J. Austral. Math. Soc. (Series A) **56** (1994), 64–116

ON THE FOURIER TRANSFORM OF A COMPACT SEMISIMPLE LIE GROUP

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(27 May 1991)

Communicated by I. Raeburn

Abstract

We develop a concrete Fourier transform on a compact Lie group by means of a symbol calculus, or $*$-product, on each integral co-adjoint orbit. These $*$-products are constructed by means of a moment map defined for each irreducible representation. We derive integral formulae for these algebra structures and discuss the relationship between two naturally occurring inner products on them. A global Kirillov-type character is obtained for each irreducible representation. The case of $SU(2)$ is treated in some detail, where some interesting connections with classical spherical trigonometry are obtained.

1991 *Mathematics subject classification* (*Amer. Math. Soc.*): 22 E 46, 58 F 06.

Introduction

Classical harmonic analysis on the real line $\mathbb{R}$ and the torus T is largely the study of the Fourier transform. This is a map from functions on the group G to functions on the dual $\hat{G}$, the set of unitary characters of G. It is given by the formula

$$(*) \qquad \phi^\wedge(\chi) = \int_G \phi(g)\chi(g)\,dg$$

where ϕ is a function on G and $\chi \in \hat{G}$. Of course this is a formal definition whose meaning must be investigated for ϕ belonging to various spaces of functions (or

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distributions) on G. The point here is that for these groups, the dual $\hat{G}$ is itself a topological group and the Fourier transform defined by $(*)$ is a very explicit and computable object. The resulting theory has a well-known generalisation to locally compact abelian groups. For non-abelian groups, the situation is considerably more complicated. We will restrict our attention here to real Lie groups. Then the proper definition of the dual $\hat{G}$ is the set of equivalence classes of irreducible unitary representations. One then defines the Fourier transform for G by generalising $(*)$. That is for ϕ a function on G and $\rho \in \hat{G}$ one defines

$$(**) \qquad \phi^\wedge(\rho) = \int_G \phi(g)\rho(g)\, dg.$$

This is then an operator-valued function of $\hat{G}$, with the operators $\phi^\wedge(\rho)$ acting on (generally) different Hilbert spaces for different $\rho \in \hat{G}$. As such it also really depends on our choice of representation in each equivalence class of $\hat{G}$. It is also clear that a necessary preliminary to any systematic study of the Fourier transform is a description of the dual $\hat{G}$. This turns out to be a deep and difficult problem. For example, the dual of $G = GL(n, \mathbb{R})$ has only recently been classified (see Vogan [21]), and the situation for general reductive groups is not yet understood, although progress has been made (see for example, Knapp and Speh [10]).

Even with the dual $\hat{G}$ in hand however, the abstract nature of $(**)$ is an obstacle to further study. It may not be possible to even formulate analogues of classical results from the abelian theory in any natural way.

The purpose of this paper is to describe an alternate framework for the definition and study of the Fourier transform for G a compact connected semisimple Lie group. In effect we construct a new, geometric Fourier transform F for such a group and then show that it incorporates the abstract definition of $(**)$. We may motivate this construction by going back to the Fourier transform for $\mathbb{R}$ or T and rewriting $(*)$ as

$$(***) \qquad \phi^\wedge(\chi) = \int_G \phi(g)e(g, \chi)\, dg.$$

Here $e(g, \chi) = \chi(g)$ can be considered a function on $G \times \hat{G}$. Now for these groups $\hat{G}$ can be naturally identified with a subset of the dual of the Lie algebra $\mathfrak{g}$. That is $\hat{G} \simeq \mathfrak{g}^*_{\mathrm{INT}}$ where $\mathfrak{g}^*_{\mathrm{INT}}$ is the set of points in $\mathfrak{g}^*$ satisfying a certain integrality condition. Thus e may be considered to be a function on $G \times \mathfrak{g}^*_{\mathrm{INT}}$, which is naturally a subset of the cotangent bundle T^*G of G, which we may call T^*G_{INT}, the integral cotangent bundle.

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The usefulness of this point of view is that even for non-abelian groups, there is a close connection between $\hat{G}$ and the dual of the Lie algebra $\mathfrak{g}^*$. Now the group G acts nontrivially on $\mathfrak{g}^*$ however, so it is the orbits of this action, the so-called co-adjoint orbits, that play the role of the points of $\mathfrak{g}^*$ in the abelian case. It is Kirillov's fundamental observation that for nilpotent connected simply-connected Lie groups, there is a natural bijection between $\hat{G}$ and the set of co-adjoint orbits (see Kirillov [9]). The theory of geometric quantization, introduced by Kostant [11] and Souriau [16], has shown that this close connection extends to many other groups. It becomes necessary, however, to introduce an integrality condition on co-adjoint orbits and thus to consider only integral co-adjoint orbits. One consequence of this theory is that for G a compact connected semisimple Lie group, $\hat{G}$ is naturally in bijection with the set of integral orbits in $\mathfrak{g}^*$, which we call $\mathfrak{g}^*_{\mathrm{INT}}$. (This fact can be considered to be a restatement of the classical description of $\hat{G}$ by highest weights together with the Borel-Weil theorem.)

Now we may define as before, the integral contangent bundle T^*G_{INT} as $G \times \mathfrak{g}^*_{\mathrm{INT}}$. Our main result is the following. There exists on T^*G_{INT} a function e which we call the Fourier kernel of G, which defines a Fourier transform F from functions on G to functions on $\mathfrak{g}^*_{\mathrm{INT}}$ by

$$F\phi(f) = \int_G \phi(g)e(g, f)\,dg$$

for ϕ a function on G, $f \in \mathfrak{g}^*_{\mathrm{INT}}$. This Fourier transform incorporates the abstract one by means of a symbol calculus on each orbit (to be explained). The Fourier kernel is constructed in a theoretically explicit fashion, it turns out to be both unique and canonical, and in some sense contains all the representation theoretic information for the group.

To justify our terminology of Fourier transform for the map F, we show that it behaves in ways that resemble the abelian case. To describe this analogy in more detail requires some additional results and notation.

First of all, if $\rho \in \hat{G}$ and $\mathscr{O}_\rho \subset \mathfrak{g}^*_{\mathrm{INT}}$ is the associated integral orbit, we show that there is a canonical finite dimensional space of functions A_ρ on $\mathscr{O}_\rho$, and an isomorphism

$$a : \mathrm{End}\ V_\rho \to A_\rho$$

where V_ρ is the space on which ρ acts. The isomorphism 'a' establishes a symbol calculus for operators on End V_ρ. It also allows the algebra structure

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of End V_ρ to be transferred to A_ρ; we call this the $*$-product on A_ρ (or on $\mathscr{O}_\rho$). This construction generalises the $*$-products constructed on the 2-sphere S^2 by Moreno and Ortega-Navarro [14].

Now it turns out that if $U_\rho \subset C(G)$ is the space of all matrix coefficients of ρ, then $F : U_\rho \to A_\rho$ is an isomorphism. That is, for $u \in U_\rho$, Fu is supported only on the orbit $\mathscr{O}_\rho$, where it is an element of A_ρ. Thus F separates a function on G into its Fourier components. Furthermore convolution in U_ρ is taken to the $*$-product on A_ρ, so we may say that convolution on the group is taken to an orbitwise product on $\mathfrak{g}^*_{\mathrm{INT}}$ by F. We also show that the Fourier transform of a positive-definite function $u \in U_\rho$ is a positive function in A_ρ.

One of the most important results in the classical representation theory for G is the Weyl character formula. Recently it has become well-known that this formula is equivalent to Kirillov's character formula, which states that the character χ_ρ of a representation $\rho \in \hat{G}$ can be obtained by 'pushing down' from $\mathfrak{g}$ the Euclidean inverse Fourier transform of the invariant measure $d\mu$ on $\mathscr{O}_\rho$. This 'push down' involves the exponential map as well as a factor related to its Jacobian. (For a precise description, see Khalgui [8]).

Kirillov's character formula seems to break down where the exponential map is not well- behaved. We show that the Fourier transform F may be used to introduce a similar character formula which however is globally defined. The result is that

$$F\bar{\chi}_\rho = \begin{cases} (\dim V_\rho)^{-1} & \text{on } \mathscr{O}_\rho; \\ 0 & \text{elsewhere.} \end{cases}$$

In other words $\bar{\chi}_\rho$ is the 'inverse Fourier transform' in our sense of a constant function on $\mathscr{O}_\rho$.

We now describe more explicitly the techniques employed to obtain these results.

In Section 1 we study a calculus of operators on a representation space (V, ρ) with respect to a non-zero G-orbit $\mathscr{M}$. The main notion here is that of an effective orbit – one that has the property that for $T \in \operatorname{End} V$, $T = 0$ if and only if $\langle Tm, m\rangle = 0$ for all $m \in \mathscr{M}$. Then we may associate to each T its symbol $\sigma_T(m) = \langle Tm, m\rangle$. We also define an orbit to be symmetric if

$$\int_{\mathscr{M}} \langle v_1, m\rangle\langle v_2, m\rangle\langle m, v_3\rangle\langle m, v_4\rangle\, dm$$

is a multiple of $\langle v_1, v_3\rangle\langle v_2, v_4\rangle + \langle v_1, v_4\rangle\langle v_2, v_3\rangle$. Various integral formulae for symbols on a symmetric orbit are proven, and we conclude in Proposition 1.17 that a symmetric orbit is effective.

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In Section 2 we introduce some ideas to describe algebras of functions on a manifold $\mathscr{M}$ which carry a Hilbert space structure. We employ an integral-type notation to deal with certain bilinear forms that arise and use it to define the notion of a triple-kernel of the algebra (with respect to the inner product). This is a function on $\mathscr{M} \times \mathscr{M} \times \mathscr{M}$ which encodes the algebra structure in a fashion similar to the way in which the kernel of an operator encodes the operator.

If $\mathscr{M}$ is an effective orbit, the space of symbols W on $\mathscr{M}$ turns out to carry two natural Hilbert space structures and an algebra structure. In Section 3 we study the operators which relate these two inner products. For a symmetric orbit, we develop explicit formulae for the kernels of these operators. We also study the triple-kernels for W, the main result being Theorem 3.11, giving a formula for the triple-kernel with respect to one of the inner products for a general effective orbit.

In Section 4, we relate the previous discussion to orbits in $\mathfrak{g}^*$ by introducing the moment map of a representation (V, ρ). Since G acts on the projective space PV of V in a Hamiltonian way, there is a canonical map (the moment map) $\Phi' : PV \to \mathfrak{g}^*$. Now if Ω is the unit sphere in V and $\epsilon : \Omega \to PV$ the natural map, we define $\Phi = \Phi' \circ \epsilon$ and call it the moment map of ρ. Our definition is actually more direct but equivalent. We study the image of Φ and prove various functorial properties of it. These results are analogues of the functorial relationships between representations and co-adjoint orbits first discovered by Kirillov [9] for nilpotent groups. We show that the extremal set of ImΦ is a single G-orbit $\mathscr{O}_\rho$ and that $\Phi^{-1}(\mathscr{O}_\rho) = \mathscr{M}_\rho$ is a single G-orbit in Ω. Furthermore $\Phi : \mathscr{M}_\rho \to \mathscr{O}_\rho$ is an S^1 bundle, which we call the canonical bundle of ρ (Proposition 4.9). In Theorem 4.11 we show that $\mathscr{M}_\rho$ is an effective orbit.

The association of the orbit $\mathscr{O}_\rho$ to each $\rho \in \hat{G}$ is injective and is closely related to classical descriptions of $\hat{G}$. If we fix a choice of positive roots, then $\mathscr{O}_\rho$ and $\mathscr{M}_\rho$ can be considered to be the orbits of the highest weight and highest weight vector respectively. It also turns out that the canonical bundle of ρ is the same as that constructed in geometric quantization—this fact was communicated to us in conversation by I. Frenkel, to whom we are grateful.

In Section 5 we use Φ to transfer structure from $\mathscr{M}_\rho$ to $\mathscr{O}_\rho$. We show how the canonical 2-form on $\mathscr{O}_\rho$ and the connection 1-form on $\mathscr{M}_\rho$ are related to the complex structure of V. We define the space of functions A_ρ on $\mathscr{O}_\rho$.

In Section 6 the main definitions concerning the Fourier kernel e and the Fourier transform F are introduced. The main results here have already been described.

In Section 7 we turn in some detail to the case of $G = SU(2)$. Here the integral orbits are spheres in $\mathfrak{g}^*$ of radii $\ell = 0, 1/2, 1, 3/2, \ldots$. We compute the Fourier kernel e of G in Theorem 7.5 and discover that it is closely related to the Cayley transform for G. At this point we remark that in the thesis Wildberger [22] we have constructed a similar Fourier transform theory for nilpotent Lie groups using entirely different methods (see also Arnal and Cortet [2]). For the group of real upper triangular matrices we found that the Fourier kernel also involved the Cayley transform. This phenomenon has been investigated for a wide class of nilpotent groups in Howe, Ratcliff and Wildberger [7] where it is shown that the role of the Cayley transform is closely related to the theory of the oscillator representation of the symplectic group.

We also remark that our Fourier transform theory coincides in the case of $SU(2)$ with that introduced by Sherman [15] in his work on Fourier analysis on spheres (see also Helgason [6]). In Theorems 7.12 and 7.13 we give formulae for the triple-kernels of the $*$-products on the integral spheres. These are related in a surprising way to certain classical identities of spherical trigonometry, and shed new light on the $*$-products of Moreno and Ortega-Navarro [14] as well as on Sherman's work. More generally, our theory provides an explicit construction of a $*$-product on any integral co-adjoint orbit of a compact semisimple Lie group. It would be of interest to compare this with the general construction of $*$-products on symplectic manifolds studied by Lichnerowicz [13], Gutt [5] and others.

This paper is meant to be largely self-contained, the major requirement of the reader being a familiarity with elementary representation theory, for which references are Varadarajan [18] and Helgason [6].

This paper has existed in preprint form for some years now. A different version of our geometric Fourier transform construction has since appeared in Figueroa, Gracia-Bondía and Várilly [4] (see also Várilly and Gracia-Bondía [19], Arnal, Cahen and Gutt [1]). Additional work on the relationship between representations and moment maps has appeared in Wildberger [23, 24] and Arnal and Ludwig [3].

Section 1

Let G be a compact Lie group and (V, ρ) an irreducible unitary representation of G. The Hermitian form on V will be denoted by $\langle\ ,\ \rangle$, and will be taken to be linear in the first variable and conjugate linear in the second. Let $\dim\ V = n$. Let $\mathscr{M}$ be a non-zero orbit in V, so that there exists $v \neq 0$ in V with $\mathscr{M} =$

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$\{g \cdot v | g \in G\}$. Here the use of ρ is suppressed, so that $g \cdot v = \rho(g)(v)$. The orbit $\mathscr{M}$ carries a G-invariant measure.

LEMMA 1.1. *The G-invariant measure dm on $\mathscr{M}$ may be normalised so that for any $v \in V$,*

$$v = \int_{\mathscr{M}} \langle v, m \rangle m \, dm. \tag{1.1}$$

PROOF. Consider the operator $A \in \text{End}\, V$ defined by

$$A(v) = \int_{\mathscr{M}} \langle v, m \rangle m \, dm$$

for $v \in V$. Then for any $g \in G$,

$$\begin{aligned} A(g \cdot v) &= \int_{\mathscr{M}} \langle g \cdot v, m \rangle m \, dm \\ &= \int_{\mathscr{M}} \langle v, g^{-1} \cdot m \rangle m \, dm \\ &= \int_{\mathscr{M}} \langle v, m \rangle g \cdot m \, dm \\ &= g \cdot A(v). \end{aligned}$$

Thus A commutes with all $g \in G$ and so by Schur's Lemma is a multiple of the identity. By an appropriate normalisation of dm we may choose the multiple to be 1.

Fix the measure dm on $\mathscr{M}$ so that (1.1) holds. Introduce $L^2(\mathscr{M}, dm)$ with inner product $\langle\ ,\ \rangle_2$. Thus for $\phi_1, \phi_2 \in L^2(\mathscr{M}, dm)$,

$$\langle \phi_1, \phi_2 \rangle_2 = \int_{\mathscr{M}} \phi_1(m) \overline{\phi_2(m)} \, dm.$$

For $v \in V$, let $\tilde{v}$ be the function on $\mathscr{M}$ defined by $\tilde{v}(m) = \langle v, m \rangle$. Let $\tilde{V}$ be the space of all $\tilde{v}$, $v \in V$, and let $\tilde{}: V \to \tilde{V}$ be the map that sends v to $\tilde{v}$.

