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A LIMIT THEOREM FOR BROWNIAN MOTION IN A RANDOM SCENERY

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ABSTRACT. We find the limiting distribution of $1/a_n \int_0^{nt} V(B_u)\,du, t \in [0,1]$, where $\{B_u\}_{u\geq 0}$ is the standard Brownian motion on $\mathbb{R}^d$, V is a particular random potential and $\{a_n\}_{n\geq 1}$ is a normalizing sequence.

1. **Introduction.** In Kesten and Spitzer (1979), the authors studied among other things the limiting behavior of

$$W_n(t)/a_n = 1/a_n \sum_{0\leq k\leq nt} \xi_{S_k}, \quad t\in[0,1] \tag{1.1}$$

where $\{S_k\}_{k\geq 1}$ is the symmetric nearest neighbor random walk on $\mathbb{Z}^d$, $\{\xi_\alpha\}_{\alpha\in\mathbb{Z}^d}$ are *i.i.d* random variables independent of the random walk and normalized in such a way that $E(\xi_\alpha)=0$ and $E(\xi_\alpha^2)=1$.

They proved that when $d=1$ and $a_n=n^{3/4}$ or when $d\geq 3$ and $a_n=n^{1/2}$, $W_n(\cdot)/a_n$ converges weakly to a self- similar process; moreover the process is Gaussian if $d\geq 3$. The only open problem left was to study the case $d=2$; it was conjectured that $a_n=(n\log n)^{1/2}$ was the appropriate normalization and the limiting distribution was Gaussian.

In this article, instead of considering W_n as defined by (1.1), we study the following process

$$X_n(t)=\int_0^{nt} V(B_u)\,du, \quad t\in[0,1],$$

where $\{B_u\}_{u\geq 0}$ is a Brownian motion independent of the random scenery $\{\xi_\alpha\}_{\alpha\in\mathbb{Z}^d}$ and $V(x)=\xi_{[x+U]}, x\in\mathbb{R}^d$, where $[x]=([x_1],\ldots,[x_d])$, $[\cdot]$ being the integer part and U is uniformly distributed over $T_d=[0,1)^d$; it is also assumed that U, $\{B_u\}_{u\geq 0}$ and $\{\xi_\alpha\}_{\alpha\in\mathbb{Z}^d}$ are all independent.

REMARK. Let $(T_k\xi)_\alpha=\xi_{k+\alpha}$, $k,\alpha\in\mathbb{Z}^d$ and define

$$\tau_x(\xi,u)=(T_{[x+u]}\xi, x+u-[x+u]), \quad x\in\mathbb{R}^d, u\in T_d.$$

If $\tilde V$ is defined by $\tilde V\big((\xi,u)\big)=\xi_0$, then $V(x)$ takes the form $V(x,\xi,U)=\tilde V\big(\tau_x(\xi,U)\big)$, and it is easy to prove that $\tau_x\circ\tau_y=\tau_{x+y}$ and the joint law μ of (ξ,U) is stationary and

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ergodic with respect to $\{\tau_x\}_{x\in\mathbb{R}^d}$. Moreover the process $\{\tau_{B_t}(\xi, U)\}_{t\geq 0}$ is a reversible Markov process and μ is an invariant ergodic measure for the process.

The main result of this paper is that $X_n(\cdot)/\sqrt{n\log n}$ converges weakly to a suitably scaled Brownian motion in the open case $d = 2$. The proof that the conjecture of Kesten and Spitzer is true follows in an analogous way.

The study of the asymptotic behavior of $\int_0^t V(B_u)\,du$ is motivated by the fact that this process plays an important role in the study of large deviations for diffusion processes with random coefficients (cf. Dawson and Remillard (1989)).

In Section 2, we prove the following Theorem.

