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REDUCIBLE RATIONAL FRACTIONS OF THE TYPE OF GAUSSIAN POLYNOMIALS WITH ONLY NON-NEGATIVE COEFFICIENTS

BY
EMIL GROSSWALD

1. **Introduction.** The following problem arose in connection with the study of Poincaré polynomials for homogeneous spaces. Let

$$f(t)=\prod_{i=1}^{k}\frac{1-t^{g_i}}{1-t^{h_i}},$$

g_i, h_i positive integers and set $d=(h_1, h_2, \ldots, h_k)$, the greatest common divisors of the exponents in the denominator. Let $h_i = dr_i$ and assume that the r_i's are comprime in pairs, i.e. that $(r_i, r_j)=1$ for $i\neq j$. In this context, the following two problems arise:

Problem 1: Under what conditions on the exponents does $f(t)$ reduce to a polynomial?

Problem 2: Under what additional conditions does the polynomial $f(t)$ have only non-negative coefficients?

The classical example of such rational fractions that reduce to polynomials with non-negative integers is that of the Gaussian polynomials $\begin{bmatrix} n \\ m \end{bmatrix}=\prod_{j=1}^{m}\frac{1-x^{n-j+1}}{1-x^{j}}$. The fact that these reduce to polynomials can easily be generalized to the present situation. The fact that the coefficients of $\begin{bmatrix} n \\ m \end{bmatrix}$ are non-negative can be proved by showing (see F. Franklin's proof presented in [7]) that they can be interpreted as the number of partitions of n into m parts, so that no part is larger than $n-m$.

This idea cannot be completely adapted in any obvious way to the present, general case, although also the present results are shown to depend on precisely such partition numbers.

In what follows, Problem 1 is answered completely by Theorem 1.

In principle also Problem 2 is answered, by Theorem 2, which states a necessary and sufficient condition for $f(t)$ to have only non-negative coefficients. This condition is, however, of a rather difficult application.

Next, the polynomial $f(t)$ is factored into polynomials $g_m(t)$ of a specific, simple structure, which we shall call elementary polynomials, so that $f(t)=\prod_{m=1}^{q} g_m(t)$.

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A practical necessary and sufficient condition for the non-negativity of coefficients is stated in Theorem 3, but it is applicable only to certain elementary polynomials $g(t)$, so that it leads only to a sufficient condition for $f(t)$, even if $f(t)$ is factorable into those special elementary polynomials. Finally, Theorem 4 gives a simple necessary condition for the general elementary polynomial and Theorem 5 a rather trivial sufficient condition for the general case.

It would be desirable to find a simpler necessary and sufficient condition that insures the non-negativity of the coefficients of any polynomial $f(t)$ of the type here considered.

I take this opportunity to acknowledge with gratitude the fact that this problem, originating with Professor Stephen Halperin, was brought to my attention by my colleague, Professor James Stasheff. I also thank Professor G. Andrews for suggestions and bibliographic information. Finally, an expression of gratitude to a referee who not only made excellent suggestions but also discovered some serious oversights in an earlier version of this paper.

2. **Main Results.**

THEOREM 1. (A). *Let* $f(t)=\prod_{i=1}^{k}\{(1-t^{g_i})/(1-t^{h_i})\}$ *with* $d=(h_1, h_2, \ldots, h_k)$, $h_i=dr_i$ $(i=1,2,\ldots,k)$ *and* $(r_i, r_j)=1$ *for* $i\neq j$. *Then* $f(t)$ *reduces to a polynomial if and only if the following two conditions hold*:

(i) *For every* j $(j=1,2,\ldots,k)$, $d \mid g_j$ *so that* $g_j=ds_j$; *and*

(ii) *for every* j, *there exists some* k, *such that* $r_j \mid s_k$.

(B). *If* (*perhaps after a renumbering of the* s_h'*s*) *also* $r_j \mid s_j$ *holds, then all coefficients of the polynomial* $f(t)$ *are non-negative.*

REMARKS. 1. In (A) the same s_h may serve as multiple for several r_j's.

2. The (rather trivial) sufficient condition in (B) is far from being necessary for the non-negativity of the coefficients of $f(t)$.

