcal{A}^\infty(X) = \mathcal{C}^\infty(X), \tag{4.1}$$

for any non-quasianalytic weight sequence  $M = (M_n)$ .

*Remark 4.1.* For fat closed convex sets  $X \subseteq \mathbb{R}^d$ ,  $\mathcal{A}^\infty$ -admissibility follows from a result of Kriegl [Kri97]. The statement in [Kri97] is more general: *Let  $X$  be a convex subset of a convenient vector space  $E$  with non-empty interior. Then  $f \in \mathcal{A}^\infty(X)$  if and only if  $f$  is smooth on  $\text{int}(X)$  and all Fréchet derivatives  $(f|_{\text{int}(X)})^{(n)}$  extend continuously to  $f^{(n)} : X \rightarrow L_n(E, \mathbb{R})$  with respect to the  $c^\infty$ -topology of  $X$ . In general the  $c^\infty$ -topology is finer than the given locally convex topology.*

It is evident that

$$\mathcal{C}^\infty(X) \subseteq \mathcal{A}^\infty(X) \subseteq \mathcal{A}_M^\infty(X).$$

The second inclusion is by definition; the first inclusion is a simple consequence of the chain rule. Let us prove the other inclusions.

LEMMA 4.2. *Let  $1 \leq p \leq q$  be integers. For  $x \in \mathbb{R}^d$  and  $v = (v', v_d) \in \mathbb{R}^d$  let  $c(t) = x + (t^q v', t^p v_d)$ , for  $t$  in a neighborhood of  $0 \in \mathbb{R}$ . Let  $f$  be of class  $\mathcal{C}^q$  in a neighborhood of the image of  $c$ . Then*

$$\frac{(f \circ c)^{(k)}(0)}{k!} = \begin{cases} \frac{1}{j!} f^{(j)}(x)((0, v_d)^j) & \text{if } k = jp < q, \\ f'(x)((v', 0)) & \text{if } k = q \notin p\mathbb{N}, \\ f'(x)((v', 0)) + \frac{1}{j!} f^{(j)}(x)((0, v_d)^j) & \text{if } k = jp = q. \end{cases}$$

For all other  $k < q$  we have  $(f \circ c)^{(k)}(0) = 0$ .

*Proof.* This follows easily from an inspection of the Faà di Bruno formula

$$\frac{(f \circ c)^{(k)}(t)}{k!} = \sum_{j \geq 1} \sum_{\substack{\alpha_i > 0 \\ \alpha_1 + \dots + \alpha_j = k}} \frac{f^{(j)}(c(t))}{j!} \left( \frac{c^{(\alpha_1)}(t)}{\alpha_1!}, \dots, \frac{c^{(\alpha_j)}(t)}{\alpha_j!} \right)$$

and the special form of  $c$ . □

PROPOSITION 4.3. *Let  $X \in \mathcal{H}(\mathbb{R}^d)$  and  $f \in \mathcal{A}^\infty(X)$ . Then  $f|_{\text{int}(X)}$  is smooth and its derivative  $(f|_{\text{int}(X)})'$  extends uniquely to a mapping  $f' : X \rightarrow L(\mathbb{R}^d, \mathbb{R})$  which belongs to  $\mathcal{A}^\infty(X, L(\mathbb{R}^d, \mathbb{R}))$ , that is,*

$$(f')_* \mathcal{C}^\infty(\mathbb{R}, X) \subseteq \mathcal{C}^\infty(\mathbb{R}, L(\mathbb{R}^d, \mathbb{R})). \tag{4.2}$$

*Proof.* That  $f|_{\text{int}(X)}$  is smooth follows from Boman’s Theorem 1.1.

There is  $0 < \alpha \leq 1$  such that  $X \in \mathcal{H}^\alpha(\mathbb{R}^d)$ . Let  $x \in \partial X$ . We may assume that there exist a truncated open cusp  $\Gamma = \Gamma_d^\alpha(r, h)$  and an open neighborhood  $Y$  of  $x$  in  $X$  such that for all  $y \in Y$  we have  $y + \Gamma \subseteq \text{int}(X)$ . It suffices to show that  $(f|_{Y \cap \text{int}(X)})'$  extends uniquely to a mapping  $f' : Y \rightarrow L(\mathbb{R}^d, \mathbb{R})$  which belongs to  $\mathcal{A}^\infty(Y, L(\mathbb{R}^d, \mathbb{R}))$ .

Let  $p < q$  be positive integers such that  $p/q \leq \alpha$  and  $q/p \notin \mathbb{N}$ . Let  $x \in Y$  and  $v = (v', v_d) \in \Gamma$ . Then the curve

$$c_{x,v}(t) := x + (t^{2q} v', t^{2p} v_d)$$

lies in  $\text{int}(X)$  for  $0 < |t| < 1$  and  $c_{x,v}(0) = x$ . Since  $f \in \mathcal{A}^\infty(X)$ ,  $f \circ c_{x,v}$  is  $\mathcal{C}^\infty$ .

Let  $v \in \Gamma$  be fixed. We define

$$f'(x)(v) := \frac{(f \circ c_{x,v})^{(2p)}(0)}{(2p)!} + \frac{(f \circ c_{x,v})^{(2q)}(0)}{(2q)!} \quad \text{for } x \in Y.$$

This definition turns into a correct statement if  $x \in \text{int}(X)$ , by Lemma 4.2.

We claim that

$$f'(\cdot)(v) : Y \rightarrow \mathbb{R} \text{ maps } \mathcal{C}^\infty\text{-curves to } \mathcal{C}^\infty\text{-curves.} \tag{4.3}$$

Let  $\mathbb{R} \ni s \mapsto x(s)$  be a  $\mathcal{C}^\infty$ -curve in  $Y$ . Then  $(s, t) \mapsto c_{x(s),v}(t)$  is a smooth mapping near  $(0, 0)$  with values in  $X$ . Thus  $(s, t) \mapsto f(c_{x(s),v}(t))$  is smooth, by Boman’s Theorem 1.1. So, in particular,  $s \mapsto \partial_t^k|_{t=0}(f \circ c_{x(s),v}(t))$  is smooth for all  $k$ . It follows that  $s \mapsto f'(x(s))(v)$  is smooth, which implies the claim.

Let  $s \mapsto x(s)$  be any  $C^\infty$ -curve in  $Y$  such that  $x(s) \in \text{int}(X)$  for  $0 < |s| \leq 1$  and  $x(0) = x_0$ . Then

$$\begin{aligned} f'(x_0)(v) &= \frac{(f \circ c_{x_0,v})^{(2p)}(0)}{(2p)!} + \frac{(f \circ c_{x_0,v})^{(2q)}(0)}{(2q)!} \\ &= \lim_{s \rightarrow 0} \left( \frac{(f \circ c_{x(s),v})^{(2p)}(0)}{(2p)!} + \frac{(f \circ c_{x(s),v})^{(2q)}(0)}{(2q)!} \right) = \lim_{s \rightarrow 0} f'(x(s))(v). \end{aligned}$$

Consequently, the given definition of  $f'(x_0)(v)$  is the only possible extension of  $f'(\cdot)(v)$  to  $x_0$  which is continuous on  $C^\infty$ -curves.

Now let  $v \in \mathbb{R}^d$  be arbitrary. Since  $\Gamma$  is open, there exist  $\epsilon > 0$  and  $w \in \Gamma$  such that  $\epsilon v + w \in \Gamma$ . For all  $x \in Y \cap \text{int}(X)$ , we have

$$f'(x)(v) = \frac{f'(x)(\epsilon v + w) - f'(x)(w)}{\epsilon},$$

and the right-hand side uniquely extends to points  $x_0 \in Y \cap \partial X$  and satisfies (4.3), by the arguments above.

Thus, we define  $f'(x_0)(v) := \lim_{s \rightarrow 0} f'(x(s))(v)$  for some  $C^\infty$ -curve  $s \mapsto x(s)$  in  $Y$  with  $x(0) = x_0$  and  $x(s) \in \text{int}(X)$  for  $0 < |s| \leq 1$ . Then  $f'(x_0)$  is linear as the pointwise limit of  $f'(x(s)) \in L(\mathbb{R}^d, \mathbb{R})$ . The definition does not depend on the curve  $x$ , since it is the unique extension for  $v \in \Gamma$ .

Let us finally show that  $f' : Y \rightarrow L(\mathbb{R}^d, \mathbb{R})$  belongs to  $\mathcal{A}^\infty(Y, L(\mathbb{R}^d, \mathbb{R}))$ . Let  $x : \mathbb{R} \rightarrow Y$  be a  $C^\infty$ -curve and let  $v \in \mathbb{R}^d$ . It suffices to show that  $s \mapsto f'(x(s))(v)$  is smooth. For  $v \in \Gamma$  this follows from (4.3). For general  $v$ ,  $f'(x(s))(v)$  is a linear combination of  $f'(x(s))(v_1)$  and  $f'(x(s))(v_2)$  for  $v_i \in \Gamma$  which locally is independent of  $s$ . The proof is complete.  $\square$

**COROLLARY 4.4.** *Let  $M = (M_k)$  be a non-quasianalytic weight sequence. Let  $X \in \mathcal{H}(\mathbb{R}^d)$  and  $f \in \mathcal{A}_M^\infty(X)$ . Then  $f|_{\text{int}(X)}$  is smooth and its derivative  $(f|_{\text{int}(X)})'$  extends uniquely to a mapping  $f' : X \rightarrow L(\mathbb{R}^d, \mathbb{R})$  which belongs to  $\mathcal{A}_M^\infty(X, L(\mathbb{R}^d, \mathbb{R}))$ , that is,*

$$(f')_* \mathcal{C}^M(\mathbb{R}, X) \subseteq \mathcal{C}^\infty(\mathbb{R}, L(\mathbb{R}^d, \mathbb{R})). \tag{4.4}$$

*Proof.* The proof is the same with the only difference that we use  $\mathcal{C}^M$ -curves (thanks to Remark 1.4); note that the curves  $c_{x,v}$  are polynomial and thus of class  $\mathcal{C}^M$ .  $\square$

*Proof of Theorem 1.13.* Let  $f \in \mathcal{A}^\infty(X)$  (respectively,  $f \in \mathcal{A}_M^\infty(X)$ ). Proposition 4.3 and Corollary 4.4 imply by induction that the Fréchet derivatives  $(f|_{\text{int}(X)})^{(n)}$  of all orders have unique extensions  $f^{(n)} : X \rightarrow L_n(\mathbb{R}^d, \mathbb{R})$  which satisfy

$$(f^{(n)})_* \mathcal{C}^\infty(\mathbb{R}, X) \subseteq \mathcal{C}^\infty(\mathbb{R}, L_n(\mathbb{R}^d, \mathbb{R}))$$

(respectively,  $(f^{(n)})_* \mathcal{C}^M(\mathbb{R}, X) \subseteq \mathcal{C}^\infty(\mathbb{R}, L_n(\mathbb{R}^d, \mathbb{R}))$ ). So  $f$  satisfies Lemma 1.10(3), since the  $\mathcal{C}^\infty$ -topology of  $X$  (respectively, the final topology on  $X$  with respect to all  $\mathcal{C}^M$ -curves in  $X$ ) coincides with the trace topology from  $\mathbb{R}^d$ , by Proposition 3.6 (respectively, Remark 3.7). Thus  $f \in \mathcal{C}^\infty(X)$ , by Lemma 1.10 and Proposition 3.8.  $\square$

### 5. Arc-smooth functions on subanalytic sets

The goal of this section is to prove Theorem 1.14.



### 5.1 Subanalytic sets

Let  $M$  be a real analytic manifold. A subset  $X$  of  $M$  is called *subanalytic* if for each  $x \in M$  there is an open neighborhood  $U$  of  $x$  in  $M$  such that  $X \cap U$  is the projection of a relatively compact semianalytic subset of  $M \times N$ , where  $N$  is a real analytic manifold. Recall that a subset  $X$  of a real analytic manifold  $M$  is semianalytic if for each  $x \in M$  there exist an open neighborhood  $U$  of  $x$  in  $M$  and finitely many real analytic functions  $f_{ij}, g_{ij}$  on  $U$  such that

$$X \cap U = \bigcup_i \bigcap_j \{f_{ij} = 0, g_{ij} > 0\}.$$

If  $\dim M \leq 2$ , then the family of subanalytic sets in  $M$  coincides with the family of semianalytic sets. In higher dimensions the family of subanalytic sets is essentially larger.

