Can. J. Math., Vol. XXV, No. 4, 1973, pp. 870–873

ON REAL ZEROS OF DEDEKIND ζ-FUNCTIONS

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1. Introduction. Let K be a finite normal extension of an algebraic number field k; let k_2 be the compositum of all quadratic extensions of k which are contained in K. Let $\zeta_k(s)$, $\zeta_K(s)$ and $\zeta_{k_2}(s)$ denote the Dedekind ζ-functions of these fields. The main purpose of this paper is to prove

THEOREM 1. *Any real simple zero of $\zeta_K(s)$ is a zero of $\zeta_{k_2}(s)$.*

In particular, if k is the rational field, any real simple zero of $\zeta_K(s)$ is a zero of an L-series

$$L_\Delta(s) = \sum_{n=1}^{\infty} (\Delta/n)n^{-s}$$

where Δ is a rational integral divisor of disc (K/Q).

The motivation arises from the following well-known facts. Let C be a number field, d its absolute discriminant, κ the residue of its ζ-function $\zeta_C(s)$ at $s = 1$. Then either $\kappa^{-1} = O(\log|d|)$ or $\zeta_C(s_0) = 0$ for some $s_0 < 1$ with $\log|d| = O((1 - s_0)^{-1},)$ in which case the lower bound for κ may be very poor indeed. Moreover, the zero s_0 is simple and unique.

Now let K be the normal closure of C over $\mathbf{Q}$, of absolute discriminant D such that

$$|D| \leqq |d|^{n!},\ n = \text{degr } C.$$

Then s_0 will also be a zero of $\zeta_K(s)$. The application of Theorem 1 to K yields

THEOREM 2. *Let C any number field of degree n and discriminant d. Then either*

$$\kappa^{-1} = O(n! \log |d|),$$

or there exists a divisor Δ of d such that

$$L_\Delta(s_0) = 0, \quad 1 - s_0 = O(\kappa).$$

Thus the task to find an effective realistic lower bound for κ is, at least in principle, reduced to the same problem for quadratic number fields. In the case where C is a totally complex quadratic extension of a totally real field, J. Sunley [**4**] and L. Goldstein [**3**] have already obtained results of this nature. I wish to record my gratitude to Prof. L. Goldstein who made me familiar with these researches, and thus provided the stimulus which led me to the present investigation.

Received April 20, 1972 and in revised form, July 10, 1972.

2. Proof of Theorem 1. The proof is based on the use of Artin L-series. We shall make use of two fundamental results of R. Brauer [**1**; **2**].

B.1. The Artin L-series are meromorphic functions of s.

B.2. If K is a normal extension of k, then

$$(\zeta_k(s))^{-1}\,\zeta_K(s)$$

is an integral function of s.

Let k_a be the maximal abelian extension of k contained in K, so that

$$k \subset k_2 \subset k_a \subset K.$$

Let $G = \mathrm{Gal}(K/k)$, so that $G' = \mathrm{Gal}\ (K/k_a)$ is the commutator group of G. Then

$$\zeta_{k_a}(s) = \zeta_{k_2}(s) \prod_{\gamma} L(s; k, \gamma)$$

where γ runs through the complex characters of G/G'.

Because the γ are abelian characters, the $L(s; k, \gamma)$ are integral functions. Because $L(s; k, \gamma) = 0 \Rightarrow L(s; k, \bar{\gamma}) = 0$ for real s, any real zero s_0 of $\zeta_{k_a}(s)$ is either a zero of $\zeta_{k_2}(s)$ or a zero of multiplicity $\geqq 2$ of $\zeta_{k_a}(s)$. By B.2 the last case is impossible, hence we assume from now on that $\zeta_{k_a}(s) \neq 0$.

Let χ_b run through all irreducible characters of G. Then

$$\zeta_K(s) = \prod_{b} L(s; k, \chi)^{\chi_b(1)},$$

where $\chi_b(1)$ denotes the dimension of the character which equals its value for the unit element of G.

It follows from B.1 that $L(s; k, \chi_b)$ has a zero of order m_b at $s = s_0$, where $m_b \in \mathbf{Z}$, and nothing may be assumed about the sign of m_b. We now define the general character

$$\phi = \sum_{b} m_b \chi_b.$$

Let k_j be any field in the range $k_a \subset k_j \subset k$, and ψ_j the character of G induced by the principal character of the subgroup $G_j = \mathrm{Gal}\ (K/k_j)$. Then it is well-known that

$$\zeta_{k_j}(s) = \prod_{b} L(s; k, \chi_b)^{r_{j,b}},$$

where the non-negative rational integers $r_{j,b}$ are determined by the decomposition

$$\psi_j = \sum_{b} r_{j,b}\chi_b.$$

By virtue of the Frobenius reciprocity the $r_{j,b}$ are explicitely given by the formula

$$r_{j,b} = |G_j|^{-1} \sum_{\gamma \in G_j} \chi_b(\gamma).$$

https://doi.org/10.4153/CJM-1973-090-3 Published online by Cambridge University Press

Thus, the order of the zero of $\zeta_{k_j}(s)$ at $s = s_0$ is given by

$$S(G_j) = \sum_b r_{j,b} m_b = |G_j|^{-1} \sum_b m_b \sum_{\gamma \in G_j} \chi_b(\gamma)$$
$$= |G_j|^{-1} \sum_{\gamma \in G_j} \phi(\gamma),$$

and we know from B.2 that $S(G_j) = 0$ or 1 for all j, $S(G') = 0$, and $S(\{1\}) = 1$, where $\{1\}$ denotes the trivial subgroup of G consisting of the unit only.

