J. Austral. Math. Soc. (Series A) **46** (1989), 365–370

MEROMORPHIC PRODUCTS DETERMINING NEAR-FIELDS

PETER FUCHS and C. J. MAXSON

(Received 13 August 1987)

Communicated by B. J. Gardner

Abstract

In this paper we continue our investigations of a construction method for subnear-rings of $M(G)$ proposed by H. Wielandt. For a meromorphic product H, $H \subseteq G^k$, G finite, we obtain necessary and sufficient conditions for $M(G, k, H)$ to be a near-field.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): primary 12 K 05, 16 A 76; secondary 20 E 10.

1. Introduction

Let G be a group written additively and k a positive integer, $k \geq 2$. R. Remak has pointed out in [4] and [5] that one can construct subgroups of the direct power G^k as follows. For $j \in \{1, 2, \dots, k\}$, let B_j be a subgroup of $G, \bar{B}_j$ a normal subgroup of B_j such that $B_j/\bar{B}_j \cong B_{j+1}/\bar{B}_{j+1}$ with isomorphisms $\sigma_j, j \in \{1, \dots, k-1\}$. Let α be an ordinal, $\{b_{i\eta} | \eta < \alpha\}$ a set of coset representatives of $\bar{B}_1$ in B_1 where $b_{10} = 0$ and define a subset $H \subseteq G^k$ by

$$H = \bigcup_{\eta<\alpha} \left[(b_{1\eta} + \bar{B}_1) \times \prod_{j=1}^{k-1} (\sigma_j \circ \sigma_{j-1} \circ \cdots \circ \sigma_1 (b_{1\eta} + \bar{B}_1)) \right].$$

Then H is called a k-fold meromorphic product and will be denoted by $H = B_1/\bar{B}_1 \underset{\sim\sigma_1}{\times} B_2/\bar{B}_2 \underset{\sim\sigma_2}{\times} \cdots \underset{\sim\sigma_{k-1}}{\times} B_k/\bar{B}_k$. It is straightforward to verify that H is a subgroup of G^k. However, only for $k = 2$ can every subgroup of G^k be obtained as a meromorphic product. Let $M(G) = \{f : G \to G\}$ act on G^k componentwise. For any subgroup H of G^k we define $M(G,k,H) = \{f \in M(G) | f(H) \subseteq H\}$. These $M(G,k,H)$ are subnear-rings of $M(G)$ with identity id: $G \to G$, $\mathrm{id}(x) = x$, for all $x \in G$.

For $k = 2$ it was shown in [1] that whenever $M(G,2,H)$ is a near-field then it must be a field and H is of the form $G/\{0\} \underset{\sim}{\times} G/\{0\}$. This result does not hold for $k \geq 3$. However, in this paper we show that every finite near-field arises from a meromorphic product of the form $B_1/\{0\} \underset{\sim}{\times} \cdots \underset{\sim}{\times} B_k/\{0\}$. More generally, for an arbitrary meromorphic product H, we obtain necessary and sufficient conditions for $M(G,k,H)$ to be a near-field. For a subset S of G we let $S^* = S \backslash \{0\}$.

2. Characterization results

We first show that any finite near-field arises from a meromorphic product.

THEOREM 2.1. *Let N be a zero-symmetric finite near-field. Then there exists a group G, a positive integer k and a subgroup H of G^k where*

$$H = B_1/\{0\} \underset{\sim \sigma_1}{\times} B_2/\{0\} \underset{\sim \sigma_2}{\times} \cdots \underset{\sim \sigma_{k-1}}{\times} B_k/\{0\}$$

such that $N = M(G,k,H)$.