Henceforth we restrict to the case  $M = \mathbb{R}^d$ .

**THEOREM 5.1** (Rectilinearization of subanalytic sets [Hir73, BM88, Par94b]). *Let  $X \subseteq \mathbb{R}^d$  be closed subanalytic. There exists a locally finite collection of real analytic mappings  $\varphi_\alpha : U_\alpha \rightarrow \mathbb{R}^d$  such that each  $\varphi_\alpha$  is the composite of a finite sequence of local blow-ups with smooth centers and:*

- (i) *each  $U_\alpha$  is diffeomorphic to  $\mathbb{R}^d$  and there are compact subsets  $K_\alpha \subseteq U_\alpha$  such that  $\bigcup_\alpha \varphi_\alpha(K_\alpha)$  is a neighborhood of  $X$  in  $\mathbb{R}^d$ ;*
- (ii) *for each  $\alpha$ ,  $\varphi_\alpha^{-1}(X)$  is a union of quadrants in  $\mathbb{R}^d$ .*

A *quadrant* in  $\mathbb{R}^d$  is a set

$$Q(I_0, I_-, I_+) = \{x \in \mathbb{R}^d : x_i = 0 \text{ if } i \in I_0, x_i \leq 0 \text{ if } i \in I_-, x_i \geq 0 \text{ if } i \in I_+\},$$

where  $I_0, I_-, I_+$  is any partition of  $\{1, 2, \dots, d\}$ .

### 5.2 Bounded fat subanalytic sets are uniformly polynomially cuspidal

This is due to Pawłucki and Pleśniak [PP86]. We recall some steps of the proof which will be needed later.

**DEFINITION 5.2.** A subset  $X \subseteq \mathbb{R}^d$  is called *uniformly polynomially cuspidal* if there exist positive constants  $M, m > 0$  and a positive integer  $n$  such that for all  $x \in \bar{X}$  there is a polynomial curve  $h_x : \mathbb{R} \rightarrow \mathbb{R}^d$  of degree at most  $n$  with the following properties:

- (i)  $h_x((0, 1]) \subseteq X$  and  $h_x(0) = x$ ;
- (ii)  $\text{dist}(h_x(t), \mathbb{R}^d \setminus X) \geq Mt^m$  for all  $x \in X$  and all  $t \in (0, 1]$ .

*Remark 5.3.* Every compact set  $X \in \mathcal{H}(\mathbb{R}^d)$  is uniformly polynomially cuspidal; this is clear by Definition 5.2. The converse is not true: for instance, the sets in Example 3.5(2) and (3) are uniformly polynomially cuspidal but not in  $\mathcal{H}(\mathbb{R}^d)$ . The set  $X$  in Example 10.9 is uniformly polynomially cuspidal but neither subanalytic nor in  $\mathcal{H}(\mathbb{R}^d)$ ; cf. [PP88, p. 284].

**THEOREM 5.4** [PP86, Proposition 6.3]. *Let  $X$  be a bounded open subanalytic subset of  $\mathbb{R}^d$ . Then there is a map  $h : \bar{X} \times \mathbb{R} \rightarrow \mathbb{R}^d$  such that  $h(x, t)$  is a polynomial in  $t$  with degree  $n$  independent of  $x \in \bar{X}$  with  $h(x, 0) = x$  for all  $x \in \bar{X}$ ,  $h(\bar{X} \times (0, 1]) \subseteq X$ , and there exist positive constants  $M, m$  such that*

$$\text{dist}(h(x, t), \mathbb{R}^d \setminus X) \geq Mt^m \quad \text{for all } x \in \bar{X}, t \in [0, 1].$$

We give a sketch of the proof in order to explicate the uniformity of  $h_x$  which will be of importance later.

The following is a corollary of the rectilinearization theorem.

PROPOSITION 5.5 [PP86, Proposition 6.3]. *Let  $X$  be a relatively compact subanalytic subset of  $\mathbb{R}^d$  of pure dimension  $d$ . Then there are a finite number of real analytic maps  $\varphi_j : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}^d$  such that, for  $I^d := [-1, 1]^d$ ,*

$$\begin{aligned} \varphi_j(I^d \times (0, 1]) &\subseteq X \quad \text{for all } j, \\ \bigcup_j \varphi_j(I^d \times \{0\}) &= \overline{X}. \end{aligned}$$

Let  $X$  be a bounded open subanalytic subset of  $\mathbb{R}^d$ . Let  $\varphi_j$  be the maps provided by Proposition 5.5. Then, for each  $j$ , the function

$$I^d \times [0, 1] \ni (y, t) \mapsto \text{dist}(\varphi_j(y, t), \mathbb{R}^d \setminus X)$$

is subanalytic (cf. [BM88, Remark 3.11]). By the Łojasiewicz inequality (cf. [BM88, Theorem 6.4]), there exist positive constants  $L, m$  such that

$$\text{dist}(\varphi_j(y, t), \mathbb{R}^d \setminus X) \geq Lt^m, \quad (y, t) \in I^d \times [0, 1].$$

The constants  $L, m$  may be assumed independent of  $j$ , by taking the minimum and maximum, respectively. Choose an integer  $n \geq m$  and write

$$\varphi_j(y, t) = T_j(y, t) + t^{n+1}Q_j(y, t), \quad (y, t) \in \mathbb{R}^d \times \mathbb{R},$$

where  $T_j(y, \cdot)$  is the Taylor polynomial at 0 of degree  $n$  of  $\varphi_j(y, \cdot)$  and  $Q_j : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}^d$  is real analytic. If we choose  $\delta \in (0, 1]$  such that  $|tQ_j(y, t)| \leq L/2$  for all  $j, y \in I^d$ , and  $t \in [0, \delta]$ , then

$$\text{dist}(T_j(y, t), \mathbb{R}^d \setminus X) \geq Lt^m - \frac{L}{2}t^n \geq \frac{L}{2}t^m, \quad (y, t) \in I^d \times [0, \delta].$$

Replacing  $t$  by  $\delta t$ , we obtain

$$\text{dist}(T_j(y, \delta t), \mathbb{R}^d \setminus X) \geq Mt^m, \quad (y, t) \in I^d \times [0, 1],$$

where  $M := L\delta^m/2$ . Clearly,  $\bigcup_j T_j(I^d \times \{0\}) = \bigcup_j \varphi_j(I^d \times \{0\}) = \overline{X}$ . Theorem 5.4 follows.

### 5.3 Fat closed subanalytic sets are $m$ -regular

Another property of fat closed subanalytic sets we need is the fact that they are  $m$ -regular in the following sense.

THEOREM 5.6 ([Bie80, Theorem 6.17], [Har83], [BM88, Theorem 6.10]). *Let  $X \subseteq \mathbb{R}^d$  be a fat closed subanalytic set. For each  $a \in X$  there exist a compact neighborhood  $K$  in  $X$ , a constant  $C > 0$ , and a positive integer  $m$  such that any two points  $x, y \in K$  can be joined by a semianalytic path  $\gamma$  in  $X$  which intersects  $\partial X$  in at most finitely many points and satisfies*

$$\ell(\gamma) \leq C|x - y|^{1/m}.$$

**5.4 L-regular decomposition**

Let us recall the *L-regular decomposition* of subanalytic sets.

First we introduce sets which are *L-regular* with respect to a given system of coordinates. Let  $X \subseteq \mathbb{R}^d$  be a subanalytic set of dimension  $d$ . If  $d = 1$ , then  $X$  is called L-regular if  $X$  is a non-empty compact interval. If  $d > 1$ , then  $X$  is L-regular if it is of the form

$$X = \{(x', x_d) \in \mathbb{R}^d : f(x') \leq x_d \leq g(x'), x \in X'\}, \tag{5.1}$$

where  $X' \subseteq \mathbb{R}^{d-1}$  is L-regular and  $f, g$  are continuous subanalytic functions on  $X'$ , analytic and satisfying  $f < g$  on  $\text{int}(X')$  with bounded partial derivatives of first order. If  $\dim X = k < d$ , then  $X$  is L-regular if

$$X = \{(y, z) \in \mathbb{R}^k \times \mathbb{R}^{d-k} : z = h(y), y \in X'\}, \tag{5.2}$$

where  $X' \subseteq \mathbb{R}^k$  is L-regular,  $\dim X' = k$ , and  $h$  is continuous subanalytic on  $X'$ , analytic on  $\text{int}(X')$  with bounded partial derivatives of first order.

In general a subanalytic set  $X$  in  $\mathbb{R}^d$  is said to be *L-regular* if it is L-regular with respect to some linear (or equivalently orthogonal) system of coordinates. It is called an *L-regular cell* if it is the relative interior of an L-regular set, that is, it is  $\text{int}(X)$  in case (5.1) and the graph of  $h$  restricted to  $\text{int}(X')$  in case (5.2). By definition, every point is a zero-dimensional L-regular cell.

It is well known that L-regular sets and L-regular cells are *quasiconvex* (cf. [Kur92], [Par94a, Lemma 2.2], or [KP06]): there is a constant  $C > 0$  such that any two points  $x, y$  in the set can be joined in the set by a subanalytic path of length at most  $C|x - y|$ .

**THEOREM 5.7** [Kur92, KP06, Paw08]. *Let  $X \subseteq \mathbb{R}^d$  be a bounded subanalytic set. Then  $X$  is a finite disjoint union of L-regular cells.*

For the proof of Theorem 1.14 we need the following preparatory results.

**LEMMA 5.8.** *Let  $[a, b] \subseteq \mathbb{R}$  be a non-trivial interval such that  $[a, b] = \bigcup_{i=1}^k F_i$  for closed sets  $F_i$ . If  $a \leq \sup F_i < b$  then there exists  $j \neq i$  such that  $\sup F_i \in F_j$  and  $\sup F_i < \sup F_j$ .*

*Proof.* Fix  $i$  and suppose that  $t := \sup F_i < b$ . There is a sequence  $(t, b) \ni t_n \rightarrow t$ . After passing to a subsequence we may assume that  $t_n \in F_j$  for some fixed  $j \neq i$ . Since  $F_j$  is closed,  $t \in F_j$ .  $\square$

**LEMMA 5.9.** *Let  $X \subseteq \mathbb{R}^d$  be a fat closed subanalytic set. Let  $x \in \partial X$  and suppose there is a basis of neighborhoods  $\mathcal{U}$  of  $x$  such that  $U \cap \text{int}(X)$  is connected for all  $U \in \mathcal{U}$ . Then there exist  $U_0 \in \mathcal{U}$  and a positive constant  $C$  such that the following holds. For all  $U \in \mathcal{U}_0 := \{U \in \mathcal{U} : U \subseteq U_0\}$  and for any two points  $y, z \in U \cap \text{int}(X)$ , there exists a rectifiable path  $\gamma$  in  $\text{int}(X)$  which connects  $y$  and  $z$  and satisfies*

$$\ell(\gamma) \leq C \text{diam}(U).$$

*Proof.* We may assume that  $X$  is bounded, by intersecting with a ball centered at  $x$ . Let  $U_0$  be any member of  $\mathcal{U}$  which is contained in this ball. By Theorem 5.7,  $\text{int}(X)$  is a finite disjoint union of L-regular cells  $\{A_1, \dots, A_k\}$ .

Fix  $U \in \mathcal{U}_0$  and let  $y, z \in U \cap \text{int}(X)$ . Since  $U \cap \text{int}(X)$  is connected, there is a path  $\sigma : [0, 1] \rightarrow U \cap \text{int}(X)$  with  $\sigma(0) = y$  and  $\sigma(1) = z$ . Then we have a finite disjoint union  $[0, 1] = \bigcup_{i=1}^k E_i$ , where  $E_i := \sigma^{-1}(A_i)$ .