Now let H^* be a minimal subgroup of G', such that $S(H^*) = 0$ and $S(H) = 1$ for each genuine subgroup H of H^*. Then we have for every genuine subgroup H of H^*

$$\sum_{\gamma \in H} (-1 + \phi(\gamma)) = 0,$$

whereas $\sum_{\gamma \in H^*} \phi(\gamma) = 0$.

It is easy to verify that these relations are compatible only if H^* is cyclic. If H^* were not cyclic, we should have for each $\gamma \in H^*$ of order N

$$\sum_{n=1}^{N} (-1 + \phi(\gamma^n)) = 0;$$

and by virtue of the Möbius inversion formula

$$\sum_{n=1,(n,N)=1}^{N} (-1 + \phi(\gamma^n)) = 0.$$

We can find group elements $\gamma_1, \ldots, \gamma_q$ of order $N_1, \ldots, N_q$ respectively such that the elements $\gamma_i^{n_i}$, $1 \leqq i \leqq q$, $1 \leqq n_i \leqq N_i$, $(n_i, N_i) = 1$ represent all group elements uniquely. Thus

$$\sum_{\gamma \in H^*} (-1 + \phi(\gamma)) = 0,$$

which is a contradiction. Thus we have shown that H^* is cyclic.

Moreover, the order of H^* cannot be divisible by an odd prime p. Otherwise the field K^*, corresponding to the subgroup H^*, would have a cyclic extension of degree p, say K_p^*, and K_p^* would be a subfield of K. The function $\zeta_{K_p^*}(s)$ would have a simple 0 at s_0. But

$$\zeta_{K_p^*}(s) = \zeta_{K^*}(s) \prod_{i=1}^{p-1} L(s; K^*, \eta_i).$$

In this product η_i runs through non-principal abelian characters in K_p^* which occur in pairs of conjugate complex characters. Hence, if the product vanishes at s_0, it must have a zero of multiplicity $\geqq 2$; this contradicts our assumption. Hence the order of H^* is a power of 2, say 2^t.

Let τ be a generator of H^*, and let H^{**} denote the subgroup of H^* which is generated by τ^2. We have

$$1 = S(H^{**}) - S(H^*) = 2^{-t} \sum_{n=1}^{2^t} (-1)^n \phi(\tau^n).$$

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The general character ϕ can be decomposed into two genuine characters ϕ_+, ϕ_- by the formula $\phi = \phi_+ - \phi_-$. This decomposition is not unique, but as ϕ is real, ϕ_+ and ϕ_- can be chosen as real characters of G. We now remember that ϕ_+ and ϕ_- are the sum of the characteristic roots of the corresponding matrix representation of G. Since the characters ϕ_+ and ϕ_- are real, conjugate roots occur with equal multiplicity. The characteristic roots forming $\phi_+(\tau)$ and $\phi_-(\tau)$ are 2^tth roots of unity. Because $\tau \in G'$, the determinants of the corresponding matrices are $+1$, and the products of the characteristic roots are $+1$. As the complex roots cancel in the product, the root -1 occurs in ϕ_+ exactly a_+ times, and in ϕ_- exactly a_- times, where a_+ and a_- are even.
As

$$\begin{aligned} 2^t &= \sum_{n=1}^{2^t} (-1)^n \phi(\tau^n) = \sum_{n=1}^{2^t} (-1)^n \phi_+(\tau^n) - \sum_{n=1}^{2^t} (-1)^n \phi_-(\tau^n) \\ &= 2^t(a_+ - a_-), \\ 1 &= a_+ - a_-, \end{aligned}$$

We obtain the desired contradiction.

Postscript. The referee has kindly pointed out to me that the result B.2 quoted above was proved originally by H. Aramata, Proc. Japan Acad. *9*(1933), 31-34.

References

1. R. Brauer, *On Artin L-series with general group characters*, Ann. of Math. *48* (1947), 502–514.
2. ——— *On the zeta-functions of algebraic number fields.* Amer. J. Math. *69* (1947), 243–250.
3. L. Goldstein, *Relatively imaginary quadratic fields of class number* 1 *or* 2, Trans. Amer. Math. Soc. *165* (1972), 353–364.
4. J. S. Sunley, *Class numbers of totally imaginary quadratic extensions of totally real fields*, Ph.D. Thesis, University of Maryland, 1971.

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https://doi.org/10.4153/CJM-1973-090-3 Published online by Cambridge University Press