THEOREM. *Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, $\{\xi_k\}_{k\in Z^d}$ be i.i.d random variables with mean 0 and variance 1, U a random variable which is uniformly distributed over $[0,1)^d = T_d$ and independent of $\{\xi_k; k \in \mathbb{Z}^d\}$. Further let P be the Wiener measure over $\mathcal{X} = C([0,\infty); \mathbb{R}^d)$ starting from 0 at time 0. We write B_t to designate the canonical Wiener process; in the following $\mathcal{W}_{\sigma^2}$ stands for the law of σB.*

Set $V(x) = \xi_{[x+U]}, x \in \mathbb{R}^d, X_n(t) = \int_0^{nt} V(B_u)\,du, n \in \mathbb{N}, t \in [0,1]$. Then under $P\otimes\mu$

a) Case $d = 1$: $X_n/n^{3/4} \Rightarrow Z \in C([0,1];\mathbb{R})$ where Z has the following representation: $Z(t) = \int_0^\infty \ell_t(x)\,dZ_1(x) + \int_0^\infty \ell_t(-x)\,dZ_2(x)$, where B, Z_1, Z_2 are 3 independent 1-dimensional Wiener processes and $\ell_t(\cdot)$ is the local time of B i.e.

$$\int_0^t 1_A(B_u)\,du = \int_A \ell_t(x)\,dx, \quad A \in \mathcal{B}(\mathbb{R}), t \geq 0.$$

b) Case $d = 2$: $P\otimes\mu \circ (X_n/\sqrt{n\log n})^{-1} \Rightarrow \mathcal{W}_{1/\pi}$ on $C([0,1];\mathbb{R})$

c) Case $d \geq 3$: $P\otimes\mu \circ (X_n/\sqrt{n})^{-1} \Rightarrow \mathcal{W}_{\sigma_d^2}$ on $C([0,1];\mathbb{R})$, where $\sigma_d^2 = \int_{\mathbb{R}^d} \frac{4}{|x|^2} f_d(x)\,dx$ and $f_d(x) = \prod_{i=1}^d \left(\frac{1-\cos x_i}{\pi x_i^2}\right), x \in \mathbb{R}^d$.

2. Proof of the theorem.

PROOF. Let $X(t) = \int_0^t V(B_u)\,du, t \geq 0$, and let $\sigma^2(t) = E\big(X^2(t)\big)$, where E stands for the expectation with respect to $P\otimes\mu$. Then

$$E\big(X(t)\big) = 0 \quad \forall t \text{ and } \sigma^2(t) = 2\int_0^t\int_0^s E\big(V(B_u)V(B_s)\big)\,du\,ds. \tag{2.1}$$

Now for $x, y \in \mathbb{Z}^d$

$$\begin{aligned} &E\big(V(x)V(y)\big) = E\big(V(0)V(y-x)\big) = E\big(\xi_0\xi_{[y-x+U]}\big) = \\ &\int_{\mathbb{R}^d} 1_{[u]=0}1_{[y-x+u]=0}\,du = h_d(y-x) = \int_{\mathbb{R}^d} e^{i<\lambda,y-x>} f_d(\lambda)\,d\lambda, \end{aligned} \tag{2.2}$$

where f_d is defined in (c) above and

$$h_d(x) = \prod_{i=1}^d (1-|x_i|)1_{\{|x_i|\leq 1\}}, \quad d \geq 1.$$

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Therefore

$$\sigma^2(t) = 2\int_0^t \int_0^s \int_{\mathbb{R}^d} E\big(e^{i<\lambda, B_s - B_u>}\big) f_d(\lambda)\,d\lambda\,du\,ds$$
$$= 2\int_0^t \int_0^s \int_{\mathbb{R}^d} e^{-\frac{|\lambda|^2}{2}u} f_d(\lambda)\,d\lambda\,du\,ds.$$

Next $\lim_{t\to\infty} \sigma^2(t)/t = +\infty$ if $d = 1, 2$ and $\lim_{t\to\infty} \frac{\sigma^2(t)}{t} = \sigma_d^2, \sigma_d^2 = \int_{\mathbb{R}^d} \frac{4}{|\lambda|^2} f_d(\lambda)\,d\lambda \in (0,\infty)$ when $d \geq 3$. Adapting a result of Kipnis and Varadhan (1986), we obtain c).

Next a) follows from an adaptation of the results of H. Kesten and F. Spitzer [1979], so we only indicate how to prove it.

PROOF OF a). By stationarity

$$E\Big(\big(X_n(t) - X_n(s)\big)^2 \Big/ n^{3/2}\Big) =$$
$$E\big(X_n^2(|t-s|)/n^{3/2}\big) = \sum_{k\in Z} E\bigg(\frac{1}{n^{3/2}}\Big(\int_{k-u}^{k+1-u} \ell_{n|t-s|}(x)\,dx\Big)^2\bigg)$$
$$\leq \frac{1}{n^{3/2}} \int_{\mathbb{R}} E\big(\ell^2_{n|t-s|}(x)\big)dx,$$

where $\ell_t(\cdot)$ is the local time of B.