Let  $E'_i$  be the set of limit points of  $E_i$ . Then  $[0, 1] = \bigcup_{i=1}^k E'_i$ . Let  $i_1 \in \{1, \dots, k\}$  be such that  $t_0 := 0 \in E'_{i_1}$ . If  $t_1 < 1$ , then there exists  $i_2 \in \{1, \dots, k\} \setminus \{i_1\}$  such that  $t_1 \in E'_{i_2}$  and  $t_2 := \sup E'_{i_2} > t_1$ , by Lemma 5.8. Moreover,  $[t_1, b] = \bigcup_{j \neq i_1} E'_j \cap [t_1, b]$ . If  $t_2 < b$  we may apply Lemma 5.8 again and find  $i_3 \in \{1, \dots, k\} \setminus \{i_1, i_2\}$  such that  $t_2 \in E'_{i_3}$  and  $t_3 := \sup E'_{i_3} > t_2$ . This procedure ends after finitely many steps and gives a finite partition  $0 = t_0 < t_1 < \dots < t_{h-1} < t_h = 1$  of  $[0, 1]$ . The points  $y = z_0, z_1, \dots, z_h = z$ , where  $z_j = \sigma(t_j)$ , all lie in  $U \cap \text{int}(X)$ . Let  $\epsilon > 0$  be sufficiently small such that the balls  $B_j := B(z_j, \epsilon)$  are all contained in  $U \cap \text{int}(X)$ . For all  $j = 1, 2, \dots, h - 1$ , there exist  $z_j^- \in B_j \cap A_{i_j}$  and  $z_j^+ \in B_j \cap A_{i_{j+1}}$ , by construction. Additionally, there exist  $z_0^+ \in B_0 \cap A_{i_1}$  and  $z_h^- \in B_h \cap A_{i_h}$ .

Since the cells are quasiconvex, for all  $j = 1, 2, \dots, h$ , there exist rectifiable paths  $\gamma_j \in A_{i_j}$  joining  $z_{j-1}^+$  and  $z_j^-$  such that

$$\ell(\gamma_j) \leq C_j |z_{j-1}^+ - z_j^-|,$$

where the constant  $C_j$  depends only on  $A_{i_j}$ . Joining the paths  $\gamma_j$  with the line segments  $[z_0, z_0^+]$ ,  $[z_j^-, z_j^+]$ , for  $j = 1, \dots, h - 1$ , and  $[z_h^-, z_h]$ , we obtain a rectifiable path  $\gamma$  in  $\text{int}(X)$  which connects  $y$  and  $z$  and has length

$$\ell(\gamma) \leq C \text{diam}(U),$$

for a constant  $C$  which depends only on the  $C_j$  and the number of cells  $k$ , since all points  $z_j, z_j^-, z_j^+$  lie in  $U$ . □

### 5.5 Proof of Theorem 1.14

The inclusion  $\mathcal{C}^\infty(X) \subseteq \mathcal{A}^\infty(X)$  is clear.

Let  $f \in \mathcal{A}^\infty(X)$ . Then  $f$  is smooth in  $\text{int}(X)$ , by Result 1.1. We must show that  $f \in \mathcal{C}^\infty(X)$ . This is a local problem, so we may assume without loss of generality that  $X$  is compact (by intersecting with a suitable ball). By Lemma 1.10 and Theorem 5.6, it suffices to show that  $f$  satisfies Lemma 1.10(3).

Fix  $x \in \partial X$ . By Theorem 5.4, there is a polynomial curve  $h_x : \mathbb{R} \rightarrow \mathbb{R}^d$  of degree at most  $n$  with the properties:

- (i)  $h_x((0, 1]) \subseteq \text{int}(X)$  and  $h_x(0) = x$ ,
- (ii)  $\text{dist}(h_x(t), \mathbb{R}^d \setminus X) \geq Mt^m$  for all  $t \in (0, 1]$ ,

where  $n, M, m$  are independent of  $x$  and  $t$ . Then there is a positive integer  $k = k(x)$  such that  $h_x(t) - x = t^k \tilde{h}_x(t)$ , where  $\tilde{h}_x(0) \neq 0$ . Set  $v_1 := \tilde{h}_x(0)/|\tilde{h}_x(0)| \in S^{d-1}$ . Choose  $d - 1$  directions  $v_2, \dots, v_d \in S^{d-1}$  such that  $v_1, v_2, \dots, v_d$  are linearly independent and define

$$\Psi_{x,v}(t_1, t_2, \dots, t_d) := h_x(t_1) + t_2 v_2 + \dots + t_d v_d$$

for  $t = (t_1, \dots, t_d)$  in the set

$$Y := \{(t_1, \dots, t_d) \in \mathbb{R}^d : t_1 \in (0, \delta), |t_j| < Ct_1^m \text{ for } 2 \leq j \leq d\}.$$

If  $C := M/(2(d - 1))$  and  $\delta > 0$  is chosen small enough, then  $\Psi_{x,v}$  is a diffeomorphism of  $Y$  onto the open subset  $H_{x,v} := \Psi_{x,v}(Y)$  of  $\text{int}(X)$  and it extends to a homeomorphism between  $Y \cup \{0\}$  and  $H_{x,v} \cup \{x\}$ ; indeed, by (2),

$$\begin{aligned} \text{dist}(\Psi_{x,v}(t), \mathbb{R}^d \setminus X) &\geq \text{dist}(h_x(t_1), \mathbb{R}^d \setminus X) - |t_2| - \dots - |t_d| \\ &> Mt_1^m - (d - 1)Ct_1^m = \frac{M}{2}t_1^m > 0, \end{aligned}$$

for  $t \in Y$ . Since  $f$  is smooth in  $\text{int}(X)$ , we have

$$\partial_{t_2}^{\alpha_2} \cdots \partial_{t_d}^{\alpha_d} (f \circ \Psi_{x,v})(t) = d_{v_2}^{\alpha_2} \cdots d_{v_d}^{\alpha_d} f(\Psi_{x,v}(t)) \quad \text{for all } t \in Y, \alpha_j \geq 0. \tag{5.3}$$

The left-hand side of (5.3) extends continuously to  $t = 0$ , since  $f \circ \Psi_{x,v} \in \mathcal{A}^\infty(\bar{Y})$  and  $\mathcal{A}^\infty(\bar{Y}) = \mathcal{C}^\infty(\bar{Y})$ , by Theorem 1.13, as  $\bar{Y}$  is a Hölder set. Since  $\Psi_{x,v}$  is a homeomorphism  $Y \cup \{0\} \rightarrow H_{x,v} \cup \{x\}$ , we may conclude that the directional derivatives  $d_{v_2}^{\alpha_2} \cdots d_{v_d}^{\alpha_d} f$ ,  $\alpha_j \geq 0$ , extend continuously from  $H_{x,v}$  to  $x$ .

If we perturb the directions  $v_2, \dots, v_d$  a little such that  $v_1, v_2, \dots, v_d$  remain linearly independent and take the intersection  $H_x$  of the corresponding sets  $H_{x,v}$ , then  $H_x$  still is an open subset of  $\text{int}(X)$  with  $h_x(t) \in H_x$  for small  $t > 0$  and  $x \in \bar{H}_x$ . Then  $d_{w_2}^{\alpha_2} \cdots d_{w_d}^{\alpha_d} f$ ,  $\alpha_j \geq 0$ , extend continuously from  $H_x$  to  $x$  for all  $w_2, \dots, w_d$  near  $v_2, \dots, v_d$ . By Lemma 1.10, we infer that the Fréchet derivatives  $f^{(p)}$  of all orders of  $f$  extend continuously from  $H_x$  to  $x$ .

Thus for all  $x \in \partial X$  and  $p \in \mathbb{N}$ , we have a candidate for the Fréchet derivative  $f^{(p)}(x)$  of  $f$  at  $x$  and an open set  $H_x \subseteq \text{int}(X)$  on which  $f^{(p)}(y)$  tends to this candidate as  $y \rightarrow x$ . It remains to prove that the extension of  $f^{(p)}$  to  $X$  thus defined is continuous on  $X$ . First we show that it is bounded.

CLAIM 1. For all  $p \in \mathbb{N}$ ,  $f^{(p)}$  is bounded on  $X$ .

Let  $p \in \mathbb{N}$  be fixed. It suffices to show that  $f^{(p)}$  is bounded on  $\text{int}(X)$  (since  $X$  is fat). For contradiction suppose that there is a sequence  $(x_\ell)$  in  $\text{int}(X)$  such that  $\|f^{(p)}(x_\ell)\|_{L_p} \rightarrow \infty$ . Since  $X$  is compact, we may assume that  $x_\ell \rightarrow x$ . Then  $x \in \partial X$ , since we already know that  $f$  is smooth on  $\text{int}(X)$ .

By Proposition 5.5, there are a finite number of real analytic maps  $\varphi_j : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}^d$  such that

$$\begin{aligned} \varphi_j(I^d \times (0, 1]) &\subseteq \text{int}(X) \quad \text{for all } j, \\ \bigcup_j \varphi_j(I^d \times \{0\}) &= X, \end{aligned}$$

where  $I^d := [-1, 1]^d$ . After passing to a subsequence we may assume that  $x_\ell \in \varphi_{j_0}(I^d \times \{0\})$  for all  $\ell$  and some  $j_0$ . Choose  $y_\ell \in I^d$  such that  $\varphi_{j_0}(y_\ell, 0) = x_\ell$ . Since  $I^d$  is compact, after passing to a subsequence we may assume that  $y_\ell \rightarrow y$  and in turn that this convergence is faster than any polynomial. The infinite polygon through the points  $y_\ell$  and  $y$  can be parameterized by a smooth curve  $c : \mathbb{R} \rightarrow I^d$  such that  $c(1/\ell) = y_\ell$  and  $c(0) = y$  (cf. [KM97, Lemma 2.8]). Then  $s \mapsto \varphi_{j_0}(c(s), 0)$  is a smooth curve in  $X$  through the points  $x_\ell$  and  $x$ .

Since  $\varphi_{j_0}$  is real analytic, for small  $t_1$  we have  $\varphi_{j_0}(y, t_1) = x + t_1^k \tilde{\varphi}_{j_0}(y, t_1)$  for some positive integer  $k$  and a real analytic map  $\tilde{\varphi}_{j_0}$  with  $\tilde{\varphi}_{j_0}(y, 0) \neq 0$ . Then  $\tilde{\varphi}_{j_0}(z, t_1) \neq 0$  for  $(z, t_1)$  in a neighborhood of  $(y, 0)$ . Thus,

$$v_1(z, t_1) := \begin{cases} \frac{\partial_{t_1} \varphi_{j_0}(z, t_1)}{|\partial_{t_1} \varphi_{j_0}(z, t_1)|} = \frac{k \tilde{\varphi}_{j_0}(z, t_1) + t_1 \tilde{\varphi}'_{j_0}(z, t_1)}{|k \tilde{\varphi}_{j_0}(z, t_1) + t_1 \tilde{\varphi}'_{j_0}(z, t_1)|} & \text{if } t_1 > 0, \\ \frac{\tilde{\varphi}_{j_0}(z, 0)}{|\tilde{\varphi}_{j_0}(z, 0)|} & \text{if } t_1 = 0, \end{cases}$$

is continuous in  $(z, t_1)$ , where  $t_1 \geq 0$ , near  $(y, 0)$ . It follows that we can find an open set of directions  $v \in S^{d-1}$  such that  $v_1(c(s), 0)$  and  $v$  are linearly independent for  $s$  near 0. For such  $v$ ,

$$(s, t_1, t_2) \rightarrow f(\varphi_{j_0}(c(s), t_1) + t_2 v)$$

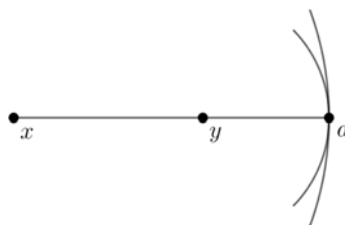


FIGURE 2. If  $y \in (x, a]$  and  $X \cap B(x, |x - a|) = \emptyset$  then  $X \cap B(y, |y - a|) = \emptyset$ .

is smooth for small  $s \in \mathbb{R}$ ,  $t_1 \geq 0$ , and  $|t_2| \leq Ct_1^m$ , by the arguments in § 5.2 and the considerations in the first part of the proof. But this implies that  $d_v^p f(x_\ell)$  is bounded for all such  $v$ , and hence  $f^{(p)}(x_\ell)$  is bounded, a contradiction. Claim 1 is proved.