PROOF. Let G be a finite group such that N is a subnear-field of $M(G)$. If $G^* = \{x_1, x_2, \ldots, x_k\}$ then we know $Nx_i \cong Nx_j$ as N-subgroups via $\sigma_{ij}: nx_i \mapsto nx_j, i, j \in \{1, 2, \ldots, k\}$. Let

$$H = Nx_1 \underset{\sim \sigma_{12}}{\times} Nx_2 \underset{\sim \sigma_{23}}{\times} \cdots \underset{\sim \sigma_{k-1k}}{\times} Nx_k.$$

Clearly $N \subseteq M(G,k,H)$. On the other hand, for $(x_1, x_2, \ldots, x_k) \in H$ and $m \in M(G,k,H), m(x_1, \ldots, x_k) = (m(x_1), \ldots, m(x_k)) \in H$. Now $m(x_1) \in Nx_1$ so $m(x_1) = f(x_1)$ for some $f \in N$. But the only k-tuple in H with $f(x_1)$ as first component is $(f(x_1), f(x_2), \ldots, f(x_k))$. Hence $f(x_i) = m(x_i)$ for all $x_i \in G^*$ and so $m = f \in N$.

We have shown that every finite near-field can be represented using a meromorphic product without quotients, that is, by using a meromorphic product of the form $B_1/\{0\} \underset{\sim}{\times} \cdots \underset{\sim}{\times} B_k/\{0\}$. Conversely one would like to characterize those meromorphic products without question that determine near-fields. In fact we consider the more general situation of meromorphic products with quotients, $H = B_1/\bar{B}_1 \underset{\sim}{\times} \cdots \underset{\sim}{\times} B_k/\bar{B}_k$ and determine, in terms of properties of H, when $M(G,k,H)$ is a near-field. The "without quotients" case then follows as a corollary.

https://doi.org/10.1017/S1446788700030858 Published online by Cambridge University Press

Throughout this section all structures are finite. We first fix some notation and give some definitions. Let $H = B_1/\bar{B}_1 \underset{\sim\sigma_1}{\times} \cdots \underset{\sim\sigma_{k-1}}{\times} B_k/\bar{B}_k$ with $B_1/\bar{B}_1 = \{0+\bar{B}_1, b_1+\bar{B}_1, b_2+\bar{B}_1, \ldots, b_n+\bar{B}_1\}$. For $j \in \{1, 2, \ldots, k\}$ we call $B_j/\bar{B}_j$ the jth *column of* H. Let $L_0 = \{\bar{B}_1, \bar{B}_2, \ldots, \bar{B}_k\}$ and

$$L_i = \{b_i + \bar{B}_1, \sigma_1(b_i + \bar{B}_1), \sigma_2 \circ \sigma_1(b_i + \bar{B}_1), \ldots, \sigma_{k-1} \circ \cdots \circ \sigma_1(b_i + \bar{B}_1)\}, \quad i \in \{1, 2, \ldots, n\},$$

and call each L_i a *line*. Further we let $\mathscr{L} = \{L_0, L_1, \ldots, L_n\}$ and $\mathscr{L}^* = \mathscr{L} \backslash \{L_0\}$. For $L \in \mathscr{L}$ and $j \in \{1, 2, \ldots, k\}$ let L^j denote the coset in L which is in the jth column of H. For $0 \neq x \in B_1 \cup \cdots \cup B_k$, say $x \in B_i$, denote by $L_{x,i}$ the unique line L such that $x \in \bigcup L$. Further, for $0 \neq x \in B_1 \cup \cdots \cup B_k$ let $C(x) = \{i \in \{1, 2, \ldots, k\} | x \in B_i\}$ and $P(x) = \{(i, j) \in \{1, 2, \ldots, n\} \times \{1, 2, \ldots, k\} | x \in L_i^j\}$. Thus $C(x)$ denotes the columns of H in which x appears and $P(x)$ gives the coordinates of the cosets which contain x in the array of lines and columns.

For $L_i, L_j \in \mathscr{L}^*$ we write $L_i \sim L_j$ if there exists $\{i_1, \ldots, i_m\} \subseteq \{1, 2, \ldots, n\}$ and $x, x_{i_1}, \ldots, x_{i_{m-1}}, y \in B_1 \cup \cdots \cup B_k$ such that $x \in \bigcup L_i \cap \bigcup L_{i_1}$, $x_{i_1} \in \bigcup L_{i_1} \cap \bigcup L_{i_2}, \ldots, x_{i_{m-1}} \in \bigcup L_{i_{m-1}} \cap \bigcup L_{i_m}$ and $y \in \bigcup L_{i_m} \cap \bigcup L_j$. It is straightforward to verify that $\sim$ is an equivalence relation on $\mathscr{L}^*$. We call the equivalence classes *connected components* and say $\mathscr{L}^*$ is connected when $\mathscr{L}^*$ is a connected component.