If conditions (i) and (ii) of Theorem 1 (A) are satisfied, then we may set $u=t^d$, so that $f(t)=h(t^d)=h(u)$, say, and $f(t)$ is a polynomial in t if and only if

$$h(u)=\prod_{i=1}^{k}\frac{1-u^{s_i}}{1-u^{r_i}} \tag{1}$$

is a polynomial in u.

For an arbitrary sequence of integers $T=\{t_1, t_2, \ldots, t_j, \ldots\}$, finite or infinite, $p_T(n)$ denotes the unrestricted number of partitions of the positive integer n, into parts t_j from T. The number of partitions of n into distinct parts from T is denoted by $q_T(n)$ and one distinguishes such partitions without repetition, according to the parity of the number of summands. Specifically, $q_T^{(0)}(n)$ stands for the number of partitions of n into an even number of distinct parts t_j from T and $q_T^{(1)}(n)$ stands for the number of such partitions with an odd number of summands; clearly, $q_T^{(n)}=q_T^{(0)}(n)+q_T^{(1)}(n)$. It is convenient to set also $p_T(0)=q_T^{(0)}(0)=1$, $q_T^{(1)}(0)=0$.

REMARK. The elements $t_1, t_2, \ldots$ of T are not necessarily different; hence, it is possible that in a partition of n into distinct elements of T, some of them have the same numerical value.

Let $R=\{r_1, r_2, \ldots, r_k\}$ and $S=\{s_1, s_2, \ldots, s_k\}$ be the sets of exponents of the denominator, and of the numerator of $h(u)$, respectively. Also, set $M=\sum_{i=1}^k r_i$, $N=\sum_{i=1}^k s_i$ and $D=N-M$. With these notations we can now state

THEOREM 2. *Let $f(t)$ satisfy conditions* (i) *and* (ii) *of Theorem* 1; *then if $u=t^d$, $f(t)=h(u)$ reduces to a polynomial of degree D in u, with only non-negative coefficients, if and only if*

$$\sum_{m=0}^{n} p_R(n-m)q_S^{(0)}(m) \geqq \sum_{m=0}^{n} p_R(n-m)q_S^{(1)}(m)$$

holds for $0 \le n \le D$.

In view of Theorem 1 it is possible to simplify the problem, by factoring $h(u)$ as follows:

Associate all factors $(1-u^{r_j})$ in the denominator, for which r_j divides a specific s_h, with the corresponding factor $1-u^{s_h}$ in the numerator. By renumbering if necessary the r_j's, we obtain a fraction of the form

$$\frac{1-u^{s_h}}{(1-u^{r_1})\cdots(1-u^{r_m})}. \tag{2}$$

Here $s_h = cr_1r_2\cdots r_m$, as follows from $(r_i, r_j)=1$ for $i \neq j$. The remaining factors $1-u^{s_q}$ of the numerator of $h(u)$ may now be distributed among the fractions (2), so that in each one the number of factors in the numerators equals the number of factors in the denominators. This is always possible, because the total number of factors in the numerator of (1) equals the total number in the denominator. The individual fractions now are of the form

$$g(u)=\frac{(1-u^{cr_1\cdots r_m})(1-u^{a_1})\cdots(1-u^{a_{m-1}})}{(1-u^{r_1})(1-u^{r_2})\cdots(1-u^{r_m})} \tag{3}$$

and

$$h(u)=g_1(u)g_2(u)\cdots g_q(u).$$

By Theorem 1, (3) reduces to a polynomial. These fractions, with at least one exponent in the numerator divisible by all exponents of the denominator and with an equal number of factors $1-u^s$ in both, numerator and denominator, will be called elementary polynomials.

REMARK. It is clear that the factorization of $h(u)$ into elementary polynomials is not unique. It also follows from Theorem 2 that not all possible factorizations of $h(u)$ are equally desirable, but that the distribution of the factors $1-u^a$ among the different $g_j(u)$'s should be done keeping in mind the applicability of Theorem 2. If all elementary polynomials $g_j(u)$ have non-negative coefficients, then so does $h(u)$ (and, hence, also $f(t)$), but not conversely.