CLAIM 2. *The Fréchet derivatives  $f^{(p)}$ ,  $p \in \mathbb{N}$ , are continuous on  $X$ .*

Let  $x \in \partial X$  and suppose that  $(x_n)$  and  $(y_n)$  are two sequences in  $\text{int}(X)$  both converging to  $x$ . By Lemma 5.9, for each  $\epsilon > 0$  there exists  $n_0 \in \mathbb{N}$  such that for all  $n \geq n_0$  the points  $x_n$  and  $y_n$  can be joined by a rectifiable path  $\gamma_n$  in  $\text{int}(X)$  with length  $\ell(\gamma_n) \leq \epsilon$ . Since  $f$  is smooth in  $\text{int}(X)$ , we may apply the fundamental theorem of calculus and Claim 1 to conclude

$$\|f^{(p)}(x_n) - f^{(p)}(y_n)\|_{L_p} \leq \left( \sup_{z \in \gamma_n} \|f^{(p+1)}(z)\|_{L_{p+1}} \right) \ell(\gamma_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

If we assume that the sequence  $(x_n)$  lies in  $H_x$ , we obtain that  $f^{(p)}(y) \rightarrow f^{(p)}(x)$  for all  $\text{int}(X) \ni y \rightarrow x$ . Finally, suppose that  $\partial X \ni x_n \rightarrow x$ . Choose  $y_n \in H_{x_n} \cap B(x_n, 1/n)$ . Then

$$\|f^{(p)}(x) - f^{(p)}(x_n)\|_{L_p} \leq \|f^{(p)}(x) - f^{(p)}(y_n)\|_{L_p} + \|f^{(p)}(x_n) - f^{(p)}(y_n)\|_{L_p} \rightarrow 0$$

as  $n \rightarrow \infty$ . This proves Claim 2 and hence the theorem. □

### 6. The Bochnak–Siciak theorem on tame closed sets

In this section we prove Theorem 1.16. The strategy for the proof is the following. Since  $f \in \mathcal{C}^\infty(X)$ , we can associate with every  $x \in X$  the formal Taylor series  $F_x$  of  $f$  at  $x$ . Using a result of Eakin and Harris [EH77] and Gabriélov [Gab73], we show that each  $F_x$  is convergent and coincides with  $f$  on their common domain. To prove that all  $F_x$  glue together to give a global holomorphic extension we will use the following lemma.

LEMMA 6.1. *Let  $X \subseteq \mathbb{R}^d$  be closed and let  $U \subseteq \mathbb{R}^d$  be open with  $U \cap X \neq \emptyset$ . Then there is an open subset  $U_0$  of  $U$  with  $U_0 \cap X = U \cap X$  and such that for all  $x \in U_0$  and all  $a \in A_x := \{a \in X : |a - x| = \text{dist}(x, X)\}$  we have  $[x, a] \subseteq U_0$ .*

*Proof.* Set  $U_0 := \{x \in U : [x, a] \subseteq U \text{ for all } a \in A_x\}$ . Then, for all  $x \in U_0$  and all  $a \in A_x$ , we have  $[x, a] \subseteq U_0$ . To see this, let  $y \in [x, a]$ . If  $y = x$  there is nothing to prove. Otherwise  $A_y = \{a\}$  and  $[y, a] \subseteq [x, a] \subseteq U$  (as in Figure 2).

Clearly,  $U_0 \cap X = U \cap X$ . It remains to show that  $U_0$  is open. To this end we first observe that if  $x_n \rightarrow x$  and  $A_{x_n} \ni a_n \rightarrow a$ , then  $a \in A_x$ . This follows from letting  $n \rightarrow \infty$  in  $|x_n - a_n| = \text{dist}(x_n, X)$ , since  $X$  is closed.

If  $U_0$  is not open, then there exists a sequence  $x_n \rightarrow x$ , where  $x_n \notin U_0$  and  $x \in U_0$ . So, for all  $n$ , there exist  $a_n \in A_{x_n}$  and  $y_n \in [x_n, a_n] \setminus U_0$ . After passing to a subsequence, we may assume that  $a_n \rightarrow a \in A_x$ , by the observation above, and in turn that  $y_n \rightarrow y \in [x, a]$ . Since  $x \in U_0$  we have  $y \in U$ , a contradiction. □

*Proof of Theorem 1.16.* Suppose that  $X \subseteq \mathbb{R}^d$  is a fat closed subanalytic set. There exist an analytic manifold  $M$  and a proper analytic map  $\varphi : M \rightarrow \mathbb{R}^d$  such that  $X = \varphi(M)$ , by the uniformization theorem; see, for example, [BM88]. Then  $f \circ \varphi$  is  $C^\infty$  and real analytic on real analytic curves in  $M$ . By the Bochnak–Siciak theorem (Result 1.5),  $f \circ \varphi$  is analytic on  $M$ . For each  $x \in X$  there is  $y \in \varphi^{-1}(x)$  such that  $\varphi$  has generic rank  $d$  at  $y$ . By a result of Eakin and Harris [EH77] (proved earlier by Gabriélov [Gab73]), the homomorphism  $\varphi^*$  of formal power series rings given by formal composition with  $\varphi$  at  $y$  is strongly injective, that is, the formal Taylor series  $F_x$  of  $f$  at  $x$  converges. It represents a holomorphic function  $F_x$  in a neighborhood  $U_x$  of  $x$  in  $\mathbb{C}^d$  which coincides with the real analytic function  $f|_{\text{int}(X)}$  on  $\text{int}(X) \cap U_x$ .

It remains to show that the  $F_x$  piece together to give a global holomorphic extension of  $f$  to a neighborhood of  $X$  in  $\mathbb{C}^d$ . We may assume that

$$U_x = U_x^{\mathbb{R}} \times i(-r_x, r_x)^d, \tag{6.1}$$

where  $U_x^{\mathbb{R}} \subseteq \mathbb{R}^d$ . We use Lemma 6.1 to replace each  $U_x^{\mathbb{R}}$  by the connected component of  $(U_x^{\mathbb{R}})_0$  which contains  $x$  (and leave the part of  $U_x$  in  $i\mathbb{R}^d$  unchanged). Thus we may assume that the cover  $\{U_x^{\mathbb{R}}\}$  of  $X$  has the property that for each  $z \in U_x^{\mathbb{R}}$  all segments  $[z, a]$ ,  $a \in A_z$ , belong to  $U_x^{\mathbb{R}}$ . By (6.1), each  $U_x$  has the property that for  $z + iw \in U_x$  also  $z + itw \in U_x$  for all  $t \in [0, 1]$ .

Let  $V$  be a connected component of  $U_x \cap U_y$ . It follows that if  $z + iw \in V$ , then  $z \in V^{\mathbb{R}} := V \cap \mathbb{R}^d$ , and  $V^{\mathbb{R}}$  is a connected component of  $U_x^{\mathbb{R}} \cap U_y^{\mathbb{R}}$ . Moreover,  $[z, a] \subseteq V^{\mathbb{R}}$  for all  $a \in A_z \subseteq X$ . Since  $X = \overline{\text{int}(X)}$ , the intersection  $V^{\mathbb{R}} \cap \text{int}(X)$  is non-empty and on this set the holomorphic extensions  $F_x$  and  $F_y$  coincide with  $f$ . By the identity theorem,  $F_x$  and  $F_y$  coincide on  $V$ . Since the component  $V$  of  $U_x \cap U_y$  was arbitrary,  $F_x$  and  $F_y$  coincide on  $U_x \cap U_y$ .  $\square$

*Proof of Corollary 1.17.* The assumption for  $X$  clearly implies that  $X = \overline{\text{int}(X)}$ . For each boundary point  $z \in \partial X$  there is a holomorphic function  $F_z$  defined in a neighborhood  $U_z$  of  $z$  in  $\mathbb{C}^d$  which coincides with  $f$  on  $U_z \cap \text{int}(X)$ ; this follows from Theorem 1.16 applied to the subanalytic set  $X_z$ . Using Lemma 6.1 as in the proof of Theorem 1.16, one easily concludes the assertion.  $\square$

### 7. Arc- $C^M$ functions on Lipschitz sets

In this section we prove Theorem 1.19: *All  $X \in \mathcal{H}^1(\mathbb{R}^d)$  satisfy*

$$C^M(X) \subseteq \mathcal{A}^M(X) \subseteq C^{M^{(2)}}(X),$$

for any non-quasianalytic weight sequence  $M = (M_k)$ .

It can be expected that a similar statement holds for  $X \in \mathcal{H}^\alpha(\mathbb{R}^d)$ , where  $\alpha < 1$ , with a larger weight sequence  $N = N(\alpha, M)$  instead of  $M^{(2)}$ . We do not pursue this question any further for  $\alpha$ -sets, but results of this type for subanalytic sets are presented in § 8.

#### 7.1 Reduction to an open set of directions

Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be smooth. The mixed partial derivatives of order  $k$  of  $f$  at any point  $x \in \mathbb{R}^2$  can be computed from directional derivatives of order  $k$  of  $f$  at  $x$  by means of the identity

$$d_v^k f(x) = \sum_{j=0}^k \binom{k}{j} v_1^j v_2^{k-j} \partial_1^j \partial_2^{k-j} f(x), \quad v = (v_1, v_2) \in \mathbb{R}^2. \tag{7.1}$$

The next lemma guarantees that the constants which appear in the process of solving these linear equations grow at most exponentially in  $k$  and hence the class  $C^M$  is preserved; a similar lemma was proved in [Nee99].

LEMMA 7.1. Let  $-1 \leq t_0 < t_1 < \dots < t_k \leq 1$  be equidistant points such that  $t_k - t_0 = a$ . If  $x_0, x_1, \dots, x_k$  is a solution of the linear system of equations

$$\sum_{j=0}^k \binom{k}{j} t_i^j x_j = y_i, \quad i = 0, 1, \dots, k, \tag{7.2}$$

then we have

$$\max_j |x_j| \leq \left(\frac{16e^2}{a}\right)^k \max_m |y_m|. \tag{7.3}$$

*Proof.* Let  $P(t) = a_0 + a_1 t + \dots + a_k t^k$  be the polynomial with coefficients  $a_j = \binom{k}{j} x_j$ . Then the system (7.2) reads

$$P(t_i) = y_i, \quad i = 0, 1, \dots, k.$$

By Lagrange’s interpolation formula (e.g. [RS02, (1.2.5)]),

$$P(t) = \sum_{i=0}^k y_i \prod_{\substack{j=0 \\ j \neq i}}^k \frac{t - t_j}{t_i - t_j},$$

and therefore

$$a_m = (-1)^{k-m} \sum_{i=0}^k y_i \sigma_{k-m}^i \prod_{\substack{j=0 \\ j \neq i}}^k \frac{1}{t_i - t_j},$$

where  $\sigma_j^i$  is the  $j$ th elementary symmetric polynomial in  $(t_\ell)_{\ell \neq i}$ . We have

$$|t_i - t_j| = \frac{a|i - j|}{k}, \quad |t_j| = \left|t_0 + a \frac{j}{k}\right| \leq 2 \frac{k + j}{k},$$

and hence, using  $e^{-k} k^k \leq k! \leq k^k$ ,

$$\prod_{\substack{j=0 \\ j \neq i}}^k \frac{1}{|t_i - t_j|} = \frac{k^k}{a^k i!(k - i)!} \leq \left(\frac{2e}{a}\right)^k,$$

and

$$|\sigma_{k-m}^i| \leq \binom{k}{m} \left(\frac{2}{k}\right)^{k-m} \frac{(2k)!}{(k + m)!} \leq \binom{k}{m} \left(\frac{2}{k}\right)^{k-m} 4^k (k - m)! \leq \binom{k}{m} 8^k.$$

It follows that

$$|x_m| \leq \left(\frac{16e}{a}\right)^k \sum_{i=0}^k |y_i| \leq \left(\frac{16e^2}{a}\right)^k \max_i |y_i|,$$

which is (7.3). □

PROPOSITION 7.2. Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  be smooth. Let  $K \subseteq \mathbb{R}^d$  be compact and let  $M = (M_k)$  be a positive sequence. The following assertions are equivalent:

- (1)  $\exists C, \rho > 0 \forall k \in \mathbb{N} \forall x \in K \forall v \in S^{d-1} : |d_v^k f(x)| \leq C \rho^k k! M_k;$



(2) there exist  $v_0 \in S^{d-1}$  and  $r > 0$  such that

$$\exists C, \rho > 0 \forall k \in \mathbb{N} \forall x \in K \forall v \in B(v_0, r) \cap S^{d-1} : |d_v^k f(x)| \leq C \rho^k k! M_k;$$

(3)  $\exists C, \rho > 0 \forall x \in K \forall \alpha \in \mathbb{N}^d : |\partial^\alpha f(x)| \leq C \rho^{|\alpha|} |\alpha|! M_{|\alpha|}$ .