LEMMA 1.2. $\tilde{}: V \to \tilde{V} \subset L^2(\mathscr{M}, dm)$ *is an isometry.*

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PROOF. Since (V, ρ) is irreducible, $\tilde{v} = 0$ if and only if $v = 0$, so $\tilde{}$ is an isomorphism. If $v_1, v_2 \in V$, then by Lemma 1.1,

$$\begin{aligned}\langle v_1, v_2\rangle &= \int_{\mathscr{M}} \langle v_1, m\rangle\langle m, v_2\rangle\, dm \\ &= \langle \tilde{v}_1, \tilde{v}_2\rangle_2.\end{aligned}$$

DEFINITION 1.3. For $T \in \operatorname{End} V$, define $K_T : \mathscr{M} \times \mathscr{M} \to \mathbb{C}$, the kernel of T with respect to $\mathscr{M}$, by the formula

$$K_T(m', m) = \langle T \cdot m', m\rangle$$

for $m', m \in \mathscr{M}$.

LEMMA 1.4. *For $T_1, T_2 \in \operatorname{End} V$ and $m'', m \in \mathscr{M}$,*

$$K_{T_2T_1}(m'', m) = \int_{\mathscr{M}} K_{T_1}(m'', m')K_{T_2}(m', m)\, dm'.$$

PROOF. For $T \in \operatorname{End} V$, let $\tilde{T}$ be the integral operator on $L^2(\mathscr{M}, dm)$ with kernel K_T. That is, for $\phi \in L^2(\mathscr{M}, dm)$ and $m \in \mathscr{M}$,

$$\tilde{T}\phi(m) = \int_{\mathscr{M}} K_T(m', m)\phi(m')\, dm'.$$

Then for $v \in V$ and $m \in \mathscr{M}$,

$$\begin{aligned}\tilde{T}\tilde{v}(m) &= \int_{\mathscr{M}} \langle T \cdot m', m\rangle\langle v, m'\rangle\, dm' \\ &= \langle T \cdot v, m\rangle \\ &= (Tv)^{\sim}(m).\end{aligned}$$

Here we have used Lemma 1.1. Now let

$$\tilde{V}^{\perp} = \left\{\phi \in L^2(\mathscr{M}, dm) \,\middle|\, \langle \phi, \tilde{v}\rangle_2 = 0 \quad \forall v \in V\right\}.$$

Then for $\phi \in \tilde{V}^{\perp}$ and $m \in \mathscr{M}$,

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$$\begin{aligned}\tilde{T}\phi(m) &= \int_{\mathscr{M}} \langle Tm', m\rangle \phi(m')\,dm' \\ &= \int_{\mathscr{M}} \phi(m')\overline{(T^*m)\tilde{\ }(m')}\,dm' \\ &= \langle \phi, (T^*m)\tilde{\ }\rangle_2 \\ &= 0.\end{aligned}$$

Here T^* is the adjoint of T. Now the facts that $\tilde{T}\tilde{v} = (Tv)\tilde{\ }$ for all $v \in V$ and $\tilde{T}\phi = 0$ for all $\phi \in \tilde{V}^{\perp}$ imply that the kernel of T_2T_1 with respect to $\mathscr{M}$ is just the kernel of the integral operator $\tilde{T}_2\tilde{T}_1$ on $L^2(\mathscr{M}, dm)$. But it is a standard fact that this is

$$K_{T_2T_1}(m'', m) = \int_{\mathscr{M}} K_{T_1}(m'', m')K_{T_2}(m', m)\,dm'.$$

PROPOSITION 1.5. *For $T \in \operatorname{End} V$, $\operatorname{tr} T = \int_{\mathscr{M}} K_T(m, m)\,dm$.*

PROOF. It follows by the argument in the previous Lemma that $\operatorname{tr} T = \operatorname{tr} \tilde{T}$ since $L^2(\mathscr{M}, dm) = \tilde{V} \oplus \tilde{V}^{\perp}$. But $\operatorname{tr} \tilde{T} = \int_{\mathscr{M}} K_T(m, m)\,dm$.

Let $|\mathscr{M}|$, the modulus of $\mathscr{M}$, be the number $\sqrt{\langle m, m\rangle}$ for any $m \in \mathscr{M}$.

COROLLARY 1.6.

$$\int_{\mathscr{M}} dm = n/|\mathscr{M}|^2.$$

PROOF. Apply Proposition 1.5 to the identity $I \in \operatorname{End} V$. Then

$$\operatorname{tr} I = n = \int_{\mathscr{M}} \langle m, m\rangle\,dm = |\mathscr{M}|^2 \int_{\mathscr{M}} dm$$

so the result follows.

COROLLARY 1.7. *If $|\mathscr{M}| = 1$, then $\int_{\mathscr{M}} dm = n$.*

DEFINITION 1.8. For $T \in \operatorname{End} V$, define $\sigma_T : \mathscr{M} \to \mathbb{C}$, the *symbol* of T with respect to $\mathscr{M}$, by the formula

$$\sigma_T(m) = K_T(m, m).$$

Also let W be the space of all σ_T, $T \in \operatorname{End} V$, and let $\sigma : \operatorname{End} V \to W$ be the map that sends T to σ_T.

We are primarily interested in orbits $\mathscr{M}$ for which the map σ can be used to establish a symbol calculus.

DEFINITION 1.9. An orbit $\mathscr{M} \subset V$ is called *effective* if the map σ is an isomorphism from End V to W.

EXAMPLE 1.10. Let $G = SU(n)$ and (V, ρ) be the standard (or defining) representation of G. Thus dim $V = n$, and the orbits of G are simply spheres. Let $\mathscr{M}$ be the unit sphere. Then $\mathscr{M}$ is an effective orbit. This is a restatement of the fact from elementary linear algebra that if $T \in$ End V and $\langle Tm, m\rangle = 0$ for all $m \in \mathscr{M}$, then $T = 0$. This fact follows from the formula

$$\begin{aligned}\langle Tv, w\rangle = \tfrac{1}{4}\big(&\langle T(v+w), v+w\rangle - \langle T(v-w), v-w\rangle \\ &+ i\langle T(v+iw), v+iw\rangle - i\langle T(v-iw), v-iw\rangle\big)\end{aligned}$$

for all $v, w \in v$. For $v_1, v_2, v_3, v_4 \in v$ define

$$P_{\mathscr{M}}(v_1, v_2, v_3, v_4) = \int_{\mathscr{M}} \langle v_1, m\rangle\langle v_2, m\rangle\langle m, v_3\rangle\langle m, v_4\rangle\, dm.$$

DEFINITION 1.11. An orbit $\mathscr{M} \subset V$ is called *symmetric* if there exists a constant $c = c(\mathscr{M})$ such that for all $v_1, v_2, v_3, v_4 \in V$,

$$P_{\mathscr{M}}(v_1, v_2, v_3, v_4) = c\,(\langle v_1, v_3\rangle\langle v_2, v_4\rangle + \langle v_1, v_4\rangle\langle v_2, v_3\rangle)\,.$$

PROPOSITION 1.12. *Let $\mathscr{M} \subset V$ be the unit sphere, considered as an orbit of $SU(n)$ acting on V. Then $\mathscr{M}$ is symmetric and for $v_1, v_2, v_3, v_4 \in V$,*

$$P_{\mathscr{M}}(v_1, v_2, v_3, v_4) = (1/(n+1))\big(\langle v_1, v_3\rangle\langle v_2, v_4\rangle + \langle v_1, v_4\rangle\langle v_2, v_3\rangle\big).$$

PROOF. By multilinearity if suffices to prove the result for v_1, v_2, v_3, v_4 members of an orthonormal basis of V. This is then an exercise in advanced calculus.

LEMMA 1.13. *Let $\mathscr{M} \subset V$ be a symmetric orbit with constant $c = c(\mathscr{M})$. Then for $T_1, T_2 \in$ End V,*

$$\int_{\mathscr{M}} \langle T_1 m, m\rangle\langle T_2 m, m\rangle dm = c\,(\operatorname{tr} T_1 \operatorname{tr} T_2 + \operatorname{tr} T_1 T_2)\,.$$

PROOF. Fix an orthonormal basis $\{v_1, \dots, v_n\}$ of V. For $1 \leq i, j \leq n$ define $T_{ij} \in \operatorname{End} V$ by the equation $T_{ij}(v) = \langle v, v_j \rangle v_i$ for $v \in V$. These form a basis of End V so we may find constants α_{ij}, β_{ij} such that $T_1 = \sum_{i,j} \alpha_{ij} T_{ij}$ and $T_2 = \sum_{i,j} \beta_{ij} T_{ij}$. Then

$$\begin{aligned}
\int_{\mathscr{M}} \langle T_1 m, m \rangle \langle T_2 m, m \rangle \, dm &= \sum_{i,j} \sum_{k,l} \alpha_{ij} \beta_{kl} \int_{\mathscr{M}} \langle T_{ij} m, m \rangle \langle T_{kl} m, m \rangle \, dm \\
&= \sum_{i,j} \sum_{k,l} \alpha_{ij} \beta_{kl} P_{\mathscr{M}} \left(v_i, v_k, v_j, v_l \right) \\
&= c \sum_{i,j} \sum_{k,l} \alpha_{ij} \beta_{kl} \left(\langle v_i, v_j \rangle \langle v_k, v_l \rangle + \langle v_i, v_l \rangle \langle v_k, v_j \rangle \right) \\
&= c \sum_{i,j} \sum_{k,l} \alpha_{ij} \beta_{kl} \left(\delta_{ij} \delta_{kl} + \delta_{il} \delta_{kj} \right) \\
&= c \left(\sum_{i=1}^{n} \alpha_{ii} \sum_{k=1}^{n} \beta_{kk} + \sum_{i,j} \alpha_{ij} \beta_{ji} \right) \\
&= c \left(\operatorname{tr} T_1 \operatorname{tr} T_2 + \operatorname{tr} T_1 T_2 \right).
\end{aligned}$$

COROLLARY 1.14.

$$c = c(\mathscr{M}) = |\mathscr{M}|^2 / (n+1).$$

PROOF. Let $T_1 = T_2 = I$ in the previous lemma. Then

$$\int_{\mathscr{M}} \langle m, m \rangle^2 \, dm = c \left(n^2 + n \right).$$

But from Corollary 1.6,

$$\int_{\mathscr{M}} \langle m, m \rangle^2 \, dm = |\mathscr{M}|^4 \int_{\mathscr{M}} dm = |\mathscr{M}|^2 n$$

so that $c = |\mathscr{M}|^2 / (n+1)$

COROLLARY 1.15. *If* $|\mathscr{M}| = 1$ *then* $c = c(\mathscr{M}) = 1/(n+1)$.

The constant c is thus independent of the symmetric orbit $\mathscr{M}$ if $|\mathscr{M}| = 1$.

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PROPOSITION 1.16. *Let $\mathscr{M} \subset V$ be a symmetric orbit. Then for $T \in \operatorname{End} V$ and $v, w \in V$,*

$$\langle Tv, w\rangle = \int_{\mathscr{M}} \langle Tm, m\rangle \left(\frac{(n+1)}{|\mathscr{M}|^2}\langle v, m\rangle\langle m, w\rangle - \langle v, w\rangle\right) dm.$$

PROOF. Write $T = \sum_{i,j} \alpha_{ij} T_{ij}$ in the notation of the proof of Lemma 1.13. Then for $v, w \in V$,

$$\begin{aligned}
\int_{\mathscr{M}} \langle Tm, m\rangle\langle v, m\rangle\langle m, w\rangle\, dm &= \sum_{i,j} \alpha_{ij} P_{\mathscr{M}}(v_i, v, v_j, w)\\
&= \frac{|\mathscr{M}|^2}{n+1} \sum_{i,j} \alpha_{ij} \left(\langle v_i, v_j\rangle\langle v, w\rangle + \langle v_i, w\rangle\langle v, v_j\rangle\right)\\
&= \frac{|\mathscr{M}|^2}{n+1} \left(\langle v, w\rangle \sum_{i=1}^{n} \alpha_{ii} + \sum_{i,j} \alpha_{ij}\langle T_{ij} v, w\rangle\right)\\
&= \frac{|\mathscr{M}|^2}{n+1} \left(\langle v, w\rangle \operatorname{tr} T + \langle Tv, w\rangle\right).
\end{aligned}$$

Now since $\operatorname{tr} T = \int_{\mathscr{M}} K_T(m, m)\, dm = \int_{\mathscr{M}} \langle Tm, m\rangle\, dm$ (from Proposition 1.5), the result follows.

Proposition 1.16 shows how to recover the values $\langle Tv, w\rangle$ $(v, w \in V)$ from the symbol σ_T of T. In particular, we have the following immediate consequence.

PROPOSITION 1.17. *If $\mathscr{M} \subset V$ is a symmetric orbit, then it is also an effective orbit.*

We may also write down an integral alternative to the formula of Example 1.10.

PROPOSITION 1.18. *Let $\mathscr{M} \subset V$ be the unit sphere. Let $T \in \operatorname{End} V$ and σ_T be its symbol with respect to $\mathscr{M}$. Then for $v, w \in V$,*

$$\langle Tv, w\rangle = \int_{\mathscr{M}} \sigma_T(m)\left((n+1)\langle v, m\rangle\langle m, w\rangle - \langle v, w\rangle\right) dm.$$

Section 2

Let $\mathscr{M}$ be a compact manifold and W a finite-dimensional space of continuous functions on $\mathscr{M}$, closed under conjugation and containing the constants. Suppose further that W is a Hilbert space with inner product $\langle\ ,\ \rangle_\alpha$. In the application we have in mind, α will be an index for several Hilbert space structures on W.

For a positive integer k, define $W^{(k)}$ to be the space of all functions on $\mathscr{M} \times \ldots \times \mathscr{M}$ (k factors) which belong to W in each argument. Consider the symmetric bilinear form on W defined by $(w_1, w_2)_\alpha = \langle w_1, \overline{w}_2\rangle_\alpha$ for $w_1, w_2 \in W$. We introduce the integral-like symbol $\int_\alpha dm$ for this form and write

$$(w_1, w_2)_\alpha = \int_\alpha w_1(m)w_2(m)dm.$$

The reader is warned that this symbol does not have its standard meaning here, but nevertheless behaves in a fashion similar to the usual integral under elementary operations. The following facts are immediate.

LEMMA 2.1. *For any* $w_1, w_2, w_3 \in W$ *and* $c \in \mathbb{C}$,

(a) $$\int_\alpha w_1(m)w_2(m)\,dm = \int_\alpha w_2(m)w_1(m)\,dm;$$

(b) $$\int_\alpha (cw_1)(m)w_2(m)\,dm = \int_\alpha w_1(m)(cw_2)(m)\,dm = c\int_\alpha w_1(m)w_2(m)\,dm;$$

(c) $$\int_\alpha w_1(m)(w_2+w_3)(m)\,dm = \int_\alpha w_1(m)w_2(m)\,dm + \int_\alpha w_1(m)w_3(m)\,dm;$$

(d) $$\int_\alpha \overline{w}_1(m)\overline{w}_2(m)\,dm = \overline{\int_\alpha w_1(m)w_2(m)\,dm}.$$

Now suppose that $\phi_1 \in W^{(k)}$ and $\phi_2 \in W^{(\ell)}$ with $\phi_1 = \phi_1(m_{11}, \ldots, m_{1k})$ and $\phi_2 = \phi_2(m_{21}, \ldots, m_{2\ell})$. Let $1 \le i \le k$, $1 \le j \le \ell$. Then we can extend the above notation to give meaning to the expression

$$\int_\alpha \phi_1(m_{11}, \ldots, m_{1,i-1}, m, m_{1,i+1}, \ldots, m_{1k}) \cdot \phi_2(m_{21}, \ldots, m_{2,j-1}, m, m_{2,j+1}, \ldots, m_{2\ell})\,dm.$$

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The result is an element of $W^{k+\ell-2}$ in the variables $m_{11}, \dots, \hat{m}_{1i}, \dots, m_{1k}, m_{21}, \dots, \hat{m}_{2j}, \dots, m_{2\ell}$, where $\hat{}$ means deletion. More generally we may 'integrate' in this fashion any finite product of functions belonging to $W^{(\infty)} = \bigoplus_{k=1}^{\infty} W^{(k)}$ as long as the variable of integration appears as an argument of exactly two of them. We also consider 'multiple integrals', that is, expressions such as

$$\int_\alpha \left(\int_\alpha \phi_1(m_1, m_2)\phi_2(m_1, m_2)\, dm_1 \right) dm_2 = \int_\alpha \int_\alpha \phi_1(m_1, m_2)\phi_2(m_1, m_2)\, dm_1 dm_2.$$

We leave the reader to check that we may freely interchange the order of integration in any such multiple integral. Thus the quantity in the above expression is also equal to

$$\int_\alpha \left(\int_\alpha \phi_1(m_1, m_2)\phi_2(m_1, m_2)\, dm_2 \right) dm_1.$$

Such manipulations will be made without further comment in what follows. We now show how to use the above conventions to introduce some familiar objects associated to W and End W.