For convenience we set $cr_1r_2 \cdots r_m = a_m$ and denote by $A = \{a_1, a_2, \ldots, a_{r_{..}}\}$ the set of exponents in the numerator of $g(u)$.

A positive integer n is said to be representatable by a set T of positive integers, if $n = c_1t_{j_1} + c_2t_{j_2} + \cdots + c_pt_{j_p}$ with the t_j's from T and the c_j's arbitrary positive integers. The integer n is representable by T, if and only if $p_T(n) > 0$; otherwise, n is not representable by T. If $d_T = (t_1, t_2, \ldots) = 1$, then every sufficiently large integer is representable by T. In case $T = \{t_1, t_2) = 1$, Sylvester [8] has shown that the largest non-representable integer is equal to $t_0 = t_1t_2 - t_1 - t_2$ and that there are exactly $\frac{1}{2}(t_0 + 1)$ non-representable integers. It easily follows that if $t_1 < t_2 < t_3 \cdots$ and the t_j's are coprime in pairs, then the largest nonrepresentable integers is always at most equal to t_0; however, this problem of representability, which goes back to Frobenius [3], is still far from being solved satisfactorily in the general case (see [1], [2], [4], [5] and [6], and the literature quoted there for some recent contributions).

Theorem 3. (a) *In order that the special elementary polynomial*

$$g(u) = \frac{(1-u^{cr_1r_2})(1-u^a)}{(1-u^{r_1})(1-u^{r_2})}, \ (r_1, r_2) = 1, \ c \in Z^+ \tag{4}$$

should have only non-negative coefficients, it is necessary and sufficient that the exponent a should be representable by $R = \{r_1, r_2\}$.

(b) *If a is representable, $g(u)$ is a reciprocal polynomial of degree $D = cr_1r_2 + a - r_1 - r_2$, with non-negative coefficients. None of the coefficients exceeds c and the sum of the coefficients equals ca.*

Exactly $(r_1 - 1)(r_2 - 1)$ of the coefficients vanish, namely when either r or $D - r$ is one of the $\frac{1}{2}(r_1 - 1)(r_2 - 1)$ non-representable integers.

Corollary 1. *If $c = 1$, and a is representable then $g(u)$ has exactly a coefficients equal to one, while the other $D + 1 - a = (r_1 - 1)(r_2 - 1)$ coefficients are zero; specifically, the vanishing coefficients are those of powers of r with either r or $D - r$ not representable. In fact, if a is representable then $c = 1$ is the necessary and sufficient condition for rational fractions of type* (4) *to have all a non-vanishing coefficients equal to one.*

Corollary 2. *A sufficient condition for the non-negativity of the coefficients of $h(u)$ is that it should be factorable into the special elementary polynomials of type* (4), *with exponents a representable by their respective sets R.*

Remark. Theorem 3 has been proved already by S. Halperin by an entirely different method (private communication of unpublished results).

THEOREM 4. *In the general elementary polynomial* $g(u)$ *of* (3), *let* $a_1 \le a_2 \le \cdots \le a_\mu < (a_1 + a_2)$ *where* $\mu = \max\{p \mid a_p < a_1 + a_2\}$; *clearly,* $\mu \ge 2$. *A necessary condition for the non-negativity of the coefficients of* $g(u)$ *is that* a_p *should be representable by* R *for* $p = 1, 2, \ldots, \mu$.

THEOREM 5. *A sufficient condition for* $f(t) = \prod_{i=1}^{m} \{(1 - t^{g_i})/(1 - t^{h_i})\}$ *to reduce to a polynomial with non-negative coefficients is that there exist integers n and d, such that* $h_i = di$ *and (perhaps after a renumbering of the* g_i*'s),* $g_i = d(n - i + 1)$.