The constants  $C, \rho$  may differ from item to item, but they change in a uniform way which depends only on  $r$ .

*Proof.* Let us first consider the case  $d = 2$ . In this case  $B := B(v_0, r) \cap S^{d-1}$  is an open arc  $I \subseteq S^1$ ; let  $\ell(I)$  denote the length of  $I$ .

(1)  $\Rightarrow$  (2) is trivial and (3)  $\Rightarrow$  (1) follows easily from (7.1).

(2)  $\Rightarrow$  (3) By a linear change of coordinates, we may assume that the arc  $I$  is symmetric about the  $y$ -axis and, by shrinking  $I$ , we may also assume that its projection to the  $y$ -axis is contained in  $\{(0, y) : 1/2 \leq y \leq 1\}$  and that the estimates in (2) hold also at the endpoints of  $I$ . Let  $(-a/2, a/2)$  be the projection of  $I$  to the  $x$ -axis and let  $-a/2 = t_0 < t_1 < \dots < t_k = a/2$  be an equidistant partition. Apply Lemma 7.1 to the system (7.1) with the  $k + 1$  directions  $v_i = (t_i, s_i)$ ,  $i = 0, \dots, k$ , in  $I$ ; then  $1/2 \leq s_i \leq 1$ . The statement about the uniform change of the constants follows from (7.3).

Now we consider the general case.

(1)  $\Leftrightarrow$  (2) The statement follows by applying the two-dimensional analogue to every affine 2-plane  $\pi$  containing the affine line  $x + \mathbb{R}v_0$ . The change of the constants  $C, \rho$  depends only on the length of the arcs defined by the intersection  $\pi \cap B$  which is independent of  $\pi$ .

(1)  $\Leftrightarrow$  (3) By the polarization formula [KM97, Lemma 7.13(1)], we have

$$\sup_{|v| \leq 1} |d_v^k f(x)| \leq \|d^k f(x)\|_{L^k} \leq (2e)^k \sup_{|v| \leq 1} |d_v^k f(x)|$$

which entails the assertion. □

### 7.2 Proof of Theorem 1.19

Let  $M = (M_k)$  be a non-quasianalytic weight sequence. Let  $X \in \mathcal{H}^1(\mathbb{R}^d)$ . The inclusion  $\mathcal{C}^M(X) \subseteq \mathcal{A}^M(X)$  is an easy consequence of Faà di Bruno's formula and log-convexity of  $M$  (cf. [RS14, Proposition 3.1]).

Let us prove  $\mathcal{A}^M(X) \subseteq \mathcal{C}^{M^{(2)}}(X)$ . A function  $f \in \mathcal{A}^M(X)$  belongs to  $\mathcal{C}^\infty(X)$ , by Theorem 1.13. Suppose for contradiction that  $f \notin \mathcal{C}^{M^{(2)}}(X)$ . Then there is  $a \in X$  such that for all  $\delta, C, \rho > 0$  there exist  $x \in X \cap B(a, \delta)$ ,  $v \in S^{d-1}$ , and  $k \in \mathbb{N}$  with

$$|d_v^k f(x)| > C \rho^k k! M_k^{(2)}. \tag{7.4}$$

We may assume that  $a \in \partial X$  (if  $a \in \text{int}(X)$  then the arguments in the proof of [KMR09, Theorem 3.9] lead to a contradiction). Since  $X \in \mathcal{H}^1(\mathbb{R}^d)$ , we may suppose that there exist  $\epsilon > 0$  and a truncated open cone  $\Gamma = \Gamma_d^1(r, h)$  such that

$$\text{for all } y \in X \cap B(a, \epsilon) \text{ we have } y + \Gamma \subseteq \text{int}(X). \tag{7.5}$$

By rescaling, we may assume that  $r = h = 1$ . Set  $C(y, r) := y + \Gamma_d^1(r, r)$  for  $0 < r \leq 1$ . There is a universal constant  $c > 0$  such that  $C(y_1, r_1) \cap C(y_2, r_2) \neq \emptyset$  if  $|y_1 - y_2| < c \min\{r_1, r_2\}$ .

Let  $\lambda_k \searrow 0$  be the sequence associated with the sequence  $M_k$ , by Lemma 2.4. By Proposition 7.2 and (7.4) (using  $\delta := c\lambda_{n+1}/3$ ,  $C := \lambda_n^{-1}$ ,  $\rho := \lambda_n^{-3}$ ), there exist sequences  $x_n \in X \cap B(a, c\lambda_{n+1}/3)$ ,  $v_n \in S^{d-1} \cap \mathbb{R}_+\Gamma$ ,  $k_n \in \mathbb{N}$  such that

$$|d_{v_n}^{k_n} f(x_n)| \geq \lambda_n^{-3k_n-1} k_n! M_{k_n}^{(2)} \quad \text{for all } n. \tag{7.6}$$

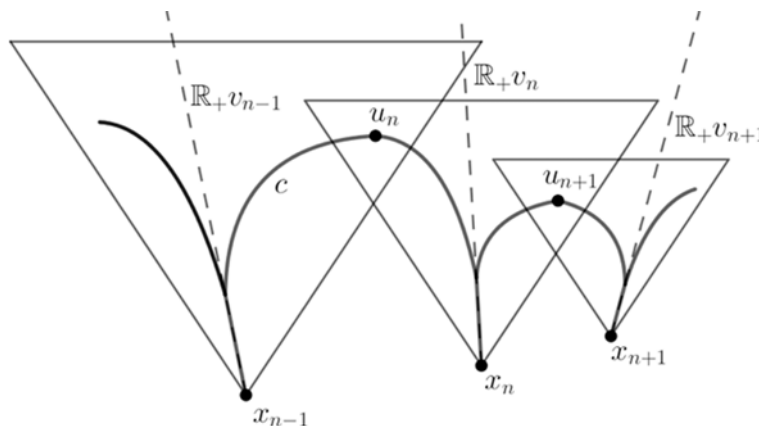


FIGURE 3. The curve  $c$  in  $C_{n-1} \cup C_n \cup C_{n+1}$ .

Let us set  $C_n := C(x_n, \lambda_n)$ . Since  $|x_n - x_{n+1}| < c\lambda_{n+1}$ , there is a sequence  $u_n$  such that  $u_{n+1} \in C_n \cap C_{n+1}$  for all  $n$ . Evidently,  $x_n$  and  $u_n$  are both  $1/\lambda_n$ -converging to  $a$ . We may assume that for all  $n \geq n_0$  we have  $C_n \subseteq \text{int}(X)$ , by (7.5).

Without loss of generality assume that  $a = 0$ . Let  $c_n(t) = x_n + t^2\lambda_nv_n$ . Let  $T_n$  and  $t_n$  be chosen as in (2.8), and let  $\varphi$  be the function used in the proof of Lemma 2.4. Define

$$c(t) = \varphi\left(\frac{t - t_n}{T_n}\right)c_n(t - t_n) + \left(1 - \varphi\left(\frac{t - t_n}{T_n}\right)\right)(u_n\mathbb{1}_{(-\infty, t_n]}(t) + u_{n+1}\mathbb{1}_{[t_n, +\infty)}(t))$$

for  $t \in [t_n - T_n, t_n + T_n]$  (here  $\mathbb{1}_A$  denotes the characteristic function of the set  $A$ ); note that  $t_n + T_n = t_{n+1} - T_{n+1}$  (see Figure 3).

Extend  $c$  by  $c = 0$  on  $[t_\infty, \infty)$ . Then  $c$  is  $\mathcal{C}^\infty$  on  $[t_{n_0} - T_{n_0}, +\infty) \setminus \{t_\infty\}$  and  $c(t_n - T_n) = u_n$  and  $c(t_n + T_n) = u_{n+1}$ . By construction,  $c(t) \in C_n$  if  $t \in [t_n - T_n, t_n + T_n]$  and thus  $c$  lies in  $X$ . Since the curves  $c_n$  as well as  $u_n$  satisfy (2.6), the proof of Lemma 2.4 implies that  $c$  is a  $\mathcal{C}^M$ -curve.

Then, since  $f \in \mathcal{A}^\infty(X)$ , for all  $k$ ,

$$(f \circ c)^{(2k)}(t_n) = \frac{(2k)!}{k!} \lambda_n^k d_{v_n}^k f(x_n).$$

Using (7.6), we may conclude

$$\left(\frac{|(f \circ c)^{(2k_n)}(t_n)|}{(2k_n)! M_{2k_n}}\right)^{1/(2k_n+1)} = \left(\frac{\lambda_n^{k_n} |d_{v_n}^{k_n} f(x_n)|}{k_n! M_{2k_n}}\right)^{1/(2k_n+1)} \geq \frac{1}{\lambda_n} \rightarrow \infty,$$

as  $n \rightarrow \infty$ , contradicting the assumption  $f \in \mathcal{A}^M(X)$ . □

### 8. Arc- $\mathcal{C}^M$ functions on subanalytic sets

Let  $M = (M_k)$  be a non-quasianalytic weight sequence. Let  $X$  be a simple fat closed subanalytic set. We will see in this section that  $\mathcal{A}^M(X) \subseteq \mathcal{C}^N(X)$  for some other non-quasianalytic weight sequence  $N$  which depends only on  $M$  and  $X$  (in an explicit way).

**8.1 Rectilinearization**

We start with some simple observations. For arbitrary sets  $Y \subseteq \mathbb{R}^e$ ,  $X \subseteq \mathbb{R}^d$ , we denote by  $\mathcal{C}^\infty(Y, X)$  the class of mappings  $\varphi : Y \rightarrow X$  such that  $\varphi_i \in \mathcal{C}^\infty(Y)$  for all components  $\varphi_i = \text{pr}_i \circ \varphi$ . Similarly for  $\mathcal{C}^M(Y, X)$  and  $\mathcal{C}^\omega(Y, X)$ .

LEMMA 8.1. *Let  $X \subseteq \mathbb{R}^d$  and  $Y \subseteq \mathbb{R}^e$ .*

- (1) *If  $\varphi \in \mathcal{C}^\infty(Y, X)$  and  $\mathcal{A}^\infty(Y) = \mathcal{C}^\infty(Y)$ , then  $\varphi^* \mathcal{A}^\infty(X) \subseteq \mathcal{C}^\infty(Y)$ .*
- (2) *If  $\varphi \in \mathcal{C}^\omega(Y, X)$  and  $\mathcal{A}^\omega(Y) = \mathcal{C}^\omega(Y)$ , then  $\varphi^* \mathcal{A}^\omega(X) \subseteq \mathcal{C}^\omega(Y)$ .*
- (3) *If  $\varphi \in \mathcal{C}^M(Y, X)$  and  $\mathcal{A}^M(Y) \subseteq \mathcal{C}^N(Y)$ , then  $\varphi^* \mathcal{A}^M(X) \subseteq \mathcal{C}^N(Y)$ .*

*Proof.* We prove (1); (2) and (3) work similarly. Let  $f \in \mathcal{A}^\infty(X)$ . Assume that  $f \circ \varphi \notin \mathcal{C}^\infty(Y)$ . Since  $\mathcal{C}^\infty(Y) = \mathcal{A}^\infty(Y)$ , there exists  $c \in \mathcal{C}^\infty(\mathbb{R}, Y)$  such that  $f \circ \varphi \circ c \notin \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ . But  $\varphi \circ c$  is a  $\mathcal{C}^\infty$ -curve in  $X$ , contradicting  $f \in \mathcal{A}^\infty(X)$ . □

Combining this lemma with the rectilinearization of subanalytic sets (see Theorem 5.1), we conclude the following result.