From the scaling property of $B, E\big(n^{-3/2}\ell^2_{nt}(x)\big) = \frac{1}{\sqrt{n}} E\big(\ell_t^2(x/\sqrt{n})\big), t \geq 0, x \in \mathbb{R}$. Hence

$$E\Big(\big(X_n(t) - X_n(s)\big)^2 \Big/ n^{3/2}\Big) \leq E\Big(\int_{\mathbb{R}} \ell^2_{|t-s|}(x)\,dx\Big) = |t-s|^{3/2} E\Big(\int_{\mathbb{R}} \ell_1^2(x)\,dx\Big)$$

It follows from Billingsley (1968) that $\{P \otimes \mu \circ (X_n/n^{3/4})^{-1}\}_{n\geq 1}$ is tight.

Next

$$P \otimes \mu\Big(\sup_k \int_k^{k+1} \ell_{nt}(x+u)\,du > \epsilon n^{3/4}\Big) \longrightarrow 0 \quad \forall t \in [0,1].$$

It follows that for every $0 = t_0 \leq t_1 \leq \cdots \leq t_m \leq 1$ and $\alpha_1 \cdots, \alpha_m \in \mathbb{R}$,

$$\lim_{n\to\infty} E\Big(\exp\Big\{i\sum_{j=1}^m \alpha_j\big(X_n(t_j) - X_n(t_{j-1})\big)/n^{3/4}\Big\}\Big)$$
$$= E\bigg(\exp\Big\{-\frac{1}{2}\int_{\mathbb{R}} \Big(\sum_1^m \alpha_j\big(\ell_{t_j}(x) - \ell_{t_{j-1}}(x)\big)\Big)^2 dx\Big\}\bigg)$$

which completes the proof.

PROOF OF b). Set $a(t) = (t\log t)^{1/2}, t \geq 2$, and let $p_t(k) = \int_0^t 1_{[B_s+U]=k}\,ds$. Further let $\theta(t) = \sum_{k\in Z^2} p_t^2(k), t \geq 0$. From now on we will write P and $\sum_k$ instead of $P \otimes \mu$ and $\sum_{k\in Z^2}$.

Assume

$$A_1\colon \lim_{t\to\infty} \sigma^2(t)/a^2(t) = \frac{1}{\pi}$$

and

$$A_2\colon \lim_{t\to\infty} E\big(\theta^2(t)/a^4(t)\big) = \frac{1}{\pi^2}, \text{ and therefore } E\big(\theta^2(t)\big) \leq t^2\alpha(t)$$

where α is increasing and $\alpha(t) \sim \frac{1}{\pi^2}(\log t)^2$, which means $\lim_{t\to\infty} \frac{\alpha(t)}{(\log t)^2} = \frac{1}{\pi^2}$. Then

$$\left\{P \circ \left(X_n/a(n)\right)^{-1}\right\}_{n\geq 2} \text{ is tight (in } C[0,1]; \mathbb{R}) \tag{2.3}$$

and

$$\lim_{n\to\infty} E\Big(e^{i\sum_{j=1}^m \alpha_j\left(X_n(t_j)-X_n(t_{j-1})\right)/a(n)}\Big) = e^{-\frac{1}{2\pi}\sum_{j=1}^m \alpha_j^2(t_j-t_{j-1})} \tag{2.4}$$

for any $0 = t_0 \leq t_1 \leq \cdots \leq t_m \leq 1$ and $\alpha_j \in \mathbb{R}$.

Clearly (2.3) and (2.4) are equivalent to b). So suppose that A_1 and A_2 hold. To prove (2.3) it suffices to prove (cf. Billingsley (1968, Theorem 8.2))

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty} P\Big(\sup_{\substack{0\leq s\leq t\leq 1\\ 0\leq t-s\leq\delta}} |X_n(t)-X_n(s)| > 3\sqrt{2}\epsilon a(n)\Big) = 0 \quad \forall \epsilon > 0. \tag{2.5}$$