3. **Proof of Theorem 1.** (A). *Necessity of* (*i*). $f(t)$ reduces to a polynomial if and only if every zero of the denominator is cancelled by a zero of at least equal multiplicity in the numerator. Each factor of the denominator furnishes $h_j = dr_j$ zeros of the form $e^{2\pi i\nu/dr_j}$ ($\nu = 0, 1, 2, \ldots, dr_j - 1$). In particular, for $\nu = r_j$, each factor has the zero $e^{2\pi i/d}$, which is, hence, a k-fold zero. The zeros of the numerator are $e^{2\pi is/g_j}$ ($j = 0, 1, \ldots, k$; $s = s^{(j)} = 0, 1, \ldots, g_j - 1$). For some values of j and $s^{(j)}$ one must have $2\pi i/d = 2\pi is^{(j)}/g_j$, so that $g_j = s^{(j)}d$. However, each factor in the numerator has $e^{2\pi i/d}$ only as a single, simple zero (namely for the unique value of $s^{(j)} = g_j/d$). Hence, all exponents of the numerators have to satisfy $g_j = s^{(j)}d$ for some integer $s^{(j)}$, so that $d \mid g_j$, and this proves the necessity of (i).

We now set $h_j = dr_j$, $g_j = ds_j$, with $(r_i, r_j) = 1$ for $i \ne j$ so that $f(t)$ becomes a function of $u = t^d$ only, say $f(t) = h(u) = \prod_{i=1}^{k} \{(1 - u^{s_i})/(1 - u^{r_i})\}$.

Necessity and sufficient of (*ii*). In $h(u)$, the numerator and the denominator have both the zero at $u = 1$ as a k-fold zero; hence, in the denominator the factor is cancelled and it is sufficient to consider only the zeros $u = e^{2\pi i\nu_j/r_j}$ with $\nu_j = 1, 2, \ldots, r_j - 1$. If none of the s_i is divisible by r_j, then the zero $u = e^{2\pi i/r_j}$ of the denominator is not cancelled, so that the condition is necessary. To prove its sufficiency, let us assume that $r_1, r_2, \ldots, r_m$ are divisors of the same exponent s in the numerator. Then, because of $(r_i, r_j) = 1$, $s = cr_1 \cdots r_m$, with $c \in Z^+$ as seen. All the zeros $e^{2\pi i\nu_j/r_j}$ ($\nu_j = 1, 2, \ldots, r_j - 1$) of the denominator occur as distinct zeros among the zeros $e^{2\pi i\lambda/s} = e^{(2\pi i/r_j)(\lambda r_j/s)}$ ($\lambda = 1, 2, \ldots, s - 1$) of the numerator. Indeed, we only have to take $\lambda r_j = \nu_j s$, or $\lambda = \nu_j cr_1 r_2 \cdots r_{j-1} r_{j+1} \cdots r_m$. As $r_j \mid s$ and as $\lambda = (\nu_j/r_j)s < s$, this coefficient λ actually does occur and the zero $e^{2\pi i\nu_j/r_j}$ is cancelled. It remains to verify that each such zero of the denominator is simple, or, equivalently, that the zeros of the denominator are distinct for different subscripts j. If not, there exist integers ν_j, ν_h, with $1 \le \nu_j \le r_j - 1$, $1 \le \nu_h \le r_h - 1$ and $e^{2\pi i\nu_j/r_j} = e^{2\pi i\nu_h/r_h}$, or $\nu_j/r_j = \nu_h/r_h$, i.e., $r_h/r_j = \nu_h/\nu_j$. However, $(r_h, r_j) = 1$, so that the rational number r_h/r_j is in reduced form and cannot be represented by a ratio of smaller integers. Hence, $r_h/r_j = \nu_h/\nu_j$ is impossible and all zeros of the denominator are distinct, simple and are cancelled by the zeros $e^{2\pi i\lambda/s}$ of the numerator. This finishes the proof of part (A).

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(B). If $h(u)=\prod_{j=1}^{k}\{(1-u^{s_j})/(1-u^{r_j})\}$ and $r_j \mid s_j$, one can set $s_j=m_jr_j$, $m_j\in Z^+$. Let $u^{r_j}=x$; then

$$\frac{1-u^{s_j}}{1-u^{r_j}}=\frac{1-x^{m_j}}{1-x}=1+x+\cdots+x^{m_j-1}=1+u^{r_j}+\cdots+u^{(m_j-1)r_j}$$

and $h(u)$ has only non-negative coefficients as product of polynomials with such coefficients; also, with t^d for u, the same holds for $f(t)$.