THEOREM 8.2. *Let  $M = (M_k)$  be a non-quasianalytic weight sequence. Let  $X \subseteq \mathbb{R}^d$  be a fat closed subanalytic set. There is a locally finite collection of real analytic mappings  $\varphi_\alpha : U_\alpha \rightarrow \mathbb{R}^d$ , where the  $U_\alpha$  are open sets in  $\mathbb{R}^d$ , such that, for all  $\alpha$ ,*

$$\varphi_\alpha^* \mathcal{A}^\infty(X) \subseteq \mathcal{C}^\infty(\varphi_\alpha^{-1}(X)), \tag{8.1}$$

$$\varphi_\alpha^* \mathcal{A}^\omega(X) \subseteq \mathcal{C}^\omega(\varphi_\alpha^{-1}(X)), \tag{8.2}$$

$$\varphi_\alpha^* \mathcal{A}^M(X) \subseteq \mathcal{C}^{M^{(2)}}(\varphi_\alpha^{-1}(X)). \tag{8.3}$$

*Proof.* We use Theorem 5.1. Since  $X = \overline{\text{int}(X)}$ , we may assume that, for the quadrants  $Q(I_0, I_-, I_+)$  whose union is  $\varphi_\alpha^{-1}(X)$ , we have  $I_0 = \emptyset$ . We claim that a union  $Y$  of quadrants  $Q(\emptyset, I_-, I_+)$  is  $\mathcal{A}^\infty$ - and  $\mathcal{A}^\omega$ -admissible. Furthermore, we claim that  $Y$  satisfies  $\mathcal{A}^M(Y) \subseteq \mathcal{C}^{M^{(2)}}(Y)$ . Then Lemma 8.1 implies the result.

*$\mathcal{A}^\infty$ -admissibility.* By Theorem 1.13, each  $Q = Q(\emptyset, I_-, I_+)$  is  $\mathcal{A}^\infty$ -admissible. Any two different quadrants  $Q_1, Q_2$  have non-empty intersection  $\pi$  which consists of a coordinate sector of dimension  $k \in \{0, \dots, d - 1\}$  (for  $k = 0$ ,  $\pi = \{0\}$ ). Suppose that  $\pi$  is a coordinate sector of dimension  $k$ . Let  $v \in Q_1 \cup Q_2$  be any vector perpendicular to  $\pi$ . Then  $\sigma_v := \pi + \mathbb{R}v$  is a  $(k + 1)$ -dimensional closed convex set contained in  $Q_1 \cup Q_2$ . We may conclude that  $f|_{\sigma_v} \in \mathcal{C}^\infty(\sigma_v)$ . Thus the directional derivatives  $d_w^n f$  of  $f$  of all orders  $n$  at points in  $\pi$  with direction  $w \in \bigcup_{v \in Q_1 \cup Q_2} \sigma_v$  exist and are unique. These suffice to compute the partial derivatives of  $f$  of all orders at points in  $\pi$ . This proves that  $Q_1 \cup Q_2$  is  $\mathcal{A}^\infty$ -admissible. The general case follows by induction. This also proves that we even have  $\mathcal{A}_M^\infty(Y) = \mathcal{C}^\infty(Y)$ .

*$\mathcal{A}^\omega$ -admissibility.* This follows from Theorem 1.16 and the fact that  $Y$  is  $\mathcal{A}^\infty$ -admissible.

*Finally, we show  $\mathcal{A}^M(Y) \subseteq \mathcal{C}^{M^{(2)}}(Y)$ .* Since we already have  $\mathcal{A}_M^\infty(Y) = \mathcal{C}^\infty(Y)$ , it suffices to check that the estimates (1.1) (for  $M^{(2)}$  instead of  $M$ ) hold for all  $f \in \mathcal{A}^M(Y)$  and for each compact  $K \subseteq Y$ . This is clear, since  $f|_Q \in \mathcal{A}^M(Q) \subseteq \mathcal{C}^{M^{(2)}}(Q)$ , by Theorem 1.19, for each of the finitely many quadrants  $Q$  which make up  $Y$ . □

**8.2 Controlled loss of regularity**

Let  $M = (M_k)$  be a weight sequence. Recall that, for positive integers  $a$ ,  $M^{(a)}$  denotes the weight sequence defined by  $M_k^{(a)} := M_{ak}$ .

PROPOSITION 8.3. *Let  $M = (M_k)$  be a non-quasianalytic weight sequence. Let  $X \subseteq \mathbb{R}^d$  be a fat compact subanalytic set. Then there is a positive integer  $a$ , independent of  $M$ , such that*

$$\mathcal{C}^\infty(X) \cap \mathcal{A}^M(X) \subseteq \mathcal{C}^{M^{(a)}}(X). \tag{8.4}$$

*Proof.* Let  $\varphi_\alpha$  be the finitely many mappings provided by Theorem 5.1. We may assume that the Jacobian determinant of each  $\varphi_\alpha$  is a monomial times a nowhere vanishing factor. Let  $f \in \mathcal{C}^\infty(X) \cap \mathcal{A}^M(X)$ . By Theorem 8.2,  $f \circ \varphi_\alpha \in \mathcal{C}^{M^{(2)}}(Y_\alpha)$  where  $Y_\alpha$  is a union of quadrants in  $\mathbb{R}^d$ . By [BBC18, Theorem 1.4], for each  $\alpha$  there is a positive integer  $a_\alpha$  such that  $f$  is of class  $\mathcal{C}^{M^{(a_\alpha)}}$  on  $\varphi_\alpha(Y_\alpha)$ . It follows that  $f \in \mathcal{C}^{M^{(a)}}(X)$ , where  $a = \max_\alpha a_\alpha$ . □

For  $a \in \mathbb{R}_{>0}$  we may define the weight sequence  $M^a$  by  $M_k^a := (M_k)^a$ . If  $M = (M_k)$  has moderate growth (see (2.4)) and  $a$  is an integer, then there exists  $C = C(a)$  such that

$$M_k^a \leq M_{ak} \leq C^k M_k^a \quad \text{for all } k,$$

that is,  $M^{(a)}$  and  $M^a$  define the same Denjoy–Carleman class. Note also that  $M^a$  has moderate growth whenever  $M$  has.

Assume that for each  $a > 0$ , the weight sequence  $M^a$  is non-quasianalytic and define

$$\widehat{\mathcal{A}}^M(X) := \bigcap_{a>0} \mathcal{A}^{M^a}(X) \quad \text{and} \quad \widehat{\mathcal{C}}^M(X) := \bigcap_{a>0} \mathcal{C}^{M^a}(X).$$

THEOREM 8.4. *Let  $M = (M_k)$  be a weight sequence of moderate growth such that  $M^a$  is non-quasianalytic for all  $a > 0$ . Let  $X \subseteq \mathbb{R}^d$  be a fat closed subanalytic set. Then*

$$\mathcal{C}^\infty(X) \cap \widehat{\mathcal{A}}^M(X) = \widehat{\mathcal{C}}^M(X). \tag{8.5}$$

If  $X$  is simple, then

$$\widehat{\mathcal{A}}^M(X) = \widehat{\mathcal{C}}^M(X). \tag{8.6}$$

*Proof.* The inclusion  $\widehat{\mathcal{C}}^M(X) \subseteq \mathcal{C}^\infty(X) \cap \widehat{\mathcal{A}}^M(X)$  is obvious. The converse inclusion follows from Proposition 8.3. □

Remark 8.5. Instead of [BBC18, Theorem 1.4] one can also use the results of [CC99].

**9. Applications**

**9.1 Solutions of real analytic equations**

THEOREM 9.1. *Let  $U \subseteq \mathbb{R}^{d+1}$  be open and let  $H : U \rightarrow \mathbb{R}$  be a real analytic function (not identically zero). Let  $X \subseteq \mathbb{R}^d$  be a closed set such that for all  $z \in \partial X$  there is a closed fat subanalytic set  $X_z$  such that  $z \in X_z \subseteq X$ ; for example,  $X$  itself is fat and subanalytic or a Hölder set. If  $f \in \mathcal{C}^\infty(X)$  satisfies  $H(x, f(x)) = 0$  for all  $x \in X$ , then  $f$  extends to a holomorphic function on a neighborhood of  $X$  in  $\mathbb{C}^d$ .*

*Proof.* Suppose first that  $X \subseteq \mathbb{R}^d$  is fat closed subanalytic. As in the proof of Theorem 1.16, there is a proper real analytic map  $\varphi : M \rightarrow \mathbb{R}^d$  with  $X = \varphi(M)$ . Then  $(z, y) \mapsto H(\varphi(z), y)$  is not identically zero. By the classical version of this theorem (cf. [Boc70, Sic70, Mal67]), we may conclude that  $z \mapsto (f \circ \varphi)(z)$  is real analytic on  $M$ . The proof of Theorem 1.16 (in §6) then yields the assertion.

In the general case, fix  $z \in \partial X$  and a closed fat subanalytic set  $X_z$  with  $z \in X_z \subseteq X$ . Then  $f|_{X_z} \in \mathcal{C}^\infty(X_z)$  satisfies  $H(x, f(x)) = 0$  for all  $x \in X_z$ . Thus, by the first part of the proof,  $f|_{X_z}$  extends to a holomorphic function on a neighborhood of  $X_z$  in  $\mathbb{C}^d$ . That these local extensions glue to the desired global extension follows from Lemma 6.1 as in the proof of Theorem 1.16.  $\square$

We obtain the following corollary for *Nash functions*, that is, real analytic functions  $f : U \rightarrow \mathbb{R}$  defined in an open semialgebraic set  $U \subseteq \mathbb{R}^d$  which satisfy a non-trivial polynomial equation  $P(x, f(x)) = 0$  for all  $x \in U$ .

**COROLLARY 9.2.** *Let  $X \subseteq \mathbb{R}^d$  be a fat closed semialgebraic set and let  $f : \text{int}(X) \rightarrow \mathbb{R}$  be a Nash function whose partial derivatives of all orders extend continuously to the boundary of  $X$ . Then  $f$  is the restriction of a Nash function on an open neighborhood of  $X$ .*

*Proof.* The extension of  $f$  clearly also satisfies the defining polynomial equation.  $\square$

### 9.2 Composite real analytic functions

Suppose that  $\varphi : M \rightarrow \mathbb{R}^d$  is a real analytic map, where  $M$  is a real analytic manifold. Assume that  $g \in \mathcal{C}^\infty(\mathbb{R}^d)$  and  $f = g \circ \varphi \in \mathcal{C}^\omega(M)$ . Let  $X := \varphi(M)$ . Our results yield a sufficient condition for  $g|_X$  to admit a real analytic extension to some open neighborhood of  $X$ .

**COROLLARY 9.3.** *Let  $\varphi : M \rightarrow \mathbb{R}^d$  be real analytic and such that:*

- (i)  $X := \varphi(M)$  is a fat closed subanalytic subset of  $\mathbb{R}^d$ ;
- (ii) each  $c \in \mathcal{C}^\omega(\mathbb{R}, X)$  admits a lifting  $\tilde{c} \in \mathcal{C}^\omega(\mathbb{R}, M)$ , that is,  $c = \varphi \circ \tilde{c}$ .