DEFINITION 2.2. A *reproducing kernel* for W with respect to $\langle\ ,\ \rangle_\alpha$ is a function $R_\alpha \in W^{(2)}$ such that for all $w \in W$,

$$w(m) = \int_\alpha R_\alpha(m', m) w(m') dm'.$$

DEFINITION 2.3. If $T \in \operatorname{End} W$, a *kernel for T with respect to* $\langle\ ,\ \rangle_\alpha$ is a function $K_\alpha(T) \in W^{(2)}$ such that for all $w \in W$,

$$Tw(m) = \int_\alpha K_\alpha(T)(m', m) w(m')\, dm'.$$

Note that a reproducing kernel for W is just a kernel for the identity $I \in \operatorname{End} W$. (When α is fixed or there is no possibility of confusion, we will drop the reference to $\langle\ ,\ \rangle_\alpha$.)

The following is a standard fact whose proof we leave to the reader.

LEMMA 2.4. *Suppose* $\dim W = p$ *and that* $\{e_1, \dots, e_p\}$ *is an orthonormal basis of* W *with respect to* $\langle\ ,\ \rangle_\alpha$. *Then for any* $T \in \operatorname{End} W$, *a kernel for* T *with respect to* $\langle\ ,\ \rangle_\alpha$ *is*

$$K_\alpha(T)(m', m) = \sum_{i=1}^{p} \overline{e_i(m')} T e_i(m).$$

Furthermore it is unique (so does not depend on the particular basis). In particular

$$R_\alpha(m', m) = \sum_{i=1}^{p} \overline{e_i(m')} e_i(m)$$

is the unique reproducing kernel for W with respect to $\langle\ ,\ \rangle_\alpha$.

Now suppose that the space W has in addition an algebra structure, denoted by $\times$. It will be useful to construct an object which will encode the algebra structure of $(W, \times)$ in a fashion similar to the way the kernel $K_\alpha(T)$ encodes the operator $T \in \operatorname{End} W$.

DEFINITION 2.5. A *triple-kernel* (or 3-*kernel*) for the algebra $(W, \times)$ with respect to $\langle\ ,\ \rangle_\alpha$ is a function $B_\alpha \in W^{(3)}$ such that for all $w_1, w_2 \in W$

$$w_1 \times w_2(m) = \int_\alpha \int_\alpha B_\alpha(m, m_1, m_2) w_1(m_1) w_2(m_2)\, dm_1 dm_2.$$

LEMMA 2.6. *Suppose as before that $\{e_1, \dots, e_p\}$ is an orthonormal basis of W with respect to $\langle\ ,\ \rangle_\alpha$ and that for $1 \le i, j \le p$,*

$$e_i \times e_j = \sum_{k=1}^{p} c_{ij}^k e_k, \qquad c_{ij}^k \in \mathbb{C}.$$

Then a triple-kernel for the algebra $(W, \times)$ with respect to $\langle\ ,\ \rangle_\alpha$ is given by

$$B_\alpha(m, m_1, m_2) = \sum_{i,j,k} c_{ij}^k \overline{e_i(m_1) e_j(m_2)} e_k(m).$$

Furthermore it is unique (so does not depend on the choice of basis).

PROOF. Let $w_1, w_2 \in W$ with $w_1 = \sum_{i=1}^p a_i e_i$, $w_2 = \sum_{i=1}^p b_i e_i$. Then

$$\begin{aligned}
&\int_\alpha \int_\alpha \sum_{i,j,k} c_{ij}^k \overline{e_i(m_1) e_j(m_2)} e_k(m) w_1(m_1) w_2(m_2)\, dm_1 dm_2 \\
&\qquad = \sum_{i,j,k} \sum_{r=1}^{p} \sum_{s=1}^{p} c_{ij}^k a_r b_s \int_\alpha \int_\alpha \overline{e_i(m_1) e_j(m_2)} e_r(m_1) e_s(m_2) e_k(m)\, dm_1 dm_2 \\
&\qquad = \sum_{i,j,k} \sum_{r,s} c_{ij}^k a_r b_s \delta_{ri} \delta_{sj} e_k(m)
\end{aligned}$$

$$= \sum_{r,s} a_r b_s \sum_k c^k_{rs} e_k(m)$$
$$= \sum_{r,s} a_r b_s e_r \times e_s(m)$$
$$= w_1 \times w_2(m).$$

Thus $B_\alpha(m, m_1, m_2) = \sum_{i,j,k} c^k_{ij} \overline{e_i(m_1)e_j(m_2)} e_k(m)$ is a triple-kernel for the algebra $(W, \times)$ with respect to $\langle \ , \ \rangle_\alpha$. Its uniqueness follows directly from the condition that $B_\alpha \in W^{(3)}$.

The next few results show how properties of the algebra $(W, \times)$ are reflected in properties of the triple-kernel B_α.

PROPOSITION 2.7. *$(W, \times)$ is associative if and only if for all $m_1, m_2, m_3, m_4 \in \mathscr{M}$,*

$$\int_\alpha B_\alpha(m, m_1, m_2) B_\alpha(m_4, m, m_3)\, dm = \int_\alpha B_\alpha(m_4, m_1, m) B_\alpha(m, m_2, m_3)\, dm.$$

PROOF. Let $w_1, w_2, w_3 \in W$. Then

$$(w_1 \times w_2) \times w_3(m_4)$$
$$= \int_\alpha \int_\alpha B_\alpha(m_4, m, m_3) \int_\alpha \int_\alpha B_\alpha(m, m_1, m_2) w_1(m_1) w_2(m_2)\, dm_1 dm_2 w_3(m_3)\, dm dm_3$$

while

$$w_1 \times (w_2 \times w_3)(m_4)$$
$$= \int_\alpha \int_\alpha B_\alpha(m_4, m_1, m) w_1(m_1) \int_\alpha \int_\alpha B_\alpha(m, m_2, m_3) w_2(m_2) w_3(m_3)\, dm_2 dm_3 dm_1 dm.$$

Thus $(W, \times)$ is associative if and only if for all $m_4 \in \mathscr{M}$ and for all $w_1, w_2, w_3 \in W$,

$$\int_\alpha \int_\alpha \int_\alpha C_\alpha(m_1, m_2, m_3, m_4) w_1(m_1) w_2(m_2) w_3(m_3)\, dm_1 dm_2 dm_3 = 0$$

where

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$$C_\alpha(m_1, m_2, m_3, m_4) = \int_\alpha B_\alpha(m_4, m, m_3) B_\alpha(m, m_1, m_2) - B_\alpha(m_4, m_1, m) B_\alpha(m, m_2, m_3)\, dm.$$

But since $B_\alpha \in W^{(3)}$, it is clear that $C_\alpha \in W^{(4)}$. Thus C_α must be identically zero, and we are done.

PROPOSITION 2.8. *$(W, \times)$ is commutative if and only if*

$$B_\alpha(m, m_1, m_2) = B_\alpha(m, m_2, m_1).$$

PROOF. Immediate from the definitions and a change of variable.

PROPOSITION 2.9. *$1 \in W$ is the identity of $(W, \times)$ if and only if both*

$$\int_\alpha B_\alpha(m, m_1, m_2)\, dm_2 = R_\alpha(m_1, m) \qquad \text{and}$$
$$\int_\alpha B_\alpha(m, m_1, m_2)\, dm_1 = R_\alpha(m_2, m).$$

PROOF. This follows immediately from the fact that

$$w \times 1(m) = \int_\alpha \int_\alpha B_\alpha(m, m_1, m_2) w(m_1)\, dm_1 dm_2 \qquad \text{and}$$
$$1 \times w(m) = \int_\alpha \int_\alpha B_\alpha(m, m_1, m_2) w(m_2)\, dm_1 dm_2$$

since of course $1(m) = 1$.

PROPOSITION 2.10. *The algebra $(W, \times)$ has the property that $\overline{w_1 \times w_2} = \overline{w}_2 \times \overline{w}_1$ if and only if*

$$B_\alpha(m, m_2, m_1) = \overline{B_\alpha(m, m_1, m_2)}.$$

PROOF. Left to the reader.

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PROPOSITION 2.11. *The algebra* $(W, \times)$ *has the property that for all* $w_1, w_2 \in W$

$$\int_\alpha w_1 \times \overline{w}_2(m)\, dm = \langle w_1, w_2\rangle_\alpha$$

if and only if

$$\int_\alpha B_\alpha(m, m_1, m_2)\, dm = R_\alpha(m_2, m_1).$$

PROOF. We first remark that the left hand side of both expressions is valid since $1 \in W$. For $w_1, w_2 \in W$,

$$\int_\alpha w_1 \times \overline{w}_2(m)\, dm = \int_\alpha \int_\alpha \int_\alpha B_\alpha(m, m_1, m_2) w_1(m_1)\overline{w}_2(m_2)\, dm_1 dm_2 dm$$

while

$$\langle w_1, w_2\rangle_\alpha = \int_\alpha w_1(m_2)\overline{w}_2(m_2)\, dm_2.$$

Thus $(W, \times)$ has the required property if and only if for all $w_1 \in W$,

$$w_1(m_2) = \int_\alpha \int_\alpha B_\alpha(m, m_1, m_2) dm\, w_1(m_1) dm_1,$$

that is, if and only if

$$\int_\alpha B_\alpha(m, m_1, m_2) dm = R_\alpha(m_1, m_2).$$

These propositions show that in general an asymmetry exists between the first argument of a triple-kernel $B_\alpha(m, m_1, m_2)$ and the last two. We will say that a triple-kernel B_α is *symmetric* if

$$B_\alpha(m, m_1, m_2) = B_\alpha(m_1, m_2, m) = B_\alpha(m_2, m, m_1)$$

for all $m, m_1, m_2 \in \mathscr{M}$. An algebra $(W, \times)$ will be called *symmetric with respect to* $\langle\ ,\ \rangle_\alpha$ if its triple-kernel B_α is symmetric.

PROPOSITION 2.12. *The algebra* $(W, \times)$ *is symmetric with respect to* $\langle\ ,\ \rangle_\alpha$ *if and only if*

$$\langle w_1 \times w_2, \overline{w}_3\rangle_\alpha = \langle w_2 \times w_3, \overline{w}_1\rangle_\alpha = \langle w_3 \times w_1, \overline{w}_2\rangle_\alpha$$

for all $w_1, w_2, w_3 \in W$.

PROOF. This follows along the same lines as the previous arguments.

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Section 3

The definitions and notation of Section 2 will now be applied to the situation in Section 1. Thus $\mathscr{M} \subset V$ will be taken to be an effective orbit of the compact group G, and to each $T \in \operatorname{End} V$ we have the unique symbol σ_T. The space W of all symbols on $\mathscr{M}$ is a finite dimensional space of continuous functions on $\mathscr{M}$ which contains the constants and is closed under conjugation. Let $\langle\ ,\ \rangle_2$ be the inner product on W which is the restriction of the inner product from $L^2(\mathscr{M}, dm)$, so that for $w_1, w_2 \in W$,

$$\langle w_1, w_2\rangle_2 = \int_{\mathscr{M}} w_1(m)\overline{w_2(m)}\, dm.$$

We may also consider another inner product on W, arising from the Hilbert space structure of End V.

DEFINITION 3.1. For $w_1, w_2 \in W$ with $\sigma^{-1}(w_1) = T_1, \sigma^{-1}(w_2) = T_2$, define

$$\langle w_1, w_2\rangle_1 = \operatorname{tr}(T_1 T_2^*).$$

The two Hilbert space structures $\langle\ ,\ \rangle_1$ and $\langle\ ,\ \rangle_2$ are in general not the same. Let $\tilde{\eta} : W \to W$ be the unique invertible operator such that

$$\langle \tilde{\eta}(w_1), w_2\rangle_1 = \langle w_1, w_2\rangle_2$$

for all $w_1, w_2 \in W$. Equivalently consider the operator $\eta : \operatorname{End} V \to \operatorname{End} V$ such that $\sigma_{\eta(T)} = \tilde{\eta}(\sigma_T)$ for all $T \in \operatorname{End} V$.

PROPOSITION 3.2. *For $T \in \operatorname{End} V$ and $v \in V$,*

$$\eta(T)(v) = \int_{\mathscr{M}} \sigma_T(m)\langle v, m\rangle m\, dm.$$

PROOF. Let $\mu : \operatorname{End} V \to \operatorname{End} V$ be defined by

$$\mu(T)(v) = \int_{\mathscr{M}} \sigma_T(m)\langle v, m\rangle m\, dm$$

for $T \in \operatorname{End} V$ and $v \in V$. The kernel of $\mu(T)$ with respect to $\mathscr{M}$ is

$$K_{\mu(T)}(m_1, m_2) = \int_{\mathscr{M}} \sigma_T(m)\langle m_1, m\rangle\langle m, m_2\rangle m\, dm.$$

Thus for $T_1, T_2 \in \operatorname{End} V$, $\mu(T_1)T_2^*$ has kernel

$$\begin{aligned}
K_{\mu(T_1)T_2^*}(m_1, m_2) &= \int_{\mathcal{M}} K_{T_2^*}(m_1, m) K_{\mu(T_1)}(m, m_2) dm \\
&= \int_{\mathcal{M}} \int_{\mathcal{M}} \langle m_1, T_2 m\rangle \sigma_{T_1}(m') \langle m, m'\rangle \langle m', m_2\rangle \, dm' dm \\
&= \int_{\mathcal{M}} \sigma_{T_1}(m') \langle m_1, T_2 m'\rangle \langle m', m_2\rangle \, dm'
\end{aligned}$$

from Lemma 1.4.

Thus by Proposition 1.5,

$$\begin{aligned}
\operatorname{tr} \mu(T_1)T_2^* &= \int_{\mathcal{M}} \int_{\mathcal{M}} \sigma_{T_1}(m') \langle m_1, T_2 m'\rangle \langle m', m_1\rangle \, dm' dm_1 \\
&= \int_{\mathcal{M}} \sigma_{T_1}(m') \langle m', T_2 m'\rangle \, dm' \\
&= \int_{\mathcal{M}} \sigma_{T_1}(m') \overline{\sigma_{T_2}(m')} \, dm' \\
&= \langle \sigma_{T_1}, \sigma_{T_2}\rangle_2 .
\end{aligned}$$

COROLLARY 3.3.

(a) *For* $T \in \operatorname{End} V$, $\operatorname{tr} \eta(T) = |\mathcal{M}|^2 \operatorname{tr} T$

(b) *For* $I =$ Identity, $\eta I = |\mathcal{M}|^2 I$.

PROOF.

(a)
$$\begin{aligned}
\operatorname{tr} \eta(T) &= \int_{\mathcal{M}} K_{\eta(T)}(m, m) \, dm \\
&= \int_{\mathcal{M}} \int_{\mathcal{M}} \sigma_T(m') \langle m, m'\rangle \langle m', m\rangle \, dm' dm \\
&= \int_{\mathcal{M}} \sigma_T(m') \langle m', m'\rangle \, dm' \\
&= |\mathcal{M}|^2 \operatorname{tr} T
\end{aligned}$$

(b)
$$\begin{aligned}
\eta(I)(v) &= \int_{\mathcal{M}} \langle m, m\rangle \langle v, m\rangle m \, dm \\
&= |\mathcal{M}|^2 v.
\end{aligned}$$

LEMMA 3.4. *η commutes with the action of G on* End V *by conjugation.*

PROOF. For $g \in G$, $v \in V$ and $T \in$ End V we use Proposition 3.2 to obtain

$$\eta(T)(g \cdot v) = \int_{\mathcal{M}} \sigma_T(m)\langle g \cdot v, m\rangle m\, dm$$
$$= \int_{\mathcal{M}} \sigma_T(g \cdot m)\langle v, m\rangle g \cdot m\, dm$$

But $\sigma_T(g \cdot m) = \langle g^{-1} \cdot T \cdot g \cdot m, m\rangle$ so

$$g^{-1} \cdot (\eta(T)(g \cdot v)) = \eta(g^{-1}Tg)(v).$$

PROPOSITION 3.5. *The kernel for $\tilde{\eta} : W \to W$ with respect to $\langle\ ,\ \rangle_2$ is the function $N \in W^{(2)}$ given by*

$$N(m', m) = |\langle m', m\rangle|^2.$$

PROOF. We must show that $N(m', m) \in W^{(2)}$ and that for all $w \in W$ and $m \in \mathcal{M}$,

$$\tilde{\eta}(w)(m) = \int_{\mathcal{M}} N(m', m) w(m')\, dm.$$

Now for fixed $m' \in \mathcal{M}$ the operator $T_{m'}, \in$ End V defined by $T_{m'}(v) = \langle v, m'\rangle m'$ has symbol $\sigma_{T_{m'}}(m) = |\langle m', m\rangle|^2$ so by the symmetry of N, $N \in W^{(2)}$.