Put

$$A(n,\epsilon,\delta) = \Big\{\sup_{\substack{0\leq s\leq t\leq 1\\ 0\leq t-s\leq\delta}} \sum_k \xi_k^2\big(p_{nt}(k)-p_{ns}(k)\big)^2 \leq \epsilon^2 a^2(n)\Big\}$$

and set $m = [1/\delta]$. Let $u > 0$ be given and set $\xi_{k,u} = \xi_k 1_{\{|\xi_k|\leq u\}}$, $\zeta_{k,u} = \xi_k - \xi_{k,u}$. Since $p_t(k)$ is increasing in t for fixed k,

$$\begin{aligned} &P\big(A(n,\epsilon,\delta)^c\big)\\ &\leq P\Big(\sup_{0\leq s\leq 1}\sum_k \xi_k^2\big(p_{n(s+\delta)}(k)-p_{ns}(k)\big)^2 > \epsilon^2 a^2(n)\Big)\\ &\leq \sum_{j=0}^{m-1} P\Big(\sum_k \xi_k^2\big(p_{n(j+2)\delta}(k)-p_{nj\delta}(k)\big)^2 > \epsilon^2 a^2(n)\Big). \end{aligned}$$

By definition $\xi_k^2 = \xi_{k,u}^2 + \zeta_{k,u}^2$ and $E\big(\xi_{k,u}^4\big) \leq u^4$. Therefore

$$P\big(A(n,\epsilon,\delta)^c\big) \leq \frac{4u^4}{\delta\epsilon^4 a^4(n)} E\big(\theta^2(2n\delta)\big) + \frac{2}{\delta\epsilon^2 a^2(n)}\sigma^2(2n\delta)E(\zeta_{0,u}^2), \tag{2.6}$$

where we have used Markov's inequality in both terms and the stationarity of $\xi_{[U+B_t],u}$ and $\zeta_{[U+B_t],u}$.

It follows from (2.6), A_1 and A_2 that

$$\limsup_{n\to\infty} P\big(A(n,\epsilon,\delta)^c\big) \leq 16u^4\delta/\epsilon^4\pi^2 + \frac{4}{\pi\epsilon^2}E(\zeta_{0,u}^2) \tag{2.7}$$

for every $\delta \in [0,1]$ and $u > 0$.

Let $\delta \to 0$ and then $u \to \infty$ in (2.7) to verify that

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty} P\big(A(n,\epsilon,\delta)^c\big) = 0.$$

Now suppose that $1 \geq \delta \geq t - s \geq 0$. Then since $X(t) = \sum_k \xi_k p_t(k)$,

$$\begin{aligned} P\Big(A(n,\epsilon,\delta) \cap \big\{|X_n(t) - X_n(s)| > \sqrt{2}\epsilon a(n)\big\}\Big) \\ \leq P\Big(\sum_{k\neq j} \xi_k\xi_j\big(p_{nt}(k) - p_{ns}(k)\big)\big(p_{nt}(j) - p_{ns}(j)\big) > \epsilon^2 a^2(n)\Big) \\ \leq \frac{1}{\epsilon^4 a^4(n)} E\Big(\Big(\sum_{k\neq j} \xi_k\xi_j\big(p_{nt}(k) - p_{ns}(k)\big)\big(p_{nt}(j) - p_{ns}(j)\big)\Big)^2\Big) \\ \leq \frac{2}{\epsilon^4 a^4(n)} E\big(\theta^2(n|t-s|)\big) \leq \frac{2|t-s|^2}{\epsilon^4} C_n \text{ and } C_n \longrightarrow 1/\pi^2, \text{ by } A_2. \end{aligned}$$

It follows from Billingsley (1968) that

$$\limsup_{n\to\infty} P\Big(A(n,\epsilon,\delta) \cap \big\{\sup_{j\delta \leq s \leq (j+1)\delta} |X_n(s) - X_n(j\delta)| > \sqrt{2}\epsilon a(n)\big\}\Big) \leq \frac{K_1\delta^2}{\epsilon^4}$$

for some finite constant K_1 (ind. from δ and ϵ). Combining the last inequality with

$$\lim_{\delta\downarrow 0} \limsup_{n\to\infty} P\big(A(n,\epsilon,\delta)^c\big) = 0$$

we get (2.5) thus completing the proof of (2.3).