*Then, for each  $g \in \mathcal{C}^\infty(\mathbb{R}^d)$  with  $g \circ \varphi \in \mathcal{C}^\omega(M)$ , there exists a holomorphic function  $G$  defined in an open neighborhood of  $X$  in  $\mathbb{C}^d$  such that  $g|_X = G|_X$ .*

*Proof.* Follows from Theorem 1.16.  $\square$

Conditions for the existence of a smooth solution  $g$  of the equation  $f = g \circ \varphi$  have been intensively studied; see [BM82, BMP96, BM98].

*Remark 9.4.* For instance, the conditions of the corollary are satisfied in the following situation. Let  $\rho : G \rightarrow O(V)$  be a coregular finite-dimensional orthogonal representation of a compact Lie group. Let  $\sigma = (\sigma_1, \dots, \sigma_d)$  be a minimal system of generators of the algebra  $\mathbb{R}[V]^G$  of  $G$ -invariant polynomials. Schwarz’s theorem [Sch75] (see also [Mat77]) holds that for each  $G$ -invariant  $f \in \mathcal{C}^\infty(V)$  there exists  $g \in \mathcal{C}^\infty(\mathbb{R}^d)$  such that  $f = g \circ \sigma$ . The set  $X = \sigma(V)$  is closed semialgebraic and fat, by the assumption that  $\rho$  is coregular; cf. [PS85]. Real analytic curves in  $X$  admit real analytic liftings to  $V$ , by [AKLM00] and [PR16, Theorem 4]. The corollary implies that every  $G$ -invariant real analytic function  $f$  on  $V$  is of the form  $f = g \circ \sigma$ , where  $g$  is a holomorphic function defined in an open neighborhood of  $X$  in  $\mathbb{C}^d$ . A more general result (with a different proof) is due to Luna [Lun76].

### 9.3 Division of smooth functions and pseudo-immersions

Statements about smooth functions on open sets can sometimes be reduced to corresponding statements for functions of one real variable, thanks to Boman’s Theorem 1.4. This principle extends to  $\mathcal{A}^\infty$ -admissible sets. We illustrate this using two selected examples. The first concerns division of smooth functions.

**THEOREM 9.5.** *Suppose that  $X$  is a Hölder set or a simple fat closed subanalytic subset of  $\mathbb{R}^d$ . If  $f, g : X \rightarrow \mathbb{C}$  satisfy*

- (i)  $g, fg, f^m \in \mathcal{C}^\infty(X, \mathbb{C})$ , and
- (ii)  $|f(x)| \leq C|g(x)|^\alpha$  for all  $x \in X$ ,

for some  $m \in \mathbb{N}_{\geq 1}$  and  $C, \alpha > 0$ , then  $f \in \mathcal{C}^\infty(X, \mathbb{C})$ .

*Proof.* This follows from [JP90, Theorem 1] (which is precisely the case  $X = \mathbb{R}$ ), Theorems 1.13, and 1.14. □

In [JP90] this theorem (for  $X = \mathbb{R}$ ) was used to prove that certain maps are pseudo-immersions. A  $\mathcal{C}^\infty$ -map  $p : N \rightarrow M$  between  $\mathcal{C}^\infty$ -manifolds is a *pseudo-immersion* if for each continuous map  $f : P \rightarrow N$ , where  $P$  is a  $\mathcal{C}^\infty$ -manifold,  $p \circ f \in \mathcal{C}^\infty$  implies  $f \in \mathcal{C}^\infty$ ; see also [JP87]. Pseudo-immersivity of a smooth map is a local property. So it is enough to consider germs of smooth maps  $p : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^m, 0)$ . By Boman’s Theorem 1.1, the defining universal property must be checked only for smooth curves:  $p$  is a pseudo-immersion if and only if for each (continuous) curve  $c : \mathbb{R} \rightarrow \mathbb{R}^n$  we have the implication  $p \circ c \in \mathcal{C}^\infty \implies c \in \mathcal{C}^\infty$ .

Theorems 1.13 and 1.14 entail the following result.

**THEOREM 9.6.** *Let  $p : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a pseudo-immersion. Then the universal property of  $p$  extends to maps  $f : X \rightarrow \mathbb{R}^n$ , where  $X \subseteq \mathbb{R}^d$  is  $\mathcal{A}^\infty$ -admissible, in particular, for  $X$  a Hölder set or a simple fat closed subanalytic subset of  $\mathbb{R}^d$ .*

For instance, if  $f : X \rightarrow \mathbb{C}$  is continuous and  $f^2, f^3 \in \mathcal{C}^\infty(X, \mathbb{C})$ , then  $f \in \mathcal{C}^\infty(X, \mathbb{C})$ . In addition, by Theorem 9.1, if at least one of  $f^2$  or  $f^3$  is real analytic, then also  $f$  is real analytic.

## 10. Complements and examples

### 10.1 $\mathcal{C}^M$ -extensions

Let  $X \subseteq \mathbb{R}^d$  be a Hölder set or a fat closed subanalytic set. By Lemma 1.10, Proposition 3.8, and Theorem 5.6, any function  $f : X \rightarrow \mathbb{R}$  which satisfies Lemma 1.10(3) extends to a  $\mathcal{C}^\infty$ -function on  $\mathbb{R}^d$ . Let us investigate this in the ultradifferentiable case. For strongly regular weight sequences  $M$  there is a  $\mathcal{C}^M$ -version of Whitney’s extension theorem [Bru80].

**LEMMA 10.1.** *Let  $X \subseteq \mathbb{R}^d$  be a fat compact set either in  $\mathcal{H}(\mathbb{R}^d)$  or subanalytic. Suppose there exist a positive integer  $m$  and a constant  $D > 0$ , such that any two points  $x, y \in X$  can be joined by a rectifiable path  $\gamma$  in  $X$  and*

$$\ell(\gamma)^m \leq D|x - y|. \tag{10.1}$$

Let  $M$  be a weight sequence. Then each  $f \in \mathcal{C}^M(X)$  defines a Whitney jet on  $X$  of class  $\mathcal{C}^N$  where  $N_k := M_{mk}$ , that is, there exist constants  $C, \rho > 0$  such that

$$|f^{(\alpha)}(x)| \leq C\rho^{|\alpha|} |\alpha|! N_{|\alpha|}, \quad \alpha \in \mathbb{N}^d, x \in X, \tag{10.2}$$

$$|(R_x^p f)^\alpha(y)| \leq C \rho^{p+1} |\alpha|! N_{p+1} |x - y|^{p+1-|\alpha|}, \quad p \in \mathbb{N}, \quad |\alpha| \leq p, \quad x, y \in X, \quad (10.3)$$

where

$$(R_x^p f)^\alpha(y) = f^{(\alpha)}(y) - \sum_{|\beta| \leq p-|\alpha|} \frac{f^{(\alpha+\beta)}(x)}{\beta!} (y-x)^\beta.$$

*Proof.* Let  $f \in \mathcal{C}^M(X)$ . Now (10.2) is clearly satisfied since we even have

$$|f^{(\alpha)}(x)| \leq C \rho^{|\alpha|} |\alpha|! M_{|\alpha|}, \quad \alpha \in \mathbb{N}^d, \quad x \in X. \quad (10.4)$$

Since  $f$  has a smooth extension to  $\mathbb{R}^d$ ,  $f$  defines a Whitney jet of class  $\mathcal{C}^\infty$  on  $X$ . We claim that

$$|(R_x^p f)^\alpha(y)| \leq \frac{(d\ell(\sigma))^{p+1-|\alpha|}}{(p+1-|\alpha|)!} \sup_{\substack{\xi \in \sigma \\ |\gamma|=p+1}} |f^{(\gamma)}(\xi)| \quad (10.5)$$

for any rectifiable path  $\sigma$  which joins  $x$  and  $y$ . Then, by (10.1) and (10.4), there are constants  $C_i, \rho_i > 0$  such that

$$\begin{aligned} |(R_x^p f)^\alpha(y)| &\leq |(R_x^{m(p+1)-1} f)^\alpha(y)| + \left| \sum_{p-|\alpha| < |\beta| < m(p+1)-|\alpha|} \frac{f^{(\alpha+\beta)}(x)}{\beta!} (y-x)^\beta \right| \\ &\leq d^{m(p+1)-|\alpha|} C \rho^{m(p+1)} |\alpha|! M_{m(p+1)} \ell(\sigma)^{m(p+1)-|\alpha|} \\ &\quad + C_1 \rho_1^{m(p+1)} |\alpha|! M_{m(p+1)} |x-y|^{p-|\alpha|+1} \\ &\leq C_2 \rho_2^{m(p+1)} |\alpha|! M_{m(p+1)} |x-y|^{p-|\alpha|+1}, \end{aligned}$$

which is (10.3). To see (10.5) notice that, with  $T_x^p f(y) := \sum_{|\beta| \leq p} ((f^{(\beta)}(x))/\beta!)(y-x)^\beta$ ,

$$(R_x^p f)^\alpha(y) = f^{(\alpha)}(y) - T_x^{p-|\alpha|} f^{(\alpha)}(y) = T_y^{p-|\alpha|} f^{(\alpha)}(y) - T_x^{p-|\alpha|} f^{(\alpha)}(y).$$

By choosing a suitable parameterization, we may assume that  $\sigma : [0, 1] \rightarrow \mathbb{R}^d$  is an absolutely continuous curve from  $x$  to  $y$  such that  $|\sigma'(t)| = \ell(\sigma)$  for almost every  $t$ . Then

$$\begin{aligned} (R_x^p f)^\alpha(y) &= \int_0^1 \partial_t (T_{\sigma(t)}^{p-|\alpha|} f^{(\alpha)}(y)) dt \\ &= \int_0^1 \sum_{|\beta|=p-|\alpha|} \frac{1}{\beta!} \sum_{i=1}^d f^{(\alpha+\beta+e_i)}(\sigma(t)) (y-\sigma(t))^\beta \sigma'_i(t) dt. \end{aligned}$$

By the Cauchy–Schwarz inequality,

$$\left| \sum_{i=1}^d f^{(\alpha+\beta+e_i)}(\sigma(t)) (y-\sigma(t))^\beta \sigma'_i(t) \right| \leq |\nabla f^{(\alpha+\beta)}(\sigma(t))| |\sigma'(t)| |y-\sigma(t)|^{|\beta|}.$$

Moreover,  $|y-\sigma(t)| = |\sigma(1)-\sigma(t)| \leq \ell(\sigma)(1-t)$ . Thus

$$|(R_x^p f)^\alpha(y)| \leq \sqrt{d} \sup_{\substack{\xi \in \sigma \\ |\gamma|=p+1}} |f^{(\gamma)}(\xi)| \ell(\sigma)^{p+1-|\alpha|} \int_0^1 \frac{(1-t)^{p-|\alpha|}}{(p-|\alpha|)!} dt \sum_{|\beta|=p-|\alpha|} \frac{|\beta|}{\beta!},$$

which is (10.5). □

**COROLLARY 10.2.** *Let  $M = (M_k)$  be a strongly regular weight sequence. For all  $X \in \mathcal{H}^1(\mathbb{R}^d)$ , the functions in  $\mathcal{C}^M(X)$  are precisely the functions which admit a  $\mathcal{C}^M$ -extension to  $\mathbb{R}^d$ .*

*Proof.* This follows from Lemma 10.1 and the  $\mathcal{C}^M$ -version of Whitney’s extension theorem [Bru80], since a bounded Lipschitz set is quasiconvex, that is, (10.1) holds with  $m = 1$ ; cf. Proposition 3.8. □

**COROLLARY 10.3.** *Let  $M = (M_k)$  be a non-quasianalytic weight sequence of moderate growth such that  $M^a$  is non-quasianalytic for each  $a > 0$ . Let  $X \subseteq \mathbb{R}^d$  be a closed fat subanalytic subset. Then the functions in  $\widehat{\mathcal{C}}^M(X)$  are precisely the functions which admit a  $\widehat{\mathcal{C}}^M$ -extension to  $\mathbb{R}^d$ . If  $X$  is simple, they are precisely the functions in  $\widehat{\mathcal{A}}^M(X)$ .*

*Proof.* This follows from Theorems 8.4, 5.6, and Lemma 10.1. Indeed, Lemma 10.1 implies that each  $f \in \widehat{\mathcal{C}}^M(X)$  defines a Whitney jet of class  $\widehat{\mathcal{C}}^M$  on  $X$  (the integer  $m$  of Lemma 10.1 is local but it is absorbed by  $\widehat{\mathcal{C}}^M$ ). The extension theorem [CC98, Theorem 8] yields the required extension to  $\mathbb{R}^d$ . □

### 10.2 Examples and counterexamples

The following examples complement the results and indicate their sharpness.