Now let $w \in W$ and suppose that $w = \sigma_T$, $T \in$ End V. For $m \in \mathcal{M}$, use Proposition 3.2 to obtain

$$\tilde{\eta}(w)(m) = \sigma_{\eta(T)}(m)$$
$$= \langle \eta(T)m, m\rangle$$
$$= \int_{\mathcal{M}} \sigma_T(m')\langle m, m'\rangle\langle m', m\rangle\, dm'$$
$$= \int_{\mathcal{M}} N(m', m) w(m')\, dm'.$$

It now follows that knowledge of η^{-1} (or equivalently, $\tilde{\eta}^{-1}$) allows one to recover an operator $T \in \operatorname{End} V$ from its symbol σ_T. In fact, from Proposition 3.2 it follows that for $v, w \in V$,

$$\langle Tv, w\rangle = \int_{\mathscr{M}} \sigma_{\eta^{-1}(T)}(m)\langle v, m\rangle\langle m, w\rangle\, dm.$$

It is thus an interesting problem to write down explicitly the kernel of $\tilde{\eta}^{-1}$. In the case of $\mathscr{M}$ a symmetric orbit we use the results of Section 1 to do this. For convenience we treat only the case $|\mathscr{M}| = 1$.

PROPOSITION 3.6. *Let $\mathscr{M}$ be a symmetric orbit with $|\mathscr{M}| = 1$. Then the kernel of $\tilde{\eta}^{-1} : W \to W$ with respect to $\langle\ ,\ \rangle_2$ is the function $M \in W^{(2)}$ given by*

$$M(m', m) = (n+1)^2|\langle m', m\rangle|^2 - (n+2).$$

PROOF. It is clear that $M \in W^{(2)}$. Thus let $\tilde{\xi} : W \to W$ be the operator whose kernel with respect to $\langle\ ,\ \rangle_2$ is $M(m', m)$. Then the kernel of $\tilde{\xi}\tilde{\eta}$ is by Lemma 1.4

$$\begin{aligned} K(m'', m) &= \int_{\mathscr{M}} N(m'', m')M(m', m)\, dm' \\ &= \int_{\mathscr{M}} |\langle m'', m'\rangle|^2 \left((n+1)^2|\langle m', m\rangle|^2 - (n+2)\right) dm' \\ &= (n+1)^2 P_{\mathscr{M}}(m'', m, m'', m) - (n+2)\int_{\mathscr{M}} |\langle m'', m'\rangle|^2\, dm' \\ &= (n+1)\left(\langle m'', m''\rangle\langle m, m\rangle + \langle m'', m\rangle\langle m, m''\rangle\right) - (n+2) \\ &= (n+1)|\langle m'', m\rangle|^2 - 1. \end{aligned}$$

Here we have used Corollary 1.15. Now for $v_1, v_2 \in V$ let $w \in W$ be defined by

$$w(m) = \langle v_1, m\rangle\langle m, v_2\rangle.$$

Then

$$\begin{aligned} \int_{\mathscr{M}} K(m', m)w(m')\, dm' &= \int_{\mathscr{M}} \left((n+1)|\langle m', m\rangle|^2 - 1\right)\langle v_1, m'\rangle\langle m', v_2\rangle\, dm' \\ &= (n+1)P_{\mathscr{M}}(m, v_1, m, v_2) - \langle v_1, v_2\rangle \\ &= \langle v_1, m\rangle\langle m, v_2\rangle \\ &= w(m). \end{aligned}$$

Now since elements of the form $\langle v_1, m\rangle\langle m, v_2\rangle$ span W, we see that $K(m', m)$ is the reproducing kernel with respect to $\langle\ ,\ \rangle_2$ and so $\tilde{\xi} = \tilde{\eta}^{-1}$.

COROLLARY 3.7. *Let $\mathscr{M}$ be a symmetric orbit with $|\mathscr{M}| = 1$. Then the reproducing kernel of W with respect to $\langle\ ,\ \rangle_2$ is*

$$R_2(m', m) = (n+1)|\langle m', m\rangle|^2 - 1.$$

Now returning to the general effective orbit $\mathscr{M}$, which we assume to be of modulus 1, we can transfer the natural algebra structure of End V to W.

DEFINITION 3.8. Let $w_1, w_2 \in W$ with $\sigma^{-1}(w_i) = T_i \in \text{End } V, i = 1, 2$. Define $w_1 \times w_2 \in W$ by

$$w_1 \times w_2(m) = \sigma(T_2T_1)(m) \quad \text{for all } m \in \mathscr{M}.$$

The algebra $(W, \times)$ thus defined is isomorphic to the full matrix algebra End V. Let $B_1(m, m_1, m_2)$ and $B_2(m, m_1, m_2)$ in $W^{(3)}$ be the triple kernels for $(W, \times)$ with respect to $\langle\ ,\ \rangle_1$ and $\langle\ ,\ \rangle_2$ respectively. We may utilise the results of Section 2 and obvious properties of the algebra End V to deduce the following properties of B_1 and B_2.

PROPOSITION 3.9.

For all $m_1, m_2, m_3, m_4 \in \mathscr{M}$, and $\alpha = 1, 2$,

(a) $\displaystyle\int_\alpha B_\alpha(m, m_1, m_2)B_\alpha(m_4, m, m_3)\, dm = \int_\alpha B_\alpha(m_4, m_1, m)B_\alpha(m, m_2, m_3)\, dm$

(b) $\displaystyle\int_\alpha B_\alpha(m, m_1, m_2)\, dm_2 = R_\alpha(m_1, m)$ *and*

$$\int_\alpha B_\alpha(m, m_1, m_2)\, dm_1 = R_\alpha(m_2, m)$$

(c) $B_\alpha(m, m_2, m_1) = \overline{B_\alpha(m, m_1, m_2)}$

(d) $\displaystyle\int_1 B_1(m, m_1, m_2)\, dm = R_1(m_2, m_1)$

(e) $B_1(m, m_1, m_2)$ *is symmetric.*

This result suggests that in general B_1 will be a more natural object than B_2. Nevertheless we can compute B_2 in the case of $\mathscr{M}$ a symmetric orbit.

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PROPOSITION 3.10. *Let $\mathscr{M}$ be a symmetric orbit with $|\mathscr{M}| = 1$. Then for $m, m_1, m_2 \in \mathscr{M}$,*

$$B_2(m, m_1, m_2) = (n+1)^2\langle m, m_1\rangle\langle m_1, m_2\rangle\langle m_2, m\rangle - (n+1)|\langle m, m_1\rangle|^2 - (n+1)|\langle m, m_2\rangle|^2 + 1.$$

PROOF. Suppose $w_1, w_2 \in W$ with $\sigma^{-1}(w_i) = T_i \in \operatorname{End} V$, $i = 1, 2$. Let $T_3 = T_2T_1$ and let K_i be the kernel of T_i with respect to $\mathscr{M}$. By Proposition 1.16

$$K_i(m_1, m_2) = \int_{\mathscr{M}} \langle T_i m, m\rangle\big((n+1)\langle m_1, m\rangle\langle m, m_2\rangle - \langle m_1, m_2\rangle\big)\, dm$$

for $i = 1, 2$. Thus

$$\begin{aligned} K_3(m'', m) &= \int_{\mathscr{M}}\int_{\mathscr{M}}\int_{\mathscr{M}} \langle T_1 m_1, m_1\rangle\big((n+1)\langle m'', m_1\rangle\langle m_1, m'\rangle - \langle m'', m'\rangle\big) \\ &\qquad \cdot \langle T_2 m_2, m_2\rangle\big((n+1)\langle m', m_2\rangle\langle m_2, m\rangle - \langle m', m\rangle\big)\, dm_1 dm_2 dm' \\ &= \int_{\mathscr{M}}\int_{\mathscr{M}} w_1(m_1) w_2(m_2)\big((n+1)^2\langle m'', m_1\rangle\langle m_1, m_2\rangle\langle m_2, m\rangle \\ &\qquad - (n+1)\langle m'', m_1\rangle\langle m_1, m\rangle - (n+1)\langle m'', m_2\rangle\langle m_2, m\rangle \\ &\qquad + \langle m'', m\rangle\big)\, dm_1 dm_2. \end{aligned}$$

Thus

$$\begin{aligned} w_1 \times w_2(m) &= K_3(m, m) \\ &= \int_{\mathscr{M}}\int_{\mathscr{M}} w_1(m_1) w_2(m_2) B_2(m, m_1, m_2)\, dm_1 dm_2 \end{aligned}$$

where

$$B_2(m, m_1, m_2) = (n+1)^2\langle m, m_1\rangle\langle m_1, m_2\rangle\langle m_2, m\rangle - (n+1)|\langle m, m_1\rangle|^2 - (n+1)|\langle m, m_2\rangle|^2 + 1.$$

In contrast, the triple-kernel B_1 has a simple form even in the general case.

THEOREM 3.11. *Let $\mathscr{M}$ be an effective orbit with $|\mathscr{M}| = 1$. Then for $m, m_1, m_2 \in \mathscr{M}$,*

$$B_1(m, m_1, m_2) = \langle m, m_1\rangle\langle m_1, m_2\rangle\langle m_2, m\rangle.$$

PROOF. Let $T_1, T_2 \in \operatorname{End} V$. Then for $v \in V$,

$$\begin{aligned}\langle T_2T_1v, v\rangle &= \int_{\mathscr{M}} \langle T_1v, m\rangle\langle T_2m, v\rangle\, dm \\ &= \int_{\mathscr{M}}\int_{\mathscr{M}} \langle \eta^{-1}(T_1)m_1, m_1\rangle\langle v, m_1\rangle\langle m_1, m\rangle\, dm_1 \\ &\qquad\qquad \cdot \int_{\mathscr{M}} \langle \eta^{-1}(T_2)m_2, m_2\rangle\langle m, m_2\rangle\langle m_2, v\rangle\, dm_2 dm \\ &= \int_{\mathscr{M}}\int_{\mathscr{M}} \langle \eta^{-1}(T_1)m_1, m_1\rangle\langle \eta^{-1}(T_2)m_2, m_2\rangle\langle v, m_1\rangle \\ &\qquad\qquad\qquad \cdot \langle m_1, m_2\rangle\langle m_2, v\rangle\, dm_1 dm_2.\end{aligned}$$

Thus for $w_1, w_2 \in W$,

$$w_1 \times w_2(m) = \int_{\mathscr{M}}\int_{\mathscr{M}} \tilde{\eta}^{-1}(w_1)(m_1)\tilde{\eta}^{-1}(w_2)(m_2)\langle m, m_1\rangle\langle m_1, m_2\rangle\langle m_2, m\rangle\, dm_1 dm_2.$$

But for $w, w' \in W$,

$$\langle \tilde{\eta}^{-1}(w), w'\rangle_2 = \langle w, w'\rangle_1,$$

so

$$w_1 \times w_2(m) = \int_1\int_1 w_1(m_1)w_2(m_2)\langle m, m_1\rangle\langle m_1, m_2\rangle\langle m_2, m\rangle\, dm_1 dm_2$$

for all $w_1, w_2 \in W$. Thus

$$B_1(m, m_1, m_2) = \langle m, m_1\rangle\langle m_1, m_2\rangle\langle m_2, m\rangle.$$

Section 4

Let G be a compact Lie group and (V, ρ) a finite-dimensional unitary representation of G (not necessarily irreducible). Let $\Omega = \{v \in V \mid \langle v, v\rangle = 1\}$ be the unit sphere, and suppose $\dim V = n$. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g}^*$ its dual. We will continue to suppress the notation ρ, so that for $X \in \mathfrak{g}$ and $v \in V$, $\rho(X)(v)$ will be denoted simply by $X \cdot v$.

DEFINITION 4.1. $\Phi : \Omega \to \mathfrak{g}^*$

$$v \mapsto \Phi(v)(X) = \frac{1}{i}\langle X \cdot v, v\rangle$$

for $v \in \Omega$, $X \in \mathfrak{g}$, is the *moment map* of (V, ρ).

LEMMA 4.2. *Φ is a G-map between Ω and $\mathfrak{g}^*$, where G acts on $\mathfrak{g}^*$ via the co-adjoint action* Ad^*.

PROOF. For $g \in G$,

$$\begin{aligned}\Phi(g \cdot v)(X) &= \frac{1}{i}\langle X \cdot (g \cdot v), g \cdot v\rangle \\ &= \frac{1}{i}\langle g^{-1} \cdot (X \cdot (g \cdot v)), v\rangle \\ &= \frac{1}{i}\langle \mathrm{Ad}\, g^{-1}(X) \cdot v, v\rangle \\ &= \Phi(v)(\mathrm{Ad}\, g^{-1}(X)) \\ &= \mathrm{Ad}^* g(\Phi(v)).\end{aligned}$$

Let $\mathrm{Im}\,\Phi = \{\Phi(v) \mid v \in \Omega\}$ be the image of Φ. This is a compact subset of $\mathfrak{g}^*$.

LEMMA 4.3. *Let (V_1, ρ_1) and (V_2, ρ_2) be two unitary representations of G with moment maps Φ_1 and Φ_2 respectively. Let $V = V_1 \oplus V_2$ and Φ be the corresponding moment map. Then*

$$\mathrm{Im}\,\Phi = \{f \in \mathfrak{g}^* \mid f = tf_1 + (1-t)f_2, \quad 0 \le t \le 1,\ f_i \in \mathrm{Im}\,\Phi_i,\ i = 1, 2\}.$$

PROOF. If Ω_1, Ω_2 and Ω are the unit spheres of V_1, V_2 and V respectively, then any $v \in \Omega$ can be written as $v = z_1 v_1 + z_2 v_2$ with $v_1 \in \Omega_1$, $v_2 \in \Omega_2$ and $z_1, z_2 \in \mathbb{C}$ satisfying $|z_1|^2 + |z_2|^2 = 1$. Then for $X \in \mathfrak{g}$,

$$\begin{aligned}\Phi(v)(X) &= \frac{1}{i}\langle X \cdot (z_1 v_1 + z_2 v_2), z_1 v_1 + z_2 v_2\rangle \\ &= |z_1|^2 \frac{1}{i}\langle X \cdot v_1, v_1\rangle + |z_2|^2 \frac{1}{i}\langle X \cdot v_2, v_2\rangle\end{aligned}$$

since $\langle X \cdot v_1, v_2\rangle = \langle X \cdot v_2, v_1\rangle = 0$. Thus

$$\Phi(v) = |z_1|^2 \Phi_1(v_1) + |z_2|^2 \Phi_2(v_2).$$

LEMMA 4.4. *Let* (V_1, ρ_1) *and* (V_2, ρ_2) *be two unitary representations of* G *with moment maps* Φ_1 *and* Φ_2 *respectively. Let* $V = V_1 \otimes V_2$ *and* $\rho = \rho_1 \otimes \rho_2$ *the representation of* $G \times G = G'$ *given by* $\rho(g_1, g_2)(v_1 \otimes v_2) = (g_1 \cdot v_1) \otimes (g_2 \cdot v_2)$, $g_1, g_2 \in G$ *and* $v_1, v_2 \in V$, *with moment map* Φ. *Then* $\mathrm{Im}\Phi \subset \mathfrak{h}^* \times \mathfrak{g}^*$ *satisfies*

$$\mathrm{Im}\Phi_1 \times \mathrm{Im}\Phi_2 \subset \mathrm{Im}\Phi \subset \mathrm{conv}\,(\mathrm{Im}\Phi_1) \times \mathrm{conv}\,(\mathrm{Im}\Phi_2)$$

where conv (S) *means the convex hull of* S.