When $\mathscr{L}^*$ is connected, one can find (after possibly reordering $\mathscr{L}^*$) a set $\{x_1, \ldots, x_m\}$ such that $\{l | (l, j) \in P(x_i)$ for some $j, 1 \leq j \leq k\} = \{l | l_{i-1} \leq l \leq l_i\}$ for $i \in \{1, 2, \ldots, m\}$, where $l_0 = 1$ and $l_m = n$. We call $\{x_1, \ldots, x_m\}$ a *set of generators*. For $l_{i-1} \leq l \leq l_i$, let $\{j_1^l, \ldots, j_{k_l}^l\}$ denote the columns $j \in \{1, 2, \ldots, k\}$ such that $x_i \in L_l^j$. From this we note that $L_{x_i, j_1^l} = L_{x_i, j_2^l} = \cdots = L_{x_i, j_{k_l}^l}$.

A sequence $A = (a_1, a_2, \ldots, a_m)$ where $a_j \in \bigcup_{i=1}^k B_i$ is a *good sequence* for $x_i \in \{x_1, \ldots, x_m\}$ if

(a) $C(a_k) = C(x_k), k \in \{1, 2, \ldots, m\}$,

(b) $\forall k \in \{1, 2, \ldots, m\}, \forall j \in \{1, 2, \ldots, k\}, x_k \in \bar{B}_j$ implies $a_k \in \bar{B}_j$,

(c) $\exists l$, $l_{i-1} \leq l \leq l_i$ and $p \in \{j_1^l, \ldots, j_{k_l}^l\}$ such that $a_i \in \bar{B}_p^*$,

(d) $\forall k, j \in \{1, 2, \ldots, m\}, \forall k_1 \in C(x_k), \forall k_2 \in C(x_j)$
$$L_{x_k, k_1} = L_{x_j, k_2} \Rightarrow L_{a_k, k_1} = L_{a_j, k_2}.$$

From $L_{x_i, j_1^l} = \cdots = L_{x_i, j_{kl}^l}$ and part (d) of the definition of good sequence we obtain $L_{a_i, j_1^l} = \cdots = L_{a_i, j_{k_l}^l}$. We give one further definition, and then we present our main characterization result.

Let $x_i \in \{x_1, \ldots, x_m\}$ and let $A = (a_1, a_2, \ldots, a_m)$ be a good sequence for x_i. For $x \in G^*$ define

$$A(x) = \bigcap_{\substack{(l,j)\in P(x)\\ 1\le l\le l_1}} L^j_{a_1, j^l_1} \cap \cdots \cap \bigcap_{\substack{(l,j)\in P(x)\\ l_{m-1}\le l\le l_m}} L^j_{a_m, j^l_1}.$$

THEOREM 2.2. *Let* $H = B_1/\bar{B}_1 \underset{\sim\sigma_1}{\times} \cdots \underset{\sim\sigma_{k-1}}{\times} B_k/\bar{B}_k$. *Then* $N = M(G, k, H)$ *is a near-field if and only if*

(1) N *is 0-symmetric,*
(2) $\forall 0 \neq x \in B_1 \cup \cdots \cup B_k$, $\bigcap_{i\in C(x)} \bar{B}_i = \{0\}$,
(3) $\bigcup_{i=1}^n \bigcup L_i = G^*$,
(4) $\mathscr{L}^*$ *is connected with a set of generators* $\{x_1, \ldots, x_m\}$,
(5) $\forall_i \in \{1, 2, \ldots, m\}$, *for all good sequences* $A = (a_1, \ldots, a_m)$ *for* x_i, $\exists x \in G^*, A(x) = \varnothing$ *or* $\exists j \in \{1, 2, \ldots, k\}, \exists x \in \bar{B}_j^*, A(x) \cap \bar{B}_j = \varnothing$.