*Example 10.4* (Infinitely flat fat cusps are not  $\mathcal{A}^\infty$ -admissible). Let  $p : [0, \infty) \rightarrow [0, \infty)$  be a strictly increasing  $\mathcal{C}^\infty$ -function which is infinitely flat at 0. Consider the set  $X := \{(x, y) \in \mathbb{R}^2 : x \geq 0, 0 \leq y \leq p(x)\}$  and the function  $f : X \rightarrow \mathbb{R}$  defined by  $f(x, y) = \sqrt{x^2 + y}$ . Clearly,  $f$  is  $\mathcal{C}^\infty$  in the interior of  $X$  but  $\partial_y f$  does not extend continuously to the origin.

On the other hand,  $f \in \mathcal{A}^\infty(X)$ . Let  $x, y : \mathbb{R} \rightarrow \mathbb{R}$  be  $\mathcal{C}^\infty$ -functions such that  $(x(t), y(t)) \in X$  for all  $t \in \mathbb{R}$ . To see that  $f \in \mathcal{A}^\infty(X)$  it suffices to prove that there is a  $\mathcal{C}^\infty$ -function  $z : \mathbb{R} \rightarrow \mathbb{R}$  such that  $y = x^2 z$ .

We use the following result due to [JP90, Theorem 7]. *Let  $\varphi, \psi : \mathbb{R} \rightarrow \mathbb{R}$  be such that  $\psi \in \mathcal{C}^\infty$ ,  $\varphi\psi \in \mathcal{C}^\infty$ , and  $|\varphi| \leq |\psi|^\alpha$  for some positive constant  $\alpha$ . Then  $\varphi \in \mathcal{C}^{[2\alpha]}$ .*

We apply this result for  $\psi = x^2$  and

$$\varphi = \begin{cases} y(t)/x(t)^2 & \text{if } x(t) \neq 0, \\ 0 & \text{if } x(t) = 0. \end{cases}$$

The assumption  $0 \leq y \leq p(x)$  implies that for each  $n \in \mathbb{N}$  there is an interval  $[0, \epsilon_n)$  such that for all  $x \in [0, \epsilon_n)$  we have  $y \leq x^{2n+2}$ . We may conclude that  $\varphi$  is  $\mathcal{C}^{2n}$  on the set  $x^{-1}([0, \epsilon_n))$ . Clearly,  $\varphi$  is  $\mathcal{C}^\infty$  on the set  $\{t \in \mathbb{R} : x(t) \neq 0\}$ . Thus  $\varphi$  is  $\mathcal{C}^\infty$  everywhere.

*Example 10.5* (Necessity of simpleness). Let  $X_1 = \{(x, y) \in \mathbb{R}^2 : x \geq 0, 0 \leq y \leq x\}$  and  $X_2 = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq y/2\}$  and set  $X = X_1 \cup X_2$ . The function  $f$  on  $X$  defined by  $f(x, y) = x$  if  $(x, y) \in X_1$  and  $f(x, y) = y$  if  $(x, y) \in X_2$  belongs to  $\mathcal{A}^\infty(X)$  but clearly not to  $\mathcal{C}^\infty(X)$ . This follows from the fact that a  $\mathcal{C}^\infty$ -curve  $c(t)$  in  $X$  must vanish of infinite order at each  $t_0$  with  $c(t_0) \in X_1 \cap X_2 = \{0\}$ . Indeed, suppose that  $c(t) = t^p \tilde{c}(t)$  with  $(a, b) := \tilde{c}(0) \neq 0$  and  $c(t) \in X_1$  if  $t \leq 0$  and  $c(t) \in X_2$  if  $t \geq 0$ . If  $p$  is even, it follows that  $b \leq a \leq b/2$  which entails  $a = b = 0$ , a contradiction. If  $p$  is odd, we conclude that  $0 \leq a \leq 0, b \leq 0, a \leq b/2$  hence  $a = b = 0$  again.

A modification of this example shows that the assumption that  $X$  is simple cannot be replaced by the weaker assumption that each  $x \in X$  has a neighborhood  $U$  such that  $U \cap \text{int}(X)$  is connected: Let  $0 < r < R$ , consider  $X := X_1 \cup X_2 \cup X_3$ , where  $X_3 = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, x^2 + y^2 \geq R^2\}$ , and multiply  $f$  with a smooth bump function which is 1 on  $B(0, r)$  and has support in  $B(0, R)$ .



Nevertheless we have the following example.

*Example 10.6.* Let  $X_1 = \{(x, 0) \in \mathbb{R}^2 : x \geq 0\}$  and  $X_2 = \{(0, y) \in \mathbb{R}^2 : y \geq 0\}$  and set  $X = X_1 \cup X_2$ . Then  $X$  is  $\mathcal{A}^\infty$ -admissible. Indeed, let  $f \in \mathcal{A}^\infty(X)$ . We may assume without loss of generality that  $f(0, 0) = 1$  (by multiplying with or adding a constant). Now  $f|_{X_1}$  (respectively,  $f|_{X_2}$ ) has a  $\mathcal{C}^\infty$ -extension  $F_1$  to  $\mathbb{R} \times \{0\}$  (respectively,  $F_2$  to  $\{0\} \times \mathbb{R}$ ), by Theorem 1.13, and  $F(x, y) := F_1(x)F_2(y)$  is a  $\mathcal{C}^\infty$ -extension of  $f$ .

*Example 10.7* (There is no analogue for finite differentiability). This is an interesting consequence of Glaeser’s inequality [Gla63]: for  $f : \mathbb{R} \rightarrow [0, \infty)$ ,

$$f'(t)^2 \leq 2f(t)\|f''\|_{L^\infty(\mathbb{R})}, \quad t \in \mathbb{R}.$$

Indeed, consider the closed half-space  $X = \{x \in \mathbb{R}^d : x_d \geq 0\}$  and the function  $f : X \rightarrow \mathbb{R}$  given by  $f(x) = x_d^{k+1/2}$ . Then all partial derivatives of  $f$  up to order  $k$  extend continuously by 0 to  $\partial X$ , and the partial derivatives of order  $k$  are  $1/2$ -Hölder continuous, but not better, near points of  $\partial X$ . On the other hand, for each  $\mathcal{C}^{k,1}$ -curve  $c$  in  $X$  with compact support, the composite  $(f \circ c)(t) = c_d(t)^{k+1/2}$  is  $\mathcal{C}^k$  with

$$(f \circ c)^{(k)}(t) = C_k(c'_d(t))^k \sqrt{c_d(t)} + D_k(c(t)),$$

where  $t \mapsto D_k(c(t))$  is Lipschitz. Since  $\sqrt{c_d}$  is Lipschitz, by Glaeser’s inequality, we conclude that  $f \circ c$  is of class  $\mathcal{C}^{k,1}$ .

We want to add that the images of pseudo-immersions (which are not immersions) yield examples of sets  $X \subseteq \mathbb{R}^d$  which are not  $\mathcal{A}^\infty$ -admissible.

*Example 10.8.* If  $\gcd(p, q) = 1$  then the map  $\varphi : \mathbb{R} \ni t \mapsto (t^p, t^q) \in \mathbb{R}^2$  is a pseudo-immersion, by [Jor82]; see also [JP87, JP90, DKP85, AM89]. Now the function  $f(x, y) = y^{1/q}$  belongs to  $\mathcal{A}^\infty(\varphi(\mathbb{R}))$  but has no smooth extension to  $\mathbb{R}^2$ .

The following example shows that there are closed fat sets  $X \subseteq \mathbb{R}^d$  which satisfy

$$\mathcal{A}^\infty(X) = \{f : X \rightarrow \mathbb{R} : f \text{ satisfies 1.10(3)}\} \neq \mathcal{C}^\infty(X). \tag{10.6}$$

*Example 10.9.* Let  $X$  be the complement in  $\mathbb{R}^2$  of the set  $\{(x, y) \in \mathbb{R}^2 : x > 0, |y| < e^{-1/x}\}$ . It is well known (cf. [Bie80, Example 2.18]) that there exist functions  $f : X \rightarrow \mathbb{R}$  which satisfy Lemma 1.10(3), but  $f \notin \mathcal{C}^\infty(X)$ .

Let us show that for this  $X$  the identity in (10.6) holds. To this end let  $h : \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $h(x) = 0$  if  $x \leq 0$  and  $h(x) = e^{-1/x}$  if  $x > 0$ . Consider

$$X_\pm := \{(x, y) \in \mathbb{R}^2 : \pm y \geq h(x)\} \cup \{(x, y) \in \mathbb{R}^2 : x \leq 0\}.$$

Then  $X_\pm$  are 1-sets and hence are  $\mathcal{A}^\infty$ -admissible, by Theorem 1.13.

Suppose  $f \in \mathcal{A}^\infty(X)$ . Then  $f$  is smooth on  $\text{int}(X)$ . The restrictions  $f|_{X_\pm}$  belong to  $\mathcal{A}^\infty(X_\pm)$ , respectively. So all their derivatives extend to the boundary arcs  $\{(x, y) \in \mathbb{R}^2 : x > 0, \pm y = h(x)\}$ , respectively. It remains to check that the extensions of the derivatives of  $f|_{X_\pm}$  coincide at the origin. But this is clear, since they are uniquely determined by the restriction of  $f$  to  $X_+ \cap X_-$ .

For the converse suppose that  $f : X \rightarrow \mathbb{R}$  satisfies 1.10(3). We have to show that  $f \circ c$  is smooth for all smooth curves  $c : \mathbb{R} \rightarrow X$ . Since  $X_\pm$  are  $\mathcal{A}^\infty$ -admissible, this is clear on the

complement of  $c^{-1}(0)$  in  $\mathbb{R}$ . Assume that  $c(0) = 0$ . We claim that  $f \circ c$  is differentiable at 0 and the chain rule  $(f \circ c)'(0) = f'(0)(c'(0))$  holds. The set  $X$  is star-shaped with respect to each point in  $(-\infty, 0]$ .

For each  $v \in X$ , the curve  $\gamma(t) := tv$  lies in  $X$  for  $0 \leq t \leq 1$ . Moreover,  $\gamma_s(t) := \gamma(t) + s^2(-1 - \gamma(t))$  lies in  $X$  for  $0 \leq t \leq 1$  and  $|s| \leq 1$ . If  $s \neq 0$ , then  $\gamma_s(t) \in \text{int}(X)$  and hence

$$\frac{f(\gamma_s(t)) - f(\gamma_s(0))}{t} = \int_0^1 (f \circ \gamma_s)'(tu) \, du = (1 - s^2) \int_0^1 f'(\gamma_s(tu))(v) \, du.$$

Letting  $s \rightarrow 0$  and using that  $f$  satisfies Lemma 1.10(3), we get

$$\frac{f(\gamma(t)) - f(\gamma(0))}{t} = \int_0^1 f'(\gamma(tu))(v) \, du.$$

This tends to  $f'(\gamma(0))(v)$  as  $t \rightarrow 0$ .

Now for  $0 \leq s \leq 1$  and  $t \in \mathbb{R}$  we have  $s \cdot c(t) \in X$ . We may apply the last paragraph for  $v = c(t)/t$  and obtain

$$\frac{f(c(t)) - f(0)}{t} = \int_0^1 f'(uc(t)) \left( \frac{c(t)}{t} \right) \, du,$$

which tends to  $f'(0)(c'(0))$ , since  $f'(uc(t)) \rightarrow f'(0)$  uniformly on the bounded set  $\{c(t)/t : t \text{ near } 0\}$ . This proves the claim.

By iteration we may conclude that  $f \circ c$  is smooth; cf. the proof of [KM97, Theorem 24.5].

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