PROOF. Let Ω_1, Ω_2 and Ω be the unit spheres of V_1, V_2 and V respectively. Then for any $v_1 \in \Omega_1$, $v_2 \in \Omega_2$, $v = v_1 \otimes v_2 \in \Omega$ so that for any $(X_1, X_2) \in \mathfrak{g} \times \mathfrak{g}$,

$$\begin{aligned}\Phi(v)(X_1, X_2) &= i^{-1}\langle (X_1 \cdot v_1) \otimes v_2 + v_1 \otimes (X_2 \cdot v_2), v_1 \otimes v_2\rangle \\ &= i^{-1}(\langle X_1 \cdot v_1, v_1\rangle\langle v_2, v_2\rangle + \langle v_1, v_1\rangle\langle X_2 \cdot v_2, v_2\rangle) \\ &= \Phi_1(v_1)(X_1) + \Phi_2(v_2)(X_2).\end{aligned}$$

Thus $\mathrm{Im}\,\Phi_1 \times \mathrm{Im}\,\Phi_2 \subset \mathrm{Im}\,\Phi$. Now let $\{v_1, \ldots, v_n\}$ be an orthonormal basis of V_1. Then if $v \in \Omega$, we may write $v = \sum_{k=1}^n v_k \otimes w_k$ where $w_k \in V_2$ with $\sum_{k=1}^n |w_k|^2 = 1$. Then

$$\Phi(v)(X_1, X_2) = i^{-1}\sum_{k,l=1}^{n}\langle (X_1 \cdot v_k) \otimes w_k + v_k \otimes (X_2 \cdot w_k), v_l \otimes w_l\rangle$$

so that

$$\begin{aligned}\Phi(v)(0, X_2) &= i^{-1}\sum_{k,l=1}^{n}\delta_{k,l}\langle (X_2 \cdot w_k), w_l\rangle \\ &= i^{-1}\sum_{k=1}^{n}\langle (X_2 \cdot w_k), w_k\rangle \\ &= i^{-1}\sum_{\substack{k=1 \\ |w_k|\neq 0}}^{n} |w_k|^2\langle X_2 \cdot \frac{w_k}{|w_k|}, \frac{w_k}{|w_k|}\rangle \\ &= \Big(\sum_{\substack{k=1 \\ |w_k|\neq 0}}^{n} |w_k|^2\Phi_2(u_k)\Big)(X_2)\end{aligned}$$

where $u_k = w_k/|w_k|$ if $|w_k| \neq 0$ so that $u_k \in \Omega_2$.

Thus if $p_2 : \mathfrak{g}^* \times \mathfrak{g}^* \to \mathfrak{g}^*$ is the projection onto the second factor, we see that $p_2(\mathrm{Im}\,\Phi) \subset \mathrm{conv}\,(\mathrm{Im}\,\Phi_2)$ so by symmetry $\mathrm{Im}\,\Phi \subset \mathrm{conv}\,(\mathrm{Im}\,\phi_1) \times \mathrm{conv}\,(\mathrm{Im}\,\Phi_2)$.

LEMMA 4.5. *Let* (V, ρ) *be a unitary representation with moment map* Φ *and let* H *be a Lie subgroup of* G *with Lie algebra* $\mathfrak{h}$. *Let* $p : \mathfrak{g}^* \to \mathfrak{h}^*$ *be the projection dual to the inclusion* $i : \mathfrak{h} \to \mathfrak{g}$. *If* $\Phi_H : \Omega \to \mathfrak{h}^*$ *is the moment map of* $(V, \rho\big|_H)$ *then* $\Phi_H = p \circ \Phi$.

PROOF. Obvious from the definitions.

In general Im Φ need not be convex. For example, if $G = SU(2)$ and (V, ρ), is the standard two-dimensional representation of G, then Im Φ is easily seen to be a single G-orbit in $\mathfrak{g}^*$, namely a 2-sphere. However we have the following.

LEMMA 4.6. *Let* T *be a torus with Lie algebra* $\mathfrak{t}$. *Let* (V, ρ) *be a finite dimensional representation of* T *with weights* $i\lambda_1, \dots, i\lambda_n \in i\mathfrak{t}^*$. *Then* Im$\Phi$ *is the convex hull of* $\{\lambda_1, \dots, \lambda_n\} \subset \mathfrak{t}^*$. *In particular* Im$\Phi$ *is convex.*

PROOF. Write $V = V_1 \oplus \dots \oplus V_n$ with V_j the one-dimensional representation with weight $i\lambda_j$. That is, for $X \in \mathfrak{t}$ and $v_j \in V_j$, $X \cdot v_j = i\lambda_j(X)v_j$. Now if Φ_j is the moment map of V_j and if $v_j \in \Omega_j$, the unit sphere in V_j, we have

$$\Phi_j(v_j)(X) = \frac{1}{i}\langle X \cdot v_j, v_j\rangle = \lambda_j(X).$$

Thus Im $\Phi_j = \lambda_j$, and the result follows from Lemma 4.3 by induction.

Now let $T \subset G$ be a maximal torus with Lie algebra $\mathfrak{t}$. Choose a G-invariant positive-definite form (,) on $\mathfrak{g}$. Let $i\lambda_1, \dots, i\lambda_n \in i\mathfrak{t}^*$ be the weights of (V, ρ) restricted to T and let $D \subset \mathfrak{t}^*$ be the convex hull of $\{\lambda_1, \dots, \lambda_n\}$. Then if $p : \mathfrak{g}^* \to \mathfrak{t}^*$ is the projection, Lemmas 4.5 and 4.6 imply immediately that

$$p(\text{Im } \Phi) = D.$$

Denote by Ext (D) those points of D which are not contained in any open line segment in D. Clearly Ext (D) is a subset of $\{\lambda_1, \dots, \lambda_n\}$.

It will be useful to sometimes identify $\mathfrak{g}$ with $\mathfrak{g}^*$ via (,) and thus to consider $\mathfrak{t}^* \subset \mathfrak{g}^*$.

LEMMA 4.7. *If* $f \in \text{Im}\Phi \subset \mathfrak{g}^*$ *and* $p(f) = \lambda \in \text{Ext}(D)$, *then* $f = \lambda$.

PROOF. Write $f = \lambda + \mu$ with $\mu \in \mathfrak{g}^*, \mu(\mathfrak{t}) = 0$. Let $\mathcal{O}$ be the co-adjoint orbit through f and let $X \cdot f \in T\mathcal{O}_f$ with $X \in \mathfrak{g}$. Then $dp(X \cdot f)$ is a tangent vector to $\mathfrak{t}^*$ at $p(f) = \lambda$. Now since $p(\mathcal{O}) \subset D$, any curve in $\mathcal{O}$ through f with tangent vector $X \cdot f$ is sent by p to a curve in D through $\lambda \in \mathrm{Ext}\,(D)$. But any such curve must be singular since D is a solid convex polyhedron in $\mathfrak{t}^*$ with a finite number of vertices and λ is such a vertex. Thus $p(X \cdot f) = 0$, so that for all $Z \in \mathfrak{t}$, $X \in \mathfrak{g}$, $X \cdot f(Z) = 0$.

Now a basic fact about Lie algebras of compact groups is that they are reductive. This implies that $\mathfrak{g} = \mathfrak{t} \oplus [\mathfrak{t}, \mathfrak{g}]$, an orthogonal direct sum with respect to the form $(\ ,\ )$. Thus since we can rewrite the above as $(\lambda + \mu)([Z, X]) = 0$, we get $\mu([Z, X]) = 0$ for all $Z \in \mathfrak{t}$ and $X \in \mathfrak{g}$ so that $\mu = 0$ and $f = \lambda$.

LEMMA 4.8. *Let $\lambda \in \mathrm{Ext}(D)$. Then $v \in \Omega$ is a weight vector for T of weight $i\lambda$ if and only if $\Phi(v) = \lambda$.*

PROOF. Suppose first that $v \in \Omega$ is a weight vector of weight $i\lambda$. Then $p(\Phi(v)) = \lambda$ so that by the previous lemma, $\Phi(v) = \lambda$.

On the other hand let $v \in \Omega$ with $\Phi(v) = \lambda$. If $\{v_1, \dots, v_n\}$ is an orthonormal basis of weight vectors with v_j of weight $i\lambda_j$ then we can write $v = \sum_{i=1}^n \alpha_i v_i$ with $\sum_{i=1}^n |\alpha_i|^2 = 1$. Then if $Z \in \mathfrak{t}$,

$$\begin{aligned}
\Phi(v)(Z) &= \frac{1}{i}\langle Z \cdot v, v\rangle \\
&= \frac{1}{i}\sum_{j=1}^n |\alpha_j|^2 \langle Z \cdot v_j, v_j\rangle \\
&= \left(\sum_{j=1}^n |\alpha_j|^2 \Phi(v_j)\right)(Z) \\
&= \left(\sum_{j=1}^n |\alpha_j|^2 \lambda_j\right)(Z).
\end{aligned}$$

But if $\Phi(v) = \lambda$ then $\lambda = \sum_{j=1}^n |\alpha_j|^2 \lambda_j$. Now since $\lambda \in \mathrm{Ext}\,(D)$ all the α_j must be zero except for one and so v is actually a weight vector of weight $i\lambda$.

We remark more generally that if $F \subset D$ is a face of D and

$$F \cap \{\lambda_1, \dots, \lambda_n\} = \{\lambda_{i_1}, \dots, \lambda_{i_k}\}$$

then the set of $v \in \Omega$ for which $p(\Phi(v)) \subset F$ is

$$\left\{ v = \sum_{j=1}^k \alpha_{i_j} v_{i_j} \,\middle|\, \sum_{j=1}^k |\alpha_{i_j}|^2 = 1 \right\}.$$

Now we assume henceforth that G is semisimple and that (V, ρ) is irreducible. Then it is well known that the weights $i\lambda_1, \dots, i\lambda_n$ of (V, ρ) are contained in the convex hull of the G-translates of one of them, say $i\lambda_1$. That is, if $\mathcal{O}$ is the co-adjoint orbit through λ_1, then $\mathcal{O}$ intersects $\mathfrak{t}$ in exactly Ext (D), which is just the set of Weyl group translates of λ_1. From this it follows that $p(\mathcal{O}) \subset D$, which is part of a classical result of Kostant [12] which in this context is $p(\mathcal{O}) = D$. The orbit $\mathcal{O}$ so determined will be called the *extremal orbit* of Im Φ, and since it is determined solely by the representation, it will be denoted $\mathcal{O}_\rho$.

The weight $i\lambda_1$ is usually called a *highest weight* and it is a standard fact that a weight vector v_1 for $i\lambda_1$ is unique up to a scalar.

PROPOSITION 4.9. *Let $\Phi^{-1}(\mathcal{O}_\rho) = \mathcal{M}_\rho \subset \Omega$. Then $\mathcal{M}_\rho \to \mathcal{O}_\rho$ is an S^1 principal bundle.*

PROOF. From the remark, $\Phi^{-1}(\lambda_1) = \{zv_1 \big| |z|^2 = 1\}$. But for $X \in \mathfrak{t}$,

$$\exp tX \cdot v_1 = e^{it\lambda_1(X)} v_1$$

so since Φ is a G-map by Lemma 4.2, we have that $\mathcal{M}_\rho$ is a single G-orbit. Both $\mathcal{M}_\rho$ and $\mathcal{O}_\rho$ are homogeneous spaces for G so $\Phi : \mathcal{M}_\rho \to \mathcal{O}_\rho$ is a fibre bundle with fibre S^1. But S^1, when regarded as the unit circle in $\mathbb{C}$, acts naturally on V and preserves the fibre. We see that the S^1 action on a fibre arises from the group action of a one-parameter subgroup.

DEFINITION 4.10. The bundle $\Phi : \mathcal{M}_\rho \to \mathcal{O}_\rho$ will be called the *canonical S^1 bundle* of (V, ρ).

THEOREM 4.11. *The orbit $\mathcal{M}_\rho \subset \Omega$ is effective.*

PROOF. We begin by recalling some basic facts about G and the representation (V, ρ).

Let $\mathfrak{g}$ be the complexification of $\mathfrak{g}$ and $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra containing $\mathfrak{t}$. Let Δ be the set of roots of $\mathfrak{g}$ with respect to the action of $\mathfrak{h}$, and fix a subset $\Delta^+ \subset \Delta$ of positive roots. For each $\alpha \in \Delta^+$, we may choose non-zero vectors X_α and Y_α in the root spaces for α and $-\alpha$ respectively such that $\langle X_\alpha \cdot v, w \rangle = \langle v, Y_\alpha \cdot w \rangle$ for all $v, w \in V$.

An element $\mu \in \mathfrak{h}^*$ is a *weight* of ρ if for some non-zero vector $v \in V$ and all $H \in \mathfrak{h}$, $H \cdot v = \mu(H)v$. The set of all weights of ρ, denoted $\Lambda(\rho)$, is partially ordered by setting $\mu > \mu'$ if $\mu = \mu' + \sum_{i=1}^k m_i\alpha_i$, where $\alpha_i \in \Delta^+$

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and the $m_i \geq 0$ are integers not all zero. Then $\Lambda(\rho)$ contains a unique highest weight μ_1 with the property that $\mu_1 > \mu$ for any other weight $\mu \in \Lambda(\rho)$. It is the extension to $\mathfrak{h}$ of the functional $i\lambda_1$ on t, for some $\lambda_1 \in \text{Ext}(D)$.

For each $\mu \in \Lambda(\rho)$, we denote the weight space of μ by V_μ so that

$$V = \oplus_{\mu \in \Lambda(\rho)} V_\mu.$$

For $v_1, v_2 \in V$, define $T_{v_1,v_2} \in \text{End}\, V$ by

$$T_{v_1,v_2}(v) = \langle v, v_2 \rangle v_1.$$

For $\mu, \mu' \in \Lambda(\rho)$ define $U_{\mu,\mu'} \subset \text{End}\, V$ to be the span of all $T_{v,v'}$ with $v \in V_\mu$, $v' \in V_{\mu'}$. Then clearly

$$\text{End}\, V = \oplus_{\mu,\mu' \in \Lambda(\rho)} U_{\mu,\mu'}.$$

The spaces $U_{\mu,\mu'}$ can be partially ordered by setting $U_{\mu,\mu'} > U_{\eta,\eta'}$ if $\mu > \eta$ or if $\mu = \eta$ and $\mu' > \eta'$.

Let π be the representation of G on End V by conjugation so that for $g \in G$, $T \in \text{End}\, V$ and $v \in V$,

$$\pi(g)(T)v = g \cdot \big(T \cdot (g^{-1} \cdot v)\big)$$

while if $X \in \mathfrak{g}$ then

$$\pi(X)(T)v = X \cdot T \cdot v - T \cdot X \cdot v.$$

For $\alpha \in \Delta^+$, $v \in V_\mu$, $v' \in V_{\mu'}$, $w \in V$,

$$\begin{aligned} \pi(X_\alpha)T_{v,v'}(w) &= X_\alpha \cdot \langle w, v' \rangle v - \langle X_\alpha \cdot w, v' \rangle v \\ &= \langle w, v' \rangle X_\alpha \cdot v - \langle w, Y_\alpha \cdot v' \rangle v \\ &= T_{X_\alpha \cdot v, v'}(w) - T_{v, Y_\alpha \cdot v'}(w). \end{aligned}$$

Thus

$$\pi(X_\alpha)U_{\mu,\mu'} \subset U_{\mu+\alpha,\mu'} \oplus U_{\mu,\mu'-\alpha}. \tag{4.1}$$

Similarly

$$\pi(Y_\alpha)T_{v,v'}(w) = T_{Y_\alpha \cdot v, v'}(w) - T_{v, X_\alpha \cdot v'}(w)$$

so that

$$\pi(Y_\alpha)U_{\mu,\mu'} \subset U_{\mu-\alpha,\mu'} \oplus U_{\mu,\mu'+\alpha}.$$

Now let

$$U = \left\{ T \in \operatorname{End} V \,|\, \langle Tm, m\rangle = 0 \quad \forall m \in \mathscr{M}_\rho \right\}.$$

We must show $U = 0$. Clearly U is invariant under $\pi(G)$ since $\mathscr{M}_\rho$ is a G-orbit, so it is also $\pi(\mathfrak{g}\ )$ invariant. For any $T \in U$, we may write T as

$$T = \sum_{\mu,\mu' \in \Lambda(\rho)} S_{\mu,\mu'}$$

where $S_{\mu,\mu'} \in U_{\mu,\mu'}$. Consider the set of all spaces $U_{\mu,\mu'}$ for which there exists a $T \in U$ with non-zero component $S_{\mu,\mu'}$ in $U_{\mu,\mu'}$.

We will show this set to be empty. If not, let $U_{\eta,\eta'}$ be a maximal element, so that there exists $T \in U$ with

$$T = S_{\eta,\eta'} + \sum_{\substack{\mu,\mu' \in \Lambda(\rho) \\ (\mu,\mu') \neq (\eta,\eta')}} S_{\mu,\mu'}$$

where $0 \neq S_{\eta,\eta'} \in U_{\eta,\eta'}$ and $S_{\mu,\mu'} \in U_{\mu,\mu'}$ for all μ, $\mu' \in \Lambda(\rho)$.