4. **Proof of Theorem 2.** Let us consider (1), where $h(u)$ comes from an $f(t)$ that satisfies the conditions (i) and (ii) of Theorem 1 and, hence, reduces to a polynomial. By definition of the $q_S^{(i)}(n)$ $(i=0,1)$ and of N,

$$\prod_{j=1}^{k}(i-u^{s_j})=\sum_{m=0}^{N}\{q_S^{(0)}(m)-q_S^{(1)}(m)\}u^m;$$

similarly,

$$\prod_{j=1}^{k}(1-u^{r_j})^{-1}=\sum_{\nu=0}^{\infty}p_R(\nu)u^\nu.$$

It follows that

$$h(u)=\left(\sum_{\nu=0}^{\infty}p_R(\nu)u^\nu\right)\sum_{m=0}^{N}\{q_S^{(0)}(m)-q_S^{(1)}(m)\}u^m$$

$$=\sum_{n=0}^{\infty}u^n\sum_{\substack{m+\nu=n\\ \nu\geq 0\\ 0\leq m\leq N}}p_R(\nu)\{q_S^{(0)}(m)-q_S^{(1)}(m)\}=\sum_{n=0}^{\infty}C_nu^n,$$

with

$$C_n=\sum_{m=0}^{K}p_R(n-m)\{q_S^{(0)}(m)-q_S^{(1)}(m)\},\ K=\min(n,N). \tag{5}$$

By Theorem 1, $h(u)$ is a polynomial of degree $D=\sum_{s\in S}s-\sum_{r\in R}r=N-\sum_{r\in R}r<N$; hence, $C_n=0$ for $n>D$. For $n<D<N$, however, $n<N$ so that $K=n$. The required condition $C_n\geq 0$ now reads

$$\sum_{m=0}^{n}p_R(n-m)q_S^{(0)}(m)\geq\sum_{m=0}^{n}p_R(n-m)q_S^{(1)}(m)$$

and this finishes the proof of Theorem 2.

5. **Proof of Theorem 3 and of its corollaries.** (i) *Proof of necessity.* Here $S=\{cr_ir_2, a\}$. If a is not representable, then clearly $a<cr_1r_2$. Also, $q_S^{(0)}(0)=1$, $q_S^{(1)}(0)=0$, $q_S^{(0)}(m)=q_S^{(1)}(m)=0$ for $0<m<a$, $q_S^{(0)}(a)=0$, $q_S^{(1)}(a)=1$, so that, by (5), the coefficient of u^a is

$$\sum_{m+\nu=a}p_R(\nu)\{q_S^{(0)}(m)-q_S^{(1)}(m)\}=p_R(0)\{q_S^{(0)}(a)-q_S^{(1)}(a)\}$$

$$+p_R(a)\{q_S^{(0)}(0)-q_S^{(1)}(0)\}=-1+p_R(a).$$

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If the exponent a is not representable, then the coefficient of u^a in $g(u)$ equals -1 and $g(u)$ does not have only non-negative coefficients.

(ii) **Proof of sufficiency.** The polynomial $g(u)$ is a reciprocal polynomial of degree $D=cr_1r_2+a-r_1-r_2$. Indeed,

$$u^Dg(u^{-1})=u^{cr_1r_2+a-r_1-r_2}\frac{(1-u^{-cr_1r_2})(1-u^{-a})}{(1-u^{-r_1})(1-u^{-r_2})}=\frac{(u^{cr_1r_2}-1)(u^a-1)}{(u^{r_1}-1)(u^{r_2}-1)}=g(u).$$

It follows, in particular, that the powers u^r and u^{D-r}, have the same coefficients, and if $g(u)=\sum_{r=0}^D a_ru^r$, it is sufficient to prove that $a_r\geq 0$ for $0\leq r\leq D/2$. By expanding the denominator we obtain

$$g(u)=(1-u^{cr_1r_2})(1-u^a)\sum_{r=0}^{\infty}p_R(r)u^r. \tag{6}$$

The case $a=cr_1r_2$ is trivial and is covered by Theorem 1 (B). The alternatives $a>cr_1r_2$ and $a<cr_1r_2$ have to be considered separately. In the first case, a is always representable (see [8]). The treatment of the two cases is essentially the same. In the slightly more difficult second case, one has