PROOF. We first show that the conditions are necessary. If N is not 0-symmetric then it is known (see [3]) that $N \cong M_C(Z_2)$. But this is impossible in our situation since the identity map is in N. Suppose now $\bigcap_{i\in C(x_0)} \bar{B}_i \neq \{0\}$ for some $0 \neq x_0 \in B_1 \cup \cdots \cup B_k$, say $0 \neq b \in \bigcap_{i\in C(x_0)} \bar{B}_i$. Define $f: G \to G$ by $f(x_0) = x_0 + b$ and $f(y) = y$ for $y \neq x_0$. Then $f \in N$, a contradiction to N being a near-field. If $\bigcup_{i=1}^n \bigcup L_i \subsetneq G^*$, define $g: G \to G$ by $g(x) = x, x \in \bigcup_{i=1}^n \bigcup L_i$ and $g(y) = 0$, otherwise. Again, $g \in N$, a contradiction.

If $\mathscr{L}^*$ is not connected let C_1 and C_2 be distinct connected components. Define $h: G \to G$ by $h(0) = 0$, $h(x) = 0$ for those x such that there exists $L \in C_1$ with $x \in \bigcup L$ and $h(y) = y$ otherwise. Once again a contradiction is obtained since $h \in N$.

To show that property (5) is necessary let $i \in \{1, 2, \ldots, m\}$ and let $A = (a_1, \ldots, a_m)$ be a good sequence for x_i such that $A(x) \neq \varnothing$, for each $x \in G^*$ and $A(x) \cap \bar{B}_j \neq \varnothing$ for all $j \in \{1, 2, \ldots, k\}$ and all $x \in \bar{B}_j^*$. Define a function $f: G \to G$ by

$$f(x_k) = a_k, \qquad k = 1, 2, \ldots, m;$$
$$f(x) = y_x \in A(x) \cap \bar{B}_j, \quad x \in \bar{B}_j^* \setminus \{x_1, \ldots, x_m\}, \qquad j = 1, 2, \ldots, k;$$
$$f(0) = 0;$$
$$f(x) = y_x \in A(x), \qquad \text{otherwise.}$$

We first show $f \in N$. Let $l \in \{1, 2, \ldots, n\}$, say $l_{i-1} \le l \le l_i$. Let $y_1, y_2 \in L_l$, say $y_1 \in L_l^{i_1}, y_2 \in L_l^{i_2}$. We must show $L_{f(y_1), i_1} = L_{f(y_2), i_2}$. However, since $f(y_i) \in A(y_i)$, $i = 1, 2$, we have $f(y_1) \in L^{i_1}_{a_i, j^l_1}, f(y_2) \in L^{i_2}_{a_i, j^l_1}$ and so $L_{f(y_1), i_1} = L_{a_i, j^l_1} = L_{f(y_2), i_2}$ as required. From this we obtain $f(L_l) \subseteq L_{f(y_1), i_1}$.

Now, since $f(L_0) \subseteq L_0$ by definition, we have $f \in N$. From property (c) of the definition of good sequence there is some line $L_i \in \mathscr{L}^*$ such that $f(L_i) \subseteq L_0$ so f cannot be invertible, contrary to N being a near-field.

For the converse let $f \in N$ and suppose that $f(x) = 0$ for some $x \in G^*$. We show f must be zero map. Consequently N has no divisors of zero and thus, since a finite near-ring without divisors of zero is a near-field, we have the result.

Let $x \in \bigcup L_l$ for some l, say $l_{i-1} \leq l \leq l_i$. Since $x_i \in L_l^j$ for $j \in \{j_1^l, \dots, j_{k_l}^l\}$, $f(x_i) \in \bar{B}_j$. If $f(x_i) \neq 0$, then $f(x_1) = b_1, \dots, f(x_i) = b_i, \dots, f(x_m) = b_m$ defines a good sequence for x_i. But in this case we have $f(x) \in A(x)$ for $x \in G^*$ and $f(x) \in A(x) \cap \bar{B}_j$ for $x \in \bar{B}_j^*$, $j \in \{1, 2, \dots, k\}$, contradicting property (5). Thus $f(x_i) = 0$. But then $f(L_l) \subseteq L_0$ for all $l_{i-1} \leq l \leq l_i$. If $1 < i < m$ then $x_{i-1} \in L_{l_{i-1}}$ and $x_{i+1} \in L_{l_i}$. Again using property (5), by repeating the same argument, we have $f(L_l) \subseteq L_0$ for all $l, l_{i-2} \leq l \leq l_{i+1}$. Continuing in this manner we obtain $f(L_l) \subseteq L_0$ for all $l \in \{1, 2, \dots, n\}$. But then $f(x) \in \bigcap_{i \in C(x)} \bar{B}_i$ for all $x \in G^*$. From property (2), $f = 0$.