We claim that $\eta = \eta' = \mu_1$. For suppose first that $\mu_1 > \eta$. Write

$$S_{\eta,\eta'} = \sum_{l=1}^{s} T_{w_l, w_l'}$$

where $\{w_1', \dots, w_s'\}$ is a basis of $V_{\eta'}$ and $w_\ell \in V_\eta$, with say $w_1 \neq 0$. Then since (V, ρ) is irreducible, we can find $\alpha \in \Delta^+$ such that $X_\alpha \cdot w_1 \neq 0$. Thus

$$\pi(X_\alpha) S_{\eta,\eta'} = \sum_{l=1}^{s} \left(T_{X_\alpha \cdot w_l, w_l'} - T_{w_l, Y_\alpha \cdot w_l'} \right)$$

has a nonzero component in $U_{\eta+\alpha,\eta'}$ since $\{w_1', \dots, w_s'\}$ is linearly independent.

Combining this with the fact that none of the elements $\pi(X_\alpha)S_{\mu,\mu'}$, $(\mu, \mu') \neq (\eta, \eta')$, can have a component in $U_{\eta+\alpha,\eta'}$ (by the maximality of $U_{\eta,\eta'}$ and (4.1)), we find that $\pi(X_\alpha)T$ has a non-zero component in $U_{\eta+\alpha,\eta'}$. But this is impossible since $\pi(X_\alpha)T \in U$, so we must have $\eta = \mu_1$.

Now suppose that $\mu_1 > \eta'$. Then we write

$$S_{\eta,\eta'} = T_{v_1, v'}$$

where $v_1 \in V_\eta = V_{\mu_1}$ and $v' \in V_{\eta'}$, with both v_1, v' non-zero. This is possible since V_{μ_1} is one-dimensional. Now again by the irreducibility of (V, ρ), we may find $\beta \in \Delta^+$ such that $X_\beta \cdot v' \neq 0$. Then

$$\pi(Y_\beta) S_{\eta,\eta'} = T_{Y_\beta \cdot v_1, v'} - T_{v_1, X_\beta \cdot v'}$$

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so that $\pi(Y_\beta)T$ has a non-zero component in $U_{\eta,\eta'+\alpha}$. But since $\pi(Y_\beta)T \in U$, we have a contradiction so that $\eta' = \mu_1$.

We have thus shown that if $U \neq 0$, then there exists $T \in U$ with

$$T = S_{\mu_1,\mu_1} + \sum_{\substack{\mu,\mu' \in \Lambda(\rho) \\ (\mu,\mu') \neq (\mu_1,\mu_1)}} S_{\mu,\mu'}$$

where $0 \neq S_{\mu_1,\mu_1} \in U_{\mu_1,\mu_1}$ and $S_{\mu,\mu'} \in U_{\mu,\mu'}$ for $(\mu, \mu') \neq (\mu_1, \mu_1)$. Now let $v_1 \in \Omega$ be a weight vector for μ_1, that is, $v_1 \in V_{\mu_1}$. Then $\Phi(v_1) = \lambda_1 \in \mathcal{O}_\rho$ so that $v_1 \in M_\rho$. But then

$$\begin{aligned} \langle Tv_1, v_1\rangle &= \langle S_{\mu_1,\mu_1} v_1, v_1\rangle + \sum_{\substack{\mu,\mu' \in \Lambda(\rho) \\ (\mu,\mu') \neq (\mu_1,\mu_1)}} \langle S_{\mu,\mu'} v_1, v_1\rangle \\ &= \langle S_{\mu_1,\mu_1} v_1, v_1\rangle \\ &\neq 0. \end{aligned}$$

This is a contradiction, so that $U = 0$ and $\mathcal{M}_\rho$ is effective.

Section 5

Recall that the co-adjoint orbit $\mathcal{O}_\rho$ is canonically a symplectic manifold, that is, it carries a closed non-degenerate 2-form ω. For $f \in \mathcal{O}_\rho$, let $X_1 \cdot f, X_2 \cdot f \in T\mathcal{O}_\rho(f)$ where $X_1, X_2 \in \mathfrak{g}$. Then

$$\omega_f(X_1 \cdot f, X_2 \cdot f) = f([X_1, X_2]). \tag{5.1}$$

On the other hand, the complex structure of V defines the 1-form θ on V by

$$\theta_v(w) = \operatorname{Im}\langle w, v\rangle$$

for $v \in V$, $w \in TV(v) \simeq V$. Then for $v \in V$, $w_1, w_2 \in TV(v)$,

$$d\theta_v(w_1, w_2) = 2\operatorname{Im}\langle w_1, w_2\rangle.$$

This is the 2-form on V which determines its symplectic structure. Both θ and $d\theta$ may be restricted to $\mathcal{M}_\rho \subset V$.

LEMMA 5.1. *On $\mathcal{M}_\rho$, $d\theta = (d\Phi)^*(\omega)$.*

PROOF. Let $m \in \mathscr{M}_\rho$ and $X_1 \cdot m, X_2 \cdot m \in T\mathscr{M}_\rho(m)$, where $X_1, X_2 \in \mathfrak{g}$. Then

$$\begin{aligned} d\theta_m(X_1 \cdot m, X_2 \cdot m) &= 2\mathrm{Im}\,\langle X_1 \cdot m, X_2 \cdot m\rangle \\ &= \frac{1}{i}(\langle X_1 \cdot m, X_2 \cdot m\rangle - \langle X_2 \cdot m, X_1 \cdot m\rangle) \\ &= \frac{1}{i}\langle [X_1, X_2] \cdot m, m\rangle \\ &= \Phi(m)([X_1, X_2]) \\ &= \omega_f(X_1 \cdot f, X_2 \cdot f) \end{aligned}$$

where $f = \Phi(m) \in \mathscr{O}_\rho$. But Φ is a G-map, so that $d\Phi(X_i \cdot m) = X_i \cdot f$, $i = 1, 2$ so that $d\theta = (d\Phi)^*(\omega)$.

Note that the 1-form θ, when restricted to $\mathscr{M}_\rho$, is a connection 1-form for the canonical bundle $\Phi : \mathscr{M}_\rho \to \mathscr{O}_\rho$ whose curvature 2-form, by the Lemma above, is ω. Thus the familiar connection and curvature on the quantum bundle in geometric quantization appear here naturally as manifestations of the complex structure of V.

The orbit $\mathscr{M}_\rho$ carries a space W of symbols where as before

$$W = \{\sigma_T \mid T \in \mathrm{End}\, V\}$$

and by Theorem 4.11, $\sigma : \mathrm{End}\, V \to W$ is an isomorphism. As in Section 3, we have two inner products $\langle\ ,\ \rangle_1$ and $\langle\ ,\ \rangle_2$ on W, as well as the algebra structure $(W, \times)$.

We now make a simple but crucial observation. Each function $\sigma_T \in W$ has the property that $\sigma_T(zm) = \sigma_T(m)$ for all $z \in S^1$, so that W really 'lives' on the orbit $\mathscr{O}_\rho$.

DEFINITION 5.2. For $T \in \mathrm{End}\, V$, let a_T be the function on $\mathscr{O}_\rho$ defined by

$$a_T(f) = \sigma_T(\Phi^{-1}(f))$$

for $f \in \mathscr{O}$. This function will be called the *symbol* of T. Let A_ρ be the space of all a_T, $T \in \mathrm{End}\, V$, and $a : \mathrm{End}\, V \to A_\rho$ the map that sends T to a_T.

LEMMA 5.3. *$a : \mathrm{End}\, V \to A_\rho$ is an isomorphism.*

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Now let $d\mu$ be the unique G-invariant measure on $\mathcal{O}_\rho$ such that

$$\int_{\mathcal{O}_\rho} \phi(f)\, d\mu(f) = \int_{\mathcal{M}_\rho} \phi(\Phi(m))\, dm$$

for all continuous functions ϕ on $\mathcal{O}_\rho$. Note that $d\mu$ may also be specified by requiring that it be G-invariant and that $\int_{\mathcal{O}_\rho} d\mu = n$ (this follows from Corollary 1.7).

Define inner products $\langle\ ,\ \rangle_\alpha, \alpha = 1, 2$ on A_ρ by

$$\langle a_{T_1}, a_{T_2}\rangle_\alpha = \langle \sigma_{T_1}, \sigma_{T_2}\rangle_\alpha$$

for all $T_1, T_2 \in \operatorname{End} V$. Let $\eta' : A_\rho \to A_\rho$ be the unique invertible operator such that for all $a_1, a_2 \in A_\rho$,

$$\langle \eta'(a_1), a_2\rangle_1 = \langle a_1, a_2\rangle_2.$$

Equivalently we may say that for all $T \in \operatorname{End} V$, $a_{\eta(T)} = \eta'(a_T)$.

We let $N' \in A_\rho^{(2)}$ be the kernel of η' with respect to $\langle\ ,\ \rangle_2$, so that for all $a \in A_\rho$ and $f \in \mathcal{O}_\rho$,

$$\eta'(a)(f) = \int_{\theta_\rho} N'(f', f) a(f') d\mu(f').$$

The kernel of $(\eta')^{-1} : A_\rho \to A_\rho$ with respect to $\langle\ ,\ \rangle_2$ will be denoted $M' \in A_\rho^{(2)}$. The reproducing kernel for A_ρ with respect to $\langle\ ,\ \rangle_\alpha$ will be denoted $R'_\alpha, \alpha = 1, 2$.

DEFINITION 5.4. For $a_{T_1}, a_{T_2} \in A_\rho$ $(T_1, T_2 \in \operatorname{End} V)$ we define the *$*$-product* of a_{T_1} and a_{T_2} to be

$$a_{T_1} * a_{T_2} = a_{T_2 T_1}.$$

The triple-kernel for the algebra $(A_\rho, *)$ with respect to $\langle\ ,\ \rangle_\alpha$ will be denoted $B_\alpha(f, f_1, f_2), \alpha = 1, 2$. The following is a direct consequence of Propositions 2.7, 2.9, 2.10, 2.11 and 2.12 and elementary properties of End V.

PROPOSITION 5.5.

For all $f_1, f_2, f_3, f_4 \in \mathcal{O}_\rho, \alpha = 1, 2,$

(a) $$\int_\alpha B'_\alpha(f, f_1, f_2) B'_\alpha(f_4, f, f_3)\, d\mu(f) = \int_\alpha B'_\alpha(f_4, f_1, f) B'_\alpha(f, f_2, f_3)\, d\mu(f)$$

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(b) $\int_\alpha B'_\alpha(f, f_1, f_2)\, d\mu(f_2) = R'_\alpha(f_1, f)$ *and* $\int_\alpha B'_\alpha(f, f_1, f_2)\, d\mu(f_1) = R'_\alpha(f_2, f)$

(c) $B'_\alpha(f, f_2, f_1) = \overline{B'_\alpha(f, f_1, f_2)}$

(d) $\int_1 B'_1(f, f_1, f_2) d\mu(f) = R'_1(f_1, f_2)$

(e) $B'_1(f, f_1, f_2)$ *is symmetric* .

Section 6

We now construct the Fourier transform for G, a compact semisimple Lie group. For each $\rho \in \hat{G}$, the unitary dual of G, we have the canonical S^1 bundle $\Phi : \mathcal{M}_\rho \to \mathcal{O}_\rho$, with $\mathcal{O}_\rho$ the extremal orbit in $\mathfrak{g}^*$. Since $\hat{G}$ is parametrized by the set of highest weights, it follows from the discussion in Section 4 that a representation $\rho \in \hat{G}$ is uniquely determined by the extremal orbit $\mathcal{O}_\rho$. Define $\mathfrak{g}^*_{\text{INT}} \subset \mathfrak{g}^*$ to be the union of all extremal orbits

$$\mathfrak{g}^*_{\text{INT}} = \cup_{\rho \in \hat{G}} \mathcal{O}_\rho .$$

We use the subscript INT since the set of extremal orbits is the same as the set of integral orbits, in the sense of geometric quantization (see Kostant [11]).

The cotangent bundle T^*G of G can be identified with $G \times \mathfrak{g}^*$ by right translation. Define T^*G_{INT}, the *integral cotangent bundle*, to be

$$T^*G_{\text{INT}} = G \times g^*_{\text{INT}} \subset T^*G.$$

DEFINITION 6.1. The *Fourier kernel* of G is the function e on T^*G_{INT} defined by

$$e(g, f) = \langle g \cdot \Phi^{-1}(f), \Phi^{-1}(f) \rangle$$

where $g \in G$, $f \in \mathcal{O}_\rho \subset \mathfrak{g}^*_{\text{INT}}$, and $\Phi : \mathcal{M}_\rho \to \mathcal{O}_\rho$ is the canonical S^1 bundle of ρ.

It is clear that e is well-defined and is a continuous function on T^*G_{INT}. Also note that

$$e(g, f) = e(g, g' \cdot f)$$

whenever g and g' in G commute. Thus e does not depend on our choice of identification of T^*G with $G \times \mathfrak{g}^*$.

DEFINITION 6.2. The *Fourier transform* of a function $\phi \in L^1(G)$ is the function $F\phi$ on $\mathfrak{g}^*_{\mathrm{INT}}$ defined by

$$F\phi(f) = \int_G \phi(g)e(g, f)\, dg$$

where $f \in \mathfrak{g}^*_{\mathrm{INT}}$.

Here we have chosen Haar measure dg on G so that $\int_G dg = 1$.

Now fix an irreducible unitary representation (V, ρ) with extremal orbit $\mathcal{O}_\rho$ and $a : \mathrm{End}\, V \to A_\rho$ as in Section 5.

LEMMA 6.3. *If $\phi \in L^1(G)$, then $F\phi|_{\mathcal{O}_\rho} = a_{\rho(\phi)}$ where*

$$\rho(\phi) = \int_G \phi(g)\rho(g)\, dg \quad \in \mathrm{End}\, V.$$

PROOF. Follows immediately from the definitions.

Now since any $T \in \mathrm{End}\, V$ can be written as $\rho(\phi)$ for some $\phi \in L^1(G)$, we see that the image of F restricted to $\mathcal{O}_\rho \subset \mathfrak{g}^*_{\mathrm{INT}}$ is exactly A_ρ.

For $v_1, v_2 \in V$, define the matrix coefficient $u_{v_1,v_2} \in L^1(G)$ by

$$u_{v_1,v_2}(g) = \langle v_1, g \cdot v_2\rangle$$

for $g \in G$, and let U_ρ be the space of all such functions.

LEMMA 6.4. *For $v_1, v_2 \in V$, let $T_{v_1,v_2} \in \mathrm{End}\, V$ be defined by $T_{v_1,v_2}(v) = \langle v, v_2\rangle v_1$. Then*

$$Fu_{v_1,v_2}|_{\mathcal{O}_\rho} = \frac{1}{n} a_{T_{v_1,v_2}}.$$

PROOF. Let $f \in \mathcal{O}_\rho$ and $m \in \mathcal{M}_\rho$ such that $\Phi(m) = f$. Then

$$\begin{aligned} Fu_{v_1,v_2}(f) &= \int_G u_{v_1,v_2}(g)\langle g \cdot m, m\rangle\, dg \\ &= \int_G \overline{\langle g \cdot v_2, v_1\rangle}\langle g \cdot m, m\rangle\, dg \\ &= \frac{1}{n}\langle v_1, m\rangle\langle m, v_2\rangle \\ &= \frac{1}{n}\langle T_{v_1,v_2}(m), m\rangle \\ &= \frac{1}{n} a_{T_{v_1,v_2}}(f) \end{aligned}$$

where we have used the Schur orthogonality relations.

LEMMA 6.5. *Let* $\phi \in L^1(G)$ *such that* $\phi \in U_\rho^\perp$. *Then* $F\phi|_{\mathscr{O}_\rho} = 0$.

PROOF. If $\phi \in U_\rho^\perp$ then for $f \in \mathscr{O}_\rho$ and $m \in \mathscr{M}_\rho$ with $\Phi(m) = f$, we have

$$\begin{aligned} F\phi(f) &= \int_G \phi(g)\langle g \cdot m, m\rangle \, dg \\ &= \int_G \phi(g)\overline{u_{m,m}(g)} \, dg \\ &= 0. \end{aligned}$$

Now let χ_ρ be the character of ρ, so that for $g \in G$,

$$\chi_\rho(g) = \operatorname{tr} \rho(g).$$

The following is our version of the Kirillov character formula in this setting.