$$\begin{aligned} g(u)=\sum_{r=0}^{a-1}p_R(r)u^r&+\sum_{r=a}^{cr_1r_2-1}(p_R(r)-p_R(r-a))u^r\\ &+\sum_{r=cr_1r_2}^{cr_1r_2+a-1}(p_R(r)-p_R(r-a)-p_R(r-cr_1r_2))u^r\\ &+\sum_{r=cr_1r_2+a}^{\infty}(p_R(r)-p_R(r-a)-p_R(r-cr_1r_2)\\ &\qquad\qquad+p_R(r-a-cr_1r_2))u^r.\end{aligned} \tag{7}$$

Here the last sum is empty, because $cr_1r_2+a>D$, the degree of $g(u)$. Next, from $a<cr_1r_2$ it follows that the exponents of the third sum are larger than $D/2$ It is, therefore, sufficient to consider only the coefficients of the first two sums. In the first sum, $p_R(r)$ is positive if r is representable, $p_R(r)=0$ otherwise. In the second sum, we claim that

$$p_R(r)\geq p_R(r-a) \tag{8}$$

with equality if r is not representable. This is trivial, if $r-a$ is not representable. Otherwise, let $a=k_1r_1+k_2r_2$; then to each different representation $r-a=m_1r_1+m_2r_2$ corresponds a different representation of $r=(k_1+m_1)r_1+(k_2+m_2)r_2$ and (8) is proved. In particular, if r is one of the $\frac{1}{2}(r_1-1)(r_2-1)$ non-representable integers then $p_R(r)=0$, and also $p_R(r-a)=0$. It is clear that all these values occur in the first two sums, as $c\geq 1$. Also in case $a>cr_1r_2$ it is sufficient to consider only the first two sums, which now read

$$g(u)=\sum_{r=0}^{cr_1r_2-1}p_R(u)u^r+\sum_{r=cr_1r_2}^{a-1}(p_R(r)-p_R(r-cr_1r_2))u^r+\cdots. \tag{7'}$$

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Indeed $a-1>\frac{1}{2}(a-1+cr_1r_2)>D/2$ and all coefficients a_r with $r\le D/2$ occur in the displayed sums. Exactly as before we verify that all coefficients are non-negative, and vanish if r is one of the $\frac{1}{2}(r_1-1)(r_2-1)$ non representable integers. In fact, these occur now only in the first sum. We have shown that in all cases, the coefficients a_r of u^r are non-negative and that they vanish if r is non-representable, which is possible only for $r\le D/2$. As $g(u)$ is a reciprocal polynomial, the coefficients a_r vanish also when $D-r$ is non-representable.

(iii) **Proof of the complementary statements.** The sum of the coefficients is obtained by letting $u\to 1$. If we divide each factor of $g(u)$ by $1-u$ and then let $u\to 1$ (or by L'Hôspital's rule) we obtain

$$\sum_{r=1}^{D} a_r=\lim_{u\to 1} g(u)=\frac{cr_1r_2a}{r_1r_2}=ca.$$

Next, one may verify that the Diophantine equation $k_1r_1+k_2r_2=r$ has $\left[\frac{r}{r_1r_2}\right]+\delta$ solutions in non-negative integers k_1, k_2 where $\delta=1$ or 0 accordingly as the least positive residue of r (mod r_1r_2) is, or is not representable. Hence, in the first sums of (7) or (7′),

$$p_R(r)\le\left[\frac{cr_1r_2-1}{r_1r_2}\right]+1=c.$$

In the second sum of (7),

$$p_R(r)-p_R(r-a)\le p_R(r)\le\left[\frac{r}{r_1r_2}\right]+1\le\left[\frac{cr_1r_2-1}{r_1r_2}\right]+1=c,$$

while in the second sum of (7′), observing that

$$r\equiv r-cr_1r_2 \pmod{r_1r_2},\ p_R(r)-p_R(r-cr_1r_2)=\left[\frac{r}{r_1r_2}\right]+\delta-\left(\left[\frac{r-cr_1r_2}{r_1r_2}\right]+\delta\right)=c.$$

