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DENSE, UNIFORM AND DENSELY SUBUNIFORM CHAINS

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Abstract

A chain, or linearly ordered set, is *densely subuniform* if it is dense and for every order type the elements whose corresponding initial sections have this order type, if any, are dense in the chain. It is *uniform* if all initial sections are isomorphic. This paper gives constructions for densely subuniform chains which are not uniform. The question arises from the study of congruence-free inverse semigroups, but may also have independent interest.

1. Introduction

This paper answers a question of W. D. Munn on chains, arising from the study of the inverse semigroup T_E^* introduced in Munn (1970). In Theorem 4.3 of that paper, T_E^* is shown to be universal for those fundamental inverse semigroups with no non-trivial group congruences whose semilattice of idempotents is E. It follows from this theorem that E is the semilattice of idempotents of some congruence-free inverse semigroup if, and only if, T_E^* itself is congruence-free. This is also a consequence of Munn (1974), Theorem 1.6.

In Munn (1966) and (1974) are formulated the definitions of uniform and densely subuniform semilattices as given below for the case where E is a chain. From Theorem 2.7 of Munn (1974) it is easily seen that when E is a chain, with other than 2 elements, if T_E^* is congruence-free then E is densely subuniform.

Densely subuniform chains are, by definition, dense, and the easily accessible examples are in fact uniform. The purpose of this paper is to construct densely subuniform chains which are not uniform.

Example A, probably the most easily constructed such chain, has the interesting property that its converse is uniform. It follows from Theorem 2.12 of Munn (1974) that if E is this chain, then T_E^* is congruence-free. This provides an example of a non-uniform chain, with other than two elements,

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which is the semilattice of a congruence-free inverse semigroup. A simple modification to Example A produces a densely subuniform chain E such that neither E nor its converse is uniform (Example B). Finally, a chain E is constructed with the property that both E and its converse are densely subuniform but not uniform (Example C).

2. Preliminaries

Notation. The following notation, which appears in the proof of Munn (1974), Theorem 2.12, will be used throughout.

Let $\underset{\sim}{A} = \langle A, \leqq \rangle$ be a chain (also sometimes called a linearly, or totally, ordered set (Sierpinski (1965)). For $a \in A$, write:–

$$(\rightarrow, a] = \{x \in A : x \leqq a\},\ [a, \rightarrow) = \{x \in A : a \leqq x\}$$

$$(\rightarrow, a) = \{x \in A : x < a\},\ (a, \rightarrow) = \{x \in A : a < x\}.$$

For $a, b \in A$ $a < b$ write:–

$(a, b] = \{x \in A : a < x \leqq b\}$ and define (a, b), $[a, b)$, $[a, b]$ analogously. It is understood that, when treated as chains, these sets inherit the ordering of $\underset{\sim}{A}$.

Definitions. *Let $\underset{\sim}{A} = \langle A, \leqq \rangle$ be chain.*

$\underset{\sim}{A}$ is dense if whenever $a, b \in A$ and $a < b$, (a, b) is non-empty.

$\underset{\sim}{A}$ is uniform if whenever $a, b \in A$, $(\rightarrow, a] \cong (\rightarrow, b]$.

$\underset{\sim}{A}$ is densely subuniform if whenever $a, b, c \in A$ and $a < b$ then for some $d \in A$, $a < d < b$ and $(\rightarrow, d] \cong (\rightarrow, c]$.

Remarks. $\underset{\sim}{A}$ may be dense with least element, but except in the trivial cases where A has one or no element, if $\underset{\sim}{A}$ is uniform or densely subuniform then $\underset{\sim}{A}$ has no least element, although the possibility of a greatest element remains.

Order types. We sketch here the arithmetic of order types (i.e. isomorphism types of chains). More detailed treatments may be found in Bachmann (1967) and Sierpinski (1965).