THEOREM 6.6. $\bar{\chi}_\rho \in U_\rho$ *and* $F\bar{\chi}_\rho(f) = 1/n$ *for all* $f \in \mathscr{O}_\rho$.

PROOF. If $\{v_1, \ldots, v_n\}$ is an orthonormal basis of V then for $g \in G$,

$$\begin{aligned} \chi_\rho(g) &= \sum_{i=1}^n \langle g \cdot v_i, v_i\rangle \\ &= \sum_{i=1}^n \overline{u_{v_i,v_i}(g)} \end{aligned}$$

Thus $\bar{\chi}_\rho \in U_\rho$. Now

$$\begin{aligned} F\bar{\chi}_\rho &= \sum_{i=1}^n F u_{v_i,v_i} \\ &= \frac{1}{n}\sum_{i=1}^n a_{T_{v_i,v_i}} \\ &= \frac{1}{n} a_I \\ &= \frac{1}{n} \end{aligned}$$

where we have used the notation and result of Lemma 6.4 and where $I \in \operatorname{End} V$ is the identity.

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THEOREM 6.7. *For any* $\phi \in L^1(G)$,

$$\operatorname{tr} \rho(\phi) = \int_{\mathcal{O}_\rho} F\phi(f) d\mu(f).$$

PROOF.

$$\begin{aligned}\operatorname{tr} \rho(\phi) &= \int_{\mathcal{M}_\rho} \langle \rho(\phi)m, m\rangle \, dm \\ &= \int_{\mathcal{O}_\rho} a_{\rho(\phi)}(f)\, d\mu(f) \\ &= \int_{\mathcal{O}_\rho} F\phi(f)\, d\mu(f)\end{aligned}$$

where we have used Proposition 1.5.

PROPOSITION 6.8. *Let* $u \in U_\rho$ *be a positive-definite function. Then* $Fu(f) \geq 0$ *for all* $f \in \mathcal{O}_\rho$.

PROOF. It is a standard fact that any $u_{v,v} \in U_\rho$ is positive-definite and any $u \in U_\rho$ which is positive-definite can be written as

$$u = \sum_{i=1}^{r} u_{w_i, w_i}$$

with $w_i \in V$, $i = 1, \dots, r$. Then for $f \in \mathcal{O}_\rho$ and $m \in \mathcal{M}_\rho$ with $\Phi(m) = f$, we have

$$\begin{aligned}Fu(f) &= \sum_{i=1}^{r} Fu_{w_i, w_i}(f) \\ &= \frac{1}{n} \sum_{i=1}^{r} a_{T_{w_i, w_i}}(f) \\ &= \frac{1}{n} \sum_{i=1}^{r} |\langle w_i, m\rangle|^2 \geq 0\end{aligned}$$

from Lemma 6.4.

Now for $\phi_1, \phi_2 \in L^1(G)$, define their *convolution*

$$\phi_1 * \phi_2(g) = \int_G \phi_1(gg_0^{-1})\phi_2(g_0)\, dg_0.$$

Then $\rho(\phi_1 * \phi_2) = \rho(\phi_2)\rho(\phi_1)$ and from Lemma 6.3 and Definition 5.4 we get

PROPOSITION 6.9. *For* $\phi_1, \phi_2 \in L^1(G)$,

$$F(\phi_1 * \phi_2) \,|_{\sigma_\rho} = F(\phi_1) \,|_{\sigma_\rho} * F(\phi_2) \,|_{\sigma_\rho} .$$

Thus the $*$-product on A_ρ is the 'Fourier transform side' of convolution in U_ρ.

Section 7

We now examine in some detail the case of $G = SU(2)$. Notation and the basic facts about the representation theory are taken from Vilenkin [20]. We first review these facts. The representations in $\hat{G}$ are indexed by $\ell = 0, 1/2, 1, \ldots$ and denoted ρ_ℓ. Let V_ℓ be the space of polynomials in one variable of degree 2ℓ. Then if $\phi \in V_\ell$ and

$$g = \begin{vmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{vmatrix}$$

then $\rho_\ell(g)\phi(x) = (\beta x + \bar{\alpha})^{2\ell}\phi((\alpha x - \bar{\beta})/(\beta x + \bar{\alpha}))$. Let

$$X_1 = \tfrac{1}{2}\begin{vmatrix} 0 & i \\ i & 0 \end{vmatrix}, \qquad X_2 = \tfrac{1}{2}\begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}, \qquad X_3 = \tfrac{1}{2}\begin{vmatrix} i & 0 \\ 0 & -i \end{vmatrix}.$$

Then $\{X_1, X_2, X_3\}$ is a basis of $\mathbf{g}$ and $[X_1, X_2] = X_3, [X_2, X_3] = X_1$ and $[X_3, X_1] = X_2$. A basis for V_ℓ consists of the monomials

$$\Psi_n(x) = \frac{x^{\ell-n}}{\sqrt{(\ell-n)!(\ell+n)!}} \qquad \text{for} \quad -\ell \leq n \leq \ell.$$

This basis is orthonormal and the action of the Lie algebra is given by the formulae

$$\rho(X_1)x^{\ell-n} = \frac{i}{2}(\ell-n)x^{\ell-n-1} + \frac{i}{2}(\ell+n)x^{\ell-n+1}$$
$$\rho(X_2)x^{\ell-n} = \frac{1}{2}(\ell-n)x^{\ell-n-1} - \frac{1}{2}(\ell+n)x^{\ell-n+1}$$
$$\rho(X_3)x^{\ell-n} = -inx^{\ell-n}.$$

We denote the unit sphere in V_ℓ by Ω_ℓ and $\Phi_\ell : \Omega_\ell \to \mathbf{g}^*$ the moment map for ρ_ℓ. From the above equations we have

LEMMA 7.1.

$$\Phi_\ell(\Psi_n) = -nX_3^* \quad \textit{for} \quad -\ell \le n \le \ell.$$

Let $f_\ell = \ell X_3^* \in \mathbf{g}^*$, and $\mathcal{O}_\ell$ the orbit through f_ℓ. Then $\mathcal{O}_\ell$ is the extremal orbit for the representation ρ_ℓ and is simply a 2-sphere. Then for $g \in G$,

$$e(g, f_\ell) \quad = \quad \langle g \cdot \Psi_{-\ell}, \Psi_{-\ell} \rangle.$$

LEMMA 7.2. *For*

$$g = \begin{vmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{vmatrix} \in G, \quad e(g, f_\ell) = \alpha^{2\ell}.$$

PROOF. From Vilenkin [20], we have

$$\begin{aligned} e(g, f_\ell) = t^\ell_{-\ell-\ell}(g) &= \frac{1}{(2\ell)!}\,\frac{1}{\alpha^{-2\ell}}\,\frac{d^{2\ell}}{dz^{2\ell}}(z^{2\ell}) \\ &= \alpha^{2\ell}. \end{aligned}$$

Introduce the positive definite form on $\mathbf{g}$

$$(X, Y) = -2\mathrm{tr}\,(XY)$$

for $X, Y \in \mathbf{g} = SU(2)$. The basis $\{X_1, X_2, X_3\}$ is thus orthonormal. The orbit $\mathcal{O}_\ell$ is the sphere of radius ℓ when $\mathbf{g}$ is identified with $\mathbf{g}^*$ by the above form.

DEFINITION 7.3. For

$$g = \begin{vmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{vmatrix} \in G,$$

define $C(g) \in \mathbf{g}$ by

$$C(g) = \begin{vmatrix} \dfrac{\bar{\alpha} - \alpha}{2} & -\beta \\ \bar{\beta} & \dfrac{\alpha - \bar{\alpha}}{2} \end{vmatrix}.$$

LEMMA 7.4. *For*

$$g = \begin{vmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{vmatrix} \in G,$$

let $d = \det(I + g) \neq 0$. *Here* I *is the identity matrix. Then*

$$C(g) = \frac{d}{2}\frac{I - g}{I + g}.$$

We note that $d = \det(I + g) = 2 + \operatorname{tr} g$. The map $g \to (I - g)/(I + g)$ is the well-known Cayley transform. Because of the possible singularity when $\operatorname{tr} g = -2$, it is defined only on a subset of G. Note however that the map $C : G \to \mathfrak{g}$ has no such problem and is defined on all of G.

We also remark that the Cayley transform has been shown to play an important role in the study of the Fourier transform for certain nilpotent Lie groups (see Howe, Ratcliff and Wildberger [7] and Wildberger [22]).

THEOREM 7.5. *For* $g \in G$ *and* $X \in \mathcal{O}_\ell \subset \mathfrak{g}$,

$$e(g, X) = \left(\frac{\operatorname{tr} g}{2} + \frac{i}{2\ell}(C(g), X)\right)^{2\ell}.$$

PROOF. Since $X \in \mathcal{O}_\ell$, $X = g' \cdot f_\ell$ for some $g' \in G$. If

$$g' = \begin{vmatrix} \alpha' & \beta' \\ -\overline{\beta'} & \overline{\alpha'} \end{vmatrix} \quad \text{then} \quad g' f_\ell g'^{-1} = \frac{i\ell}{2}\begin{vmatrix} 2|\alpha'|^2 - 1 & -2\alpha'\beta' \\ -2\overline{\alpha'\beta'} & 1 - 2|\alpha'|^2 \end{vmatrix}.$$

Now since $e(g, g' f_\ell g'^{-1}) = e(g'^{-1} g g', f_\ell)$ we apply Lemma 7.2 to obtain $e(g, g' f_\ell g'^{-1}) = (\alpha'')^{2\ell}$ where

$$g'^{-1} g g' = \begin{vmatrix} \alpha'' & \beta'' \\ -\overline{\beta''} & \overline{\alpha''} \end{vmatrix}.$$

A short calculation shows that

$$\alpha'' = \alpha|\alpha'|^2 + \bar{\alpha}|\beta'|^2 + \bar{\beta}(\alpha'\beta') - \beta(\overline{\alpha'\beta'}).$$

But $|\beta'|^2 = 1 - |\alpha'|^2$ so

$$\begin{aligned}
\alpha'' &= \left(\frac{\alpha - \bar{\alpha}}{2}\right)(2|\alpha'|^2 - 1) + \bar{\beta}(\alpha'\beta') - \beta(\overline{\alpha'\beta'}) + \left(\frac{\alpha + \bar{\alpha}}{2}\right) \\
&= \frac{i}{2\ell}\left(\begin{vmatrix} \frac{\bar{\alpha} - \alpha}{2} & -\beta \\ \bar{\beta} & \frac{\alpha - \bar{\alpha}}{2} \end{vmatrix}, \frac{i\ell}{2}\begin{vmatrix} 2|\alpha'|^2 - 1 & -2\alpha'\beta' \\ -2\overline{\alpha'\beta'} & 1 - 2|\alpha'|^2 \end{vmatrix}\right) + \frac{\alpha + \bar{\alpha}}{2} \\
&= \frac{\operatorname{tr} g}{2} + \frac{i}{2\ell}(C(g), X).
\end{aligned}$$

Thus

$$e(g, X) = (\alpha'')^{2\ell} = \left(\frac{\operatorname{tr} g}{2} + \frac{i}{2\ell}(C(g), X)\right)^{2\ell}$$

LEMMA 7.6. *$C : G \to \mathfrak{g}$ has the following properties*

(a) $C(e) = 0$;
(b) *C is a G-map under the actions of G on G and $\mathfrak{g}$ by conjugation and the adjoint action respectively;*
(c) $\operatorname{Im} C = \left\{X \in \mathfrak{g} \,\middle|\, |X|^2 = (X, X) \le 4\right\}$.

PROOF. (a) and (b) are easily checked. Now for

$$g = \begin{vmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{vmatrix},$$

we have

$$C(g) = \begin{vmatrix} -i\sin\theta & 0 \\ 0 & i\sin\theta \end{vmatrix}$$

so that

$$|C(g)|^2 = 4\sin^2\theta \le 4.$$

Let $\mathscr{M}_\ell \in \Omega_\ell$ be the extremal orbit of ρ_ℓ.

PROPOSITION 7.7. *Let $m_1, m_2 \in \mathscr{M}_\ell$ and $\Phi(m_i) = f_i, i = 1, 2$ with $f_i \in \mathscr{O}_\ell$. Then*

$$|\langle m_1, m_2\rangle|^2 = \left(\frac{1}{2} + \frac{1}{2\ell^2}(f_1, f_2)\right)^{2\ell}.$$

PROOF. Let

$$g = \begin{vmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{vmatrix} \in G.$$

A short computation shows that

$$(g f_\ell g^{-1}, f_\ell) = \ell^2(2|\alpha|^2 - 1).$$

Thus

$$|\alpha|^2 = \frac{1}{2} + \frac{1}{2\ell^2}(g f_\ell g^{-1}, f_\ell).$$

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Let $m_\ell \in \mathcal{M}_\ell$ such that $\Phi(m_\ell) = f_\ell$ (for example, we could take $m_\ell = \psi_{-\ell}$ by Lemma 7.1). Then $\langle g \cdot m_\ell, m_\ell \rangle = e(g, f_\ell) = \alpha^{2\ell}$. Thus

$$\begin{aligned} |\langle g \cdot m_\ell, m_\ell \rangle|^2 &= (|\alpha|^2)^{2\ell} \\ &= \left(\frac{1}{2} + \frac{1}{2\ell^2}(g f_\ell g^{-1}, f_\ell) \right)^{2\ell}. \end{aligned}$$

For any other $g' \in G$, we thus have

$$|\langle g \cdot m_\ell, g' \cdot m_\ell \rangle|^2 = \left(\frac{1}{2} + \frac{1}{2\ell^2}(g f_\ell g^{-1}, g' f_\ell g'^{-1}) \right)^{2\ell}.$$

But then $g f_\ell g^{-1} = f_1$ and $g' f_\ell g'^{-1} = f_2$ are arbitrary elements of $\mathcal{O}_\ell$ and $m_1 = g \cdot m_\ell, m_2 = g' \cdot m_\ell$ with $\Phi(m_i) = f_i$.

COROLLARY 7.8. *If $m_1, m_2 \in \mathcal{M}_\ell$ and $\Phi(m_i) = f_i \in \mathcal{O}_\ell$, $i = 1, 2$, then*

(a) $|\langle m_1, m_2 \rangle|^2 = 1$ *if and only if* $f_1 = f_2$;
(b) $|\langle m_1, m_2 \rangle|^2 = 0$ *if and only if* $f_1 = -f_2$;
(c) $|\langle m_1, m_2 \rangle|^2 = (1/2)^{2\ell}$ *if and only if* $(f_1, f_2) = 0$.

PROOF. These statements follow immediately from the previous proposition and the fact that for $f_1, f_2 \in \mathcal{O}_\ell$

$$-\ell^2 \leq (f_1, f_2) \leq \ell^2$$

since $\mathcal{O}_\ell$ is a sphere of radius ℓ.

COROLLARY 7.9. *For any $m \in \mathcal{M}_\ell$, if $m^\perp = \{v \in V \mid \langle v, m \rangle = 0\}$ then $m^\perp \cap \mathcal{M}_\ell$ is a circle S^1.*

PROOF. Corollary 7.8 shows that if $\langle m, m' \rangle = 0$ with $m' \in \mathcal{M}_\ell$, then $\Phi(m') = -\Phi(m)$. But then m' is determined uniquely up to a scalar of modulus one.

We now turn to the space of functions $A_\rho = A_\ell$ on $\mathcal{O}_\ell$ and the $*$-product on this space. For $g \in G$, let $e_g(f) = e(g, f)$. Then the functions e_g, $g \in G$ span A_ℓ and

$$e_{g_1} * e_{g_2} = e_{g_1 g_2}.$$

Any $X \in \mathfrak{g}$ may be regarded as a function on $\mathfrak{g}^*$ and so also on $\mathcal{O}_\ell$. If ϕ is any polynomial then $\phi(X)$ is the corresponding element in $S(\mathfrak{g})$, which we also view as a function on $\mathfrak{g}^*$ and so on $\mathcal{O}_\ell$. Note that if $g \in G$ and $C(g) = Y \in \mathfrak{g}$, then Theorem 7.5 shows that

$$e_g = \left(\frac{\operatorname{tr} g}{2} + \frac{i}{2\ell} Y\right)^{2\ell}$$

which is a polynomial function on $\mathcal{O}_\ell$.

PROPOSITION 7.10. *If $X \in \mathfrak{g}$ and ϕ is a polynomial, with $\phi(X) \in A_\ell$, then*

$$X * \phi(X) = X\phi(X) - \frac{\left(X^2 - (2\ell r)^2\right)}{2\ell} \frac{d}{dX}\phi(X)$$

where $r = |X|/2$.