We will now prove (2.4). Set $Y_n = \sup_k p_n(k)$ and $q_t(x) = \frac{1}{2\pi t} e^{-\frac{|x|^2}{2t}}$. Clearly

$$\begin{aligned} P(Y_n > \epsilon) &\leq \big(\epsilon a(n)\big)^{-j} \sum_k E\big(p_n^j(k)\big) \\ &\leq \big(\epsilon a(n)\big)^{-j} j! \int_{S_n} \int_{(T_2)^j} q_{t_2}(x_2 - x_1) \cdots q_{t_j - t_{j-1}}(x_j - x_{j-1})\, dx\, dt \end{aligned}$$

where

$$S_n = \{0 \leq t_1 \leq \cdots \leq t_j \leq n\}.$$

Define $h(t,x) = \int_{T_2} q_t(y-x)\,dy$, $x \in T_2, t > 0$. By elementary calculus $h(t,0) \leq h(t,x) \leq h(t,x_0)$ $\forall x \in T_2$, where $x_0 = \left(\frac{1}{2}, \frac{1}{2}\right)$. Therefore $p(Y_n > \epsilon) \leq \big(\epsilon a(n)\big)^{-j} j!\, n\Big(\int_0^n h(t,x_0)\,dt\Big)^{j-1}$. Now $h(t,x_0) \leq 1$ and for large t $h(t,x_0) \sim \frac{1}{2\pi t}$ so $\int_0^n h(t,x_0)\,dt \sim \frac{1}{2\pi}\log n$. Choosing $j = 3$, we get $\lim_{n\to\infty} P(Y_n > \epsilon) = 0$ $\forall \epsilon > 0$. Using this and $p_t(k) \leq p_n(k)$ $\forall 0 \leq t \leq n$, $k \in Z^2$, we can prove easily that

$$\lim_{n\to\infty} \Big| E\big(e^{i\tilde{X}_n}\big) - E\Big(e^{-\frac{1}{2}\sum_k Z_n^2(k)}\Big) \Big| = 0$$

where

$$\tilde{X}_n = \frac{1}{a(n)} \sum_{j=1}^m \alpha_j\big(X_n(t_j) - X_n(t_{j-1})\big), Z_n(k) = \frac{1}{a(n)} \sum_{j=1}^m \alpha_j\big(p_{nt_j}(k) - p_{nt_{j-1}}(k)\big),$$

and $0 = t_0 \leq t_1 \leq \cdots \leq t_m \leq 1$, $\alpha_j \in \mathbb{R}$ are fixed. Therefore (2.4) will follow if we can prove that

$$\sum_k Z_n^2(k) \xrightarrow{P_r} \frac{1}{\pi} \sum_{j=1}^m \alpha_j^2 (t_j - t_{j-1}).$$

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Suppose that $t \geq s \geq 0$. By stationarity $\sigma^2(t-s) = E\Big(\big(X(t) - X(s)\big)^2\Big)$, so $E\Big(\sum_k p_t(k)p_s(k)\Big) = \frac{1}{2}\big(\sigma^2(t) + \sigma^2(s) - \sigma^2(t-s)\big)$ and it follows from A_1 that $E\Big(\sum_k p_{nt}(k)p_{ns}(k)\Big) \sim \frac{s}{\pi}a^2(n)$. Thus

$$0 \leq E\Big(\frac{1}{a^2(n)}\sum_k \big(p_{nt_j}(k) - p_{nt_{j-1}}(k)\big)\big(p_{nt_\ell}(k) - p_{nt_{\ell-1}}(k)\big)\Big) \longrightarrow 0$$

whenever $j \neq \ell$, and

$$\sum_k Z_n^2(k) \longrightarrow \frac{1}{\pi}\sum_{j=1}^m \alpha_j^2(t_j - t_{j-1})$$

in probability since $E\Big(\Big(\frac{1}{a^2(n)}\sum_k\big(p_{nt}(k) - p_{ns}(k)\big)^2 - \frac{1}{\pi}(t-s)\Big)^2\Big) \longrightarrow 0$ by A_2 and stationarity, $0 \leq s \leq t \leq 1$. This completes the proofs of (2.3) and (2.4) assuming that A_1 and A_2 are verified.

PROOF OF A_1. By (2.1), (2.2) $\sigma^2(t) = 2\int_0^t\int_0^s\int_{\mathbb{R}^2} h_2(x)q_u(x)\,dx\,du\,ds$, and h_2 has compact support. Since $q_u(\cdot) \sim \frac{1}{2\pi u}$ uniformly on compacts, it then follows easily that $\sigma^2(t) \sim \frac{t}{\pi}\log t$.