This proves that $\max\limits_r a_r\le c$. In particular, for $c=1$, the coefficients are either 0 or 1. With $D_1=r_1r_2+a-r_1-r_2$ there are all together D_1+1 coefficients. Of these at least $(r_1-1)(r_2-1)=r_1r_2-r_1-r_2+1$ vanish, so that at most a coefficients are different from zero. As the non-vanishing ones have the value one and the sum equal to a, it follows that none of the remaining coefficients vanishes, i.e., exactly a are equal to one and exactly $(r_1-1)(r_2-1)$ vanish. When $c>1$, we may write

$$g(u)=g_c(u)=g_1(u)\sum_{j=1}^{c-1} u^{jr_1r_2}=\sum_{r=0}^{D} a_r^{(c)}u^r,$$

where

$$g_1(u)=\frac{(1-u^{r_1r_2})(1-u^a)}{(1-u^{r_1})(1-u^{r_2})}=\sum_{r=0}^{D_1} a_r^{(1)}u^r,\ D_1=r_1r_2+a-r_1-r_2.$$

A comparison of the coefficients shows that $a_r^{(c)}=\sum_{r'} a_{r'}^{(1)}$, with the summation extended over all r' with $0\le r'=r-jr_1r_2\le D_1$, $j=0,1,\ldots,c-1$. From $a_{r'}^{(1)}=0$ or 1 it follows, in particular, that $a_r^{(c)}=0$ only if all $a_{r'}^{(1)}$ in the sum vanish. For this a necessary (not always sufficient) condition is that either r' or D_1-r' should be non-representable. In particular, it is necessary to have either r' or D_1-r' non-representable. The first condition (with $j=0$) leads to the necessary condition: $a_r^{(c)}=0$ for $r<D_1$ only if r is non-representable. The second one (with $j=c-1$) requires for the vanishing of $a_r^{(c)}$, that $D_1+(c-1)r_1r_2-r=D-r$ should be non-representable. On the other hand, we already know that these coefficients in fact do vanish, and we have proved that a necessary and sufficient condition for $a_r(=a_r^{(c)})$ to vanish is that either r or $D-r$ be non-representable. It is worthwhile to observe that the number of vanishing coefficients is always $(r_1-1)(r_2-1)$ and is independent of c.

Next, we observe that if we ignore the trivial cases (covered by Theorem 1 (B)) when either $r_1\ge r_2$ is a factor of a, then, if a is representable, one has $a\ge r_1+r_2$ and $r_1r_2\le D_1$. It follows, in particular, that $a_{r_1r_2}^{(c)}=a_0^{(1)}+a_{r_1r_2}^{(1)}=2$, so that the $(c-1)r_1r_2+a$ non-vanishing coefficients are all equal to one precisely when $c=1$. This finishes the proof of Theorem 3 and of its first corollary. Corollary 2 follows trivially from Theorem 3.

7. **Proof of Theorem 4.** By (5)

$$C_{a_p}=\sum_{m=0}^{a_p} p_R(a_p-m)\{q_A^{(0)}(m)-q_A^{(1)}(m)\}.$$

For any $1\le p\le\mu$ and $1\le m\le a_p$, $q_A^{(0)}(m)=0$, while $q_A^{(1)}(m)=0$ if $m\notin A$, $q_A^{(1)}(m)=1$ if $m=a_q\in A$ $(q=1,2,\ldots,p)$; also $p_R(0)=q_A^{(0)}(0)=1$. Hence, $C_{a_p}=p_R(a_p)-\sum_{q=1}^{p} p_R(a_p-a_q)$. If, for any $p\le\mu$, a_p is not representable, then $p_R(a_p)=0$ and $C_{a_p}=-\sum_{q=1}^{p} p_R(a_p-a_q)\le -p_R(0)=-1$ and $h(u)$ does not have only non-negative coefficients.

8. **Proof of Theorem 5.** If we set $t^d=u$, then $f(t)=h(u)=\begin{bmatrix} n\\ m\end{bmatrix}$, the classical Gaussian polynomial, for which the non-negativity of the coefficients is well known (see e.g., [7]).

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MATHEMATICS DEPARTMENT
TEMPLE UNIVERSITY
COLLEGE OF LIBERAL ARTS
PHILADELPHIA, PA 19122