PROOF. We will prove this for

$$X = \begin{vmatrix} ix & 0 \\ 0 & -ix \end{vmatrix} \in \mathfrak{g}.$$

The general case will follow since an arbitrary element of $\mathfrak{g}$ is conjugate (under G) to such an X. Let

$$g_t = \exp tX = \begin{vmatrix} e^{itx} & 0 \\ 0 & e^{-itx} \end{vmatrix} \in G.$$

Then

(7.1) $$e_{g_t} * e_{g_s} = e_{g_{t+s}}$$

where $t, s \in \mathbb{R}$ and

$$e_{g_t} = \left(\cos tx + \frac{i}{2\ell} C(g_t)\right)^{2\ell}$$

with

$$C(g_t) = \begin{vmatrix} -i \sin tx & 0 \\ 0 & i \sin tx \end{vmatrix} = -\frac{\sin tx}{x} X.$$

We differentiate both sides of (7.1) with respect to t and set $t = 0$

$$2\ell\left(-\frac{i}{2\ell}X\right) * \left(\cos sx - \frac{i \sin sx}{2\ell x} X\right)^{2\ell}$$
$$= 2\ell\left(\cos sx - \frac{i \sin sx}{2\ell x} X\right)^{2\ell-1}\left(-x \sin sx - \frac{i \cos sx}{2\ell} X\right).$$

Then

$$\begin{aligned}
X * \left(\cos sx - \frac{i \sin sx}{2\ell x} X\right)^{2\ell}
&= \left(\cos sx - \frac{i \sin sx}{2\ell x} X\right)^{2\ell-1} (-2\ell i x \sin sx + \cos sx X) \\
&= \left(\cos sx - \frac{i \sin sx}{2\ell x} X\right)^{2\ell-1} \\
&\quad \times \left[X \left(\cos sx - \frac{i \sin sx}{2\ell x} X\right) - (X^2 - (2\ell x)^2) \left(-\frac{i \sin sx}{2\ell x}\right)\right] \\
&= X\left(\cos sx - \frac{i \sin sx}{2\ell x}\right)^{2\ell} - \frac{(X^2 - (2\ell x)^2)}{2\ell} \frac{d}{dX}\left(\cos sx - \frac{i \sin sx}{2\ell x} X\right)^{2\ell}.
\end{aligned}$$

Thus

$$X * e_{g_s} = X e_{g_s} - \frac{\left(X^2 - (2\ell x)^2\right)}{2\ell} \frac{d}{dX} e_{g_s}.$$

Now if ϕ is any polynomial such that $\phi(X) \in A_\ell$, then $\phi(X)$ is in the span of the e_{g_s} as $s \in \mathbb{R}$, so that

$$X * \phi(X) = X\phi(X) - \frac{\left(X^2 - (2\ell x)^2\right)}{2\ell} \frac{d}{dX} \phi(X).$$

Note that $(X, X) = 4x^2$ so $x = |X|/2 = r$.

The above proposition shows that the $*$-product we have constructed agrees in the case of $G = SU(2)$ with that obtained by Moreno and Ortega-Navarro [14] on S^2.

COROLLARY 7.11. $X_1 * X_1 + X_2 * X_2 + X_3 * X_3 = \ell(\ell + 1)$.

PROOF. From the previous proposition, we have

$$X_i * X_i = X_i^2 - \frac{(X_i^2 - \ell^2)}{2\ell} = X_i^2 \left(\frac{2\ell - 1}{2\ell}\right) + \frac{\ell}{2}.$$

Thus

$$\begin{aligned}
\sum_{i=1}^{3} X_i * X_i &= \left(\frac{2\ell - 1}{2\ell}\right) \sum_{i=1}^{3} X_i^2 + \frac{3\ell}{2} \\
&= \left(\frac{2\ell - 1}{2\ell}\right) \ell^2 + \frac{3\ell}{2} \\
&= \ell(\ell + 1).
\end{aligned}$$

The inner product $\langle\ ,\ \rangle_1$ on A_ℓ is determined by the kernel $N'(f, f')$ on $\mathcal{O}_\ell$ (see Section 5) and from Proposition 7.7 we see immediately that

$$N'(f, f') = \left(\frac{1}{2} + \frac{1}{2\ell^2}(f, f')\right)^{2\ell}$$

for $f, f' \in \mathcal{O}_\ell$. The $*$-product is determined by the triple kernel $B_1'(f_1, f_2, f_3)$ with respect to $\langle\ ,\ \rangle_1$ which we now determine.

THEOREM 7.12. *For $f_1, f_2, f_3 \in \mathcal{O}_\ell$, let $\Delta = \Delta(f_1, f_2, f_3)$ be a geodesic triangle through f_1, f_2, f_3. Then*

$$B_1'(f_1, f_2, f_3) = \left|\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\cos\frac{\theta_3}{2}\right|^{2\ell} e^{-|\Delta(f_1, f_2, f_3)|}$$

where $\theta_1, \theta_2, \theta_3$ are the angles subtended at the origin 0 by the arcs f_2f_3, f_3f_1 and f_1f_2 respectively, and $|\Delta(f_1, f_2, f_3)|$ is the (signed) area of the triangle Δ with respect to the 2-form ω.

PROOF. Let $m_i \in \mathcal{M}_\ell \subset \Omega_\ell$ such that $\Phi(m_i) = f_i$, $i = 1, 2, 3$. Then

$$\begin{aligned} B_1'(f_1, f_2, f_3) &= B_1(m_1, m_2, m_3) \\ &= \langle m_1, m_2\rangle\langle m_2, m_3\rangle\langle m_3, m_1\rangle \end{aligned}$$

from Theorem 3.11.

Thus using Proposition 3.5

$$\begin{aligned} |B_1'(f_1, f_2, f_3)|^2 &= |\langle m_1, m_2\rangle|^2|\langle m_2, m_3\rangle|^2|\langle m_3, m_1\rangle|^2 \\ &= N(m_1, m_2)N(m_2, m_3)N(m_3, m_1) \\ &= N'(f_1, f_2)N'(f_2, f_3)N'(f_3, f_1) \\ &= \left(\frac{1}{2} + \frac{\cos\theta_1}{2}\right)^{2\ell}\left(\frac{1}{2} + \frac{\cos\theta_2}{2}\right)^{2\ell}\left(\frac{1}{2} + \frac{\cos\theta_3}{2}\right)^{2\ell}. \end{aligned}$$

Thus

$$|B_1'(f_1, f_2, f_3)| = \left|\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\cos\frac{\theta_3}{2}\right|^{2\ell}$$

and we need only show that

$$\arg B_1'(f_1, f_2, f_3) = -|\Delta(f_1, f_2, f_3)|.$$

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Let the arcs $f_2 f_3$, $f_3 f_1$ and $f_1 f_2$ on $\mathcal{O}_\ell$ be denoted by γ_1, γ_2 and γ_3 where

$$\begin{aligned} \gamma_1(t) &= \exp tY_1 \cdot f_2 \qquad 0 \le t \le 1 \\ \gamma_2(t) &= \exp tY_2 \cdot f_3 \qquad 0 \le t \le 1 \\ \gamma_3(t) &= \exp tY_3 \cdot f_1 \qquad 0 \le t \le 1 \end{aligned}$$

and $Y_1, Y_2, Y_3 \in \mathfrak{g}$. The condition that the γ_i be geodesics is equivalent to the conditions

$$0 = f_2(Y_1) = f_3(Y_2) = f_1(Y_3).$$

We now choose $m_1, \dots, m_3$ as above more carefully. Begin with an arbitrary but fixed $m_1 \in \Phi^{-1}(f_1)$, and define the curve γ_3' in $\mathcal{M}_\ell$ by

$$\gamma_3'(t) = \exp tY_3 \cdot m_1, \qquad 0 \le t \le 1$$

with final endpoint $m_2 = \exp Y_3 \cdot m_1$. Then define

$$\gamma_1'(t) = \exp tY_1 \cdot m_2, \qquad 0 \le t \le 1$$

with final endpoint $m_3 = \exp Y_1 \cdot m_2$. Then define

$$\gamma_2'(t) = \exp tY_2 \cdot m_3, \qquad 0 \le t \le 1$$

and call its endpoint $m_4 = \exp Y_2 \cdot m_3$. Then by Lemma 4.2,

$$\Phi \circ \gamma_i' = \gamma_i \qquad i = 1, \dots, 3$$

and $\Phi(m_i) = f_i$, $i = 1, \dots, 3$. Since $\Phi(m_4) = f_1$, Proposition 4.9 implies that

$$m_1 = e^{i\tau} m_4$$

for some $\tau \in \mathbb{R}$. Thus define

$$\gamma_4'(t) = e^{it} m_4 \qquad 0 \le t \le \tau.$$

Denote the closed curve $\gamma_1 + \gamma_2 + \gamma_3$ in $\mathcal{O}_\ell$ by γ and the closed curve $\gamma_1' + \gamma_2' + \gamma_4' + \gamma_3'$ by γ'. Then γ' lies over γ in the S^1 bundle $\Phi : \mathcal{M}_\ell \to \mathcal{O}_\ell$. Suppose that γ is oriented positively with respect to the 2-form ω on $\mathcal{O}_\ell$. Recall from Lemma 5.1 and the remarks following it that ω is the curvature form for the connection 1-form θ on $\mathcal{M}_\ell$. Thus

$$\begin{aligned} |\Delta(f_1, f_2, f_3)| &= \int_\Delta \omega \\ &= \int_{\gamma'} \theta. \end{aligned}$$

We claim that

$$0 = \int_{\gamma_1'} \theta = \int_{\gamma_2'} \theta = \int_{\gamma_3'} \theta.$$

For let $m = \exp sX_1 \cdot m_2, 0 \le s \le 1$, be on γ_1' say. Then a tangent vector to γ_1' at m is $\eta = X_1 \cdot m$ and so

$$\begin{aligned}\theta_m(\eta) &= \operatorname{Im} \langle \eta, m\rangle \\ &= \operatorname{Im} \langle X_1 \cdot m, m\rangle \\ &= \Phi(m)(X_1) \\ &= \Phi(m_2)(\exp -sX_1 \cdot X_1) \\ &= f_2(X_1) \\ &= 0.\end{aligned}$$

This proves the claim and shows that

$$\begin{aligned}|\Delta(f_1, f_2, f_3)| &= \int_{\gamma_4'} \theta \\ &= \int_0^\tau \operatorname{Im} \langle ie^{it} m_4, e^{it} m_4\rangle dt \\ &= i\tau.\end{aligned}$$

On the other hand

$$\begin{aligned}\arg B_1'(f_1, f_2, f_3) = \arg\langle m_1, m_2\rangle + \arg\langle m_2, m_3\rangle \\ + \arg\langle m_3, m_4\rangle + \arg\langle m_4, m_1\rangle.\end{aligned}$$

We now claim that

$$\arg\langle m_1, m_2\rangle = \arg\langle m_2, m_3\rangle = \arg\langle m_3, m_4\rangle = 0.$$

To see this, it suffices to consider the model of V_ℓ described at the beginning of this section. Suppose $m_1 = \psi_{-\ell}$. Then from Lemma 7.1,

$$\Phi_\ell(\psi_{-\ell}) = \ell X_3^* = f_1.$$

Then since $f_1(Y_3) = 0$, we may take Y_3 to be

$$Y_3 = s \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}, \qquad s \in \mathbb{R}.$$

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by conjugating by an element of G if necessary. But then $\exp tY_3$ has the form

$$\begin{vmatrix} \cos st & \sin st \\ -\sin st & \cos st \end{vmatrix}.$$

and thus we find that

$$\langle \exp tY_3 \cdot m_1, m_1 \rangle \in R$$

for all $t \in \mathbb{R}$ so that $\langle m_1, m_2 \rangle \in R$. This proves the second claim since the other two cases are conjugate to this one. We thus have

$$\begin{aligned} \arg B_1'(f_1, f_2, f_3) &= \arg\langle m_4, m_1 \rangle \\ &= -i\tau \\ &= -|\Delta(f_1, f_2, f_3)| \end{aligned}$$

as required.

If γ is not oriented positively then we apply the argument to the triangle $\Delta(f_1, f_3, f_2)$ to show that

$$\begin{aligned} \arg B_1'(f_1, f_3, f_2) &= -|\Delta(f_1, f_3, f_2)| \\ &= |\Delta(f_1, f_2, f_3)|. \end{aligned}$$

But $B_1'(f_1, f_3, f_2) = \overline{B_1'(f_1, f_2, f_3)}$ so we are done.

Note that the geodesic triangle Δ was not uniquely specified in the above theorem. This is due to the integrality of the 2-form ω on $\mathcal{O}_\ell$, that is

$$\int_{\mathcal{O}_\ell} \omega$$

is a multiple of 2π. In fact we may view this as a consequence of Theorem 7.12 if we wish. Of course it is the existence of the S^1 bundle with connection over $\mathcal{O}_\ell$ that is the real reason for the integrality of $\mathcal{O}_\ell$ (see for example Kostant [11]).

It turns out that the function $B_1'(f_1, f_2, f_3)$ has a simple radial dependence. Since $f_i \in \mathcal{O}_\ell$, let

$$x_i = \frac{f_i}{\ell} \in \mathcal{O}_1, \qquad i = 1, 2, 3.$$

If $\Delta(f_1, f_2, f_3)$ is a geodesic triangle through f_1, f_2 and f_3, let $\Delta'(x_1, x_2, x_3)$ be the corresponding triangle on $\mathcal{O}_1$. Then it is an immediate consequence of (5.1) that

$$|\Delta(f_1, f_2, f_3)| = \ell |\Delta(x_1, x_2, x_3)|.$$

We may thus rewrite the formula of Theorem 7.12 as

$$B_1'(\ell x_1, \ell x_2, \ell x_3) = \left(\left|\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\cos\frac{\theta_3}{2}\right| e^{-i|\Delta(x_1,x_2,x_3)|/2}\right)^{2\ell}.$$

Thus it suffices to study B_1' on the unit sphere $\mathcal{O}_1$.

THEOREM 7.13. *For* $x_1, x_2, x_3 \in \mathcal{O}_1$,

$$4B_1'(x_1, x_2, x_3) = 1 + (x_1, x_2) + x_2, x_3) + (x_3, x_1) - i(x_1, x_2 \times x_3)$$

where $x_2 \times x_3$ *is the cross product of* x_2 *and* x_3, *and is identified with* $[x_2, x_3]$ *under* $\mathfrak{g}^* \simeq \mathfrak{g}$.

PROOF. This is a result of certain formulae in spherical trigonometry. If Δ is the spherical triangle whose sides are all no greater than π with vertices x_1, x_2 and x_3 and opposite sides a, b and c respectively then we find the following formula from Todhunter [17, Section 138]

$$\tan\frac{1}{2}E = \frac{2n}{1 + \cos a + \cos b + \cos c} \tag{7.2}$$

where E is the spherical excess of Δ, and is equal to the absolute value of the area in our case, and $2n$ is the sine of the trihedral angle subtended by the triangle at the origin. From remarks made in [17, Section 51] of the same reference we may conclude that in our notation

$$2n = (x_1, x_2 \times x_3)$$

that is, it is the triple-product of x_1, x_2 and x_3. We also find there that

$$4n^2 = 1 - \cos^2 a - \cos^2 b - \cos^2 c + 2\cos a \cos b \cos c.$$

From this we may easily calculate that

$$(2n)^2 + (1 + \cos a + \cos b + \cos c)^2 = \left(4\cos\frac{a}{2}\cos\frac{b}{2}\cos\frac{c}{2}\right)^2.$$

It follows that

$$1 + \cos a + \cos b + \cos c - i2n = 4\cos\frac{a}{2}\cos\frac{b}{2}\cos\frac{c}{2}e^{-iE/2}.$$

Our result follows.

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We cannot resist extracting a few more relevant formulae from Todhunter. From Section 154 we get Keogh's theorem, stating that

$$\sin \frac{1}{2} E = 2n'$$

where $2n'$ is the sine of the trihedral angle subtended by the triangle whose vertices are in the midpoints of Δ. Combining this with (7.2), we have

$$4 \cos \frac{1}{2} a \cos \frac{1}{2} b \cos \frac{1}{2} c = \frac{2n}{2n'}.$$

This is a purely geometric interpretation of $|B_1'|$. Another one is

$$4 \cos \frac{1}{2} a \cos \frac{1}{2} b \cos \frac{1}{2} c = \cos s + \cos(s - a) + \cos(s - b) + \cos(s - c)$$

where $2s = a + b + c$, found in [17, Section 119].

It seems an interesting program to extend these results to more general compact groups.

Acknowledgements

This work was supported by the National Science and Engineering Research Council of Canada.

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