In the "without quotients" situation, that is, when $\bar{B}_j = \{0\}$ for all $j \in \{1, \dots, k\}$, properties (2) and (5) are automatically fulfilled and here we have $\bigcap_{l=1}^n \bigcup L_l = (\bigcup_{j=1}^k B_j)$. Thus we have the following.

COROLLARY 2.3. *Let* $H = B_1/\{0\} \underset{\sim \sigma_1}{\times} \cdots \underset{\sim \sigma_{k-1}}{\times} B_k/\{0\}$. *Then* $N = M(G, k, H)$ *is a near-field if and only if*

(1) N *is* 0-*symmetric,*

(2) $\bigcup_{j=1}^k B_j = G$,

(3) $\mathscr{L}^*$ *is connected.*

We conclude the paper with an example which shows that the conditions of the above theorem need not hold. This meromorphic product fulfills (1)–(4) but not (5) and therefore determines a near-ring which is not a near-field.

EXAMPLE 2.4. Let $G = (\mathbb{Z}_2)^4$ with the usual basis $\{e_1, e_2, e_3, e_4\}$. Let $B_1 = G, \bar{B}_1 = \langle e_1 + e_2, e_3 + e_4 \rangle, B_2 = G, \bar{B}_2 = \langle e_1, e_2 + e_4 \rangle, B_3 = \langle e_1, e_2, e_4 \rangle$, $\bar{B}_3 = \langle e_1 \rangle, B_4 = \langle e_1, e_3, e_2 + e_4 \rangle, \bar{B}_4 = \langle e_1 + e_2 + e_3 + e_4 \rangle, B_5 = \langle e_1, e_3 + e_4 \rangle$, $\bar{B}_5 = \{0\}, B_6 = \langle e_1, e_2 + e_3 \rangle$ and $\bar{B}_6 = \{0\}$. The following scheme determines a meromorphic product:

$$e_1 + \bar{B}_1 \mapsto e_1 + e_2 + \bar{B}_2 \mapsto e_2 + \bar{B}_3 \mapsto e_1 + \bar{B}_4 \mapsto e_3 + e_4 + \bar{B}_5 \mapsto e_1 + \bar{B}_6,$$

$$e_1 + e_4 + \bar{B}_1 \mapsto e_4 + \bar{B}_2 \mapsto e_4 + \bar{B}_3 \mapsto e_1 + e_3 + \bar{B}_4$$
$$\mapsto e_1 + e_3 + e_4 + \bar{B}_5 \mapsto e_2 + e_3 + \bar{B}_6.$$

Using $x_1 = e_1 + e_3 + e_4, x_2 = e_4$ and $A = (e_3 + e_4, e_1 + e_2)$ as a good sequence for x_1, one defines a function in $M(G, 6, H)$ which is not invertible.

References

[1] P. Fuchs and C. J. Maxson, 'Near-fields associated with invariant linear k-relations', *Proc. Amer. Math. Soc.* **103** (1988), 729–736.

[2] J. D. P. Meldrum, *Near-rings and their links with groups* (Research Notes in Math. 134, Pitman, London, 1986).

[3] G. F. Pilz, *Near-rings* (2nd ed., North-Holland, Amsterdam, 1983).

[4] R. Remak, 'Über die Darstellung der endlichen Gruppen als Untergruppen direkter Produkte', *J. Reine Angew. Math.* **163** (1930), 1–44.

[5] R. Remak, 'Über Untergruppen direkter Produkte von drei Faktoren', *J. Reine Angew. Math.* **166** (1932), 65–100.

[6] H. Wielandt, *Permutation groups through invariant relations and invariant functions* (Lecture notes, Ohio State University, Columbus, 1969).

Department of Mathematics
Texas A & M University
College Station, Texas 77843
U.S.A.