PROOF OF A_2. Since $\theta(t)$ does not depend on $\{\xi_k\}$, let us suppose that ξ_k is Gaussian with mean 0 and variance 1. Then a simple calculation yields $E\big(\theta^2(t)\big) = \frac{1}{3}E\big(X^4(t)\big) = 8\big(C_1(t) + C_2(t) + C_3(t)\big)$, where $S_t = \{0 \leq s_1 \leq \cdots \leq s_4 \leq t\}$ and

$$C_1(t) = \sum_k \int_{S_t}\int_{(T_2)^4} q_{s_2-s_1}(x_2 - x_1)q_{s_3-s_2}(x_3 + k - x_2)q_{s_4-s_3}(x_4 - x_3)\,dx\,ds$$

$$C_2(t) = \sum_k \int_{S_t}\int_{(T_2)^4} q_{s_2-s_1}(x_2 + k - x_1)q_{s_3-s_2}(x_3 - k - x_2)q_{s_4-s_3}(x_4 + k - x_3)\,dx\,ds$$

$$C_3(t) = \sum_k \int_{S_t}\int_{(T_2)^4} q_{s_2-s_1}(x_2 + k - x_1)q_{s_3-s_2}(x_3 - x_2)q_{s_4-s_3}(x_4 - k - x_3)\,dx\,ds$$

Since $h(t,0) \leq h(t,x) \leq h(t,x_0)$ $t > 0, x \in T_2, x_0 = \big(\frac{1}{2},\frac{1}{2}\big)$, we get $\int_{S_t} h(s_2 - s_1, 0)h(s_4 - s_3, 0)\,ds \leq C_1(t) \leq \frac{t^2}{2}\Big(\int_0^t h(s,x_0)\,ds\Big)^2$, and we can easily check that $C_1(t) \sim \frac{t^2}{8\pi^2}(\log t)^2$ and $\int_0^t h(s,x_0)\,ds \sim \frac{1}{2\pi}\log t$.

Using Cauchy-Schwarz inequality, we get

$$C_2(t) \leq \int_{S_t}\Big(\sum_k \int_{(T_2)^2} q^2_{s_3-s_2}(x_3 - k - x_2)\,dx_2\,dx_3\Big)^{1/2}\cdot$$
$$\Big(\sum_k \int_{(T_2)^2} q^4_{s_2-s_1}(x_2 + k - x_1)\,dx_1\,dx_2\Big)^{1/2}\cdot$$
$$\Big(\sum_k \int_{(T_2)^2} q^4_{s_4-s_3}(x_4 + k - x_3)\,dx_3\,dx_4\Big)^{1/2}\,ds.$$

Now

$$\sum_k \int_{(T_2)^2} q_t^j(y + k - x)\,dx\,dy = \int_{\mathbb{R}^2} q_t^j(y)\,dy = \frac{1}{j(2\pi t)^{j-1}}, \quad j \geq 1.$$

By elementary calculations we obtain $C_2(t) \leq t^2 C_2'$ for some constant C_2' and similarly $C_3(t) \leq C_3' t^2 d_3(t)$, where $d_3(t) = \int_0^t \left(\int_{(T_2)^2} q_s^2(y-x)\,dx\,dy \right)^{1/2} ds \sim \frac{1}{2\pi} \log t$. Hence we can conclude that $E\big(\theta^2(t)\big) \sim \frac{1}{\pi^2}(t \log t)^2$ and $E\big(\theta^2(t)\big) \leq t^2 \alpha(t)$ for some increasing α such that $\alpha(t) \sim \frac{1}{\pi^2}(\log t)^2$ which proves A_2. ∎

REMARK. 1° Since (2.3) and (2.4) depend only on A_1 and A_2, and not on the properties of Wiener process, and also A_1 and A_2 depend only on the behaviour of $q_t(\cdot)$, we see that we can replace $\{\xi_k\}_{k \in Z^2}$, U and B_t by $\{\zeta_k\}_{k \in Z}$, $U' =$ uniform on $[0,1)$ and $x(t)$: symmetric Cauchy process, and we can easily prove that $X'(t) = \int_0^t \zeta_{[x(u)+U']}\,du$ has the same limiting distribution as $X(t)$ i.e. $X'(nt)/\sqrt{n \log n}$ converges to some Wiener process.

2°: Borodin (1980) has proved a more general result than our Theorem when B_t is replaced by a symmetric random walk on Z^2.

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