If α, β are the order types of chains $\underset{\sim}{A}, \underset{\sim}{B}$ respectively, $\alpha + \beta$ is defined to be the order type of the chain formed by placing any copy of $\underset{\sim}{B}$ after one of $\underset{\sim}{A}$. More generally, if $\underset{\sim}{X}$ is a chain and for each $x \in X$, α_x is an order type, then $\Sigma\{\alpha_x : x \in \underset{\sim}{X}\}$ is defined to be the order type of $\cup\{A_x \times \{x\} : x \in X\}$, where $\underset{\sim}{A}_x$ is any chain of order type α_x, ordered by $\langle a, x \rangle \leqq \langle a', x' \rangle$ if, and only if, $x < x'$ in $\underset{\sim}{X}$ or $x = x'$ and $a \leqq a'$ in $\underset{\sim}{A}_x$. If $\underset{\sim}{X}$ is the chain of non-negative integers, we often write this ordered sum as $\alpha_0 + \alpha_1 + \alpha_2 + \cdots$. Similarly, if $\underset{\sim}{X}$ is the chain of non-positive integers, the sum is sometimes written $\cdots + \alpha_2 + \alpha_1 + \alpha_0$.

If $\underset{\sim}{X}$ has order type β, and each $\alpha_x = \alpha$, then $\Sigma\{\alpha_x : x \in \underset{\sim}{X}\}$ is denoted by $\alpha \cdot \beta$, or $\alpha\beta$.

From these definitions the following facts may be deduced:–

(2.1) $$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma),$$

(2.2) $$(\alpha\beta)\gamma = \alpha(\beta\gamma),$$

(2.3) $$\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma, \text{ and more generally,}$$
$$\alpha(\Sigma\{\beta_x : x \in \underset{\sim}{X}\}) = \Sigma\{\alpha\beta_x : x \in \underset{\sim}{X}\}.$$

The empty set is admitted, for convenience, with order type 0. We write $\alpha \leqq \beta$ in the case where $\beta = \alpha + \gamma$ for some γ. $\leqq$ is thus reflexive and transitive but not anti-symmetric on order types.

If α is the order type of $\langle A, \leqq \rangle$, then α^* denotes the order type of $\langle A, \geqq \rangle$ (the converse of $\underset{\sim}{A}$). We write $\alpha \leqq {}^*\beta$ in the case where $\alpha^* \leqq \beta^*$, that is, $\beta = \gamma + \alpha$ for some γ. The following theorem (Lindenbaum-Tarski) is proved in Sierpinski (1965):

(2.4) *If* $\alpha \leqq \beta$ *and* $\beta \leqq {}^*\alpha$, *then* $\alpha = \beta$.

We shall have need of the corollary:–

(2.5) *If* $\alpha \leqq \alpha^*$ *then* $\alpha = \alpha^*$.

Proof. Since $\alpha \leqq \alpha^*$, we have $\alpha^* \leqq {}^*(\alpha^*)^*$, i.e. $\alpha^* \leqq {}^*\alpha$, so by 2.4, $\alpha = \alpha^*$.

Ordinals. The ordinals may be defined as the order types of well-ordered sets; if otherwise defined, we make this identification. The relation $\leqq$ on ordinals *is* anti-symmetric and is also linear. We use the relation $<$ only between ordinals, and, conveniently, the ordinal α is the order type of $\{\beta : \beta < \alpha\}$ under $\leqq$. The finite ordinals are identified with the natural numbers, and ω represents the first infinite ordinal.

Cardinals. In the presence of the Axiom of Choice[†], cardinals may be identified with *initial ordinals*, that is, ordinals whose set of predecessors is equivalent to the set of predecessors of no smaller ordinal. The infinite initial ordinals may be enumerated by the ordinals. ω_α denotes the α-th infinite initial ordinal ($\omega_0 = \omega$). By cardinality arguments, one can show:–

(2.6) If $\beta < \omega_\alpha$ then $\beta + \omega_\alpha = \omega_\alpha$ and $\beta \cdot \omega_\alpha = \omega_\alpha$.

[†] All sets used in the examples of this paper may be well-ordered, so the use of the Axiom of Choice may be eliminated.

$\{n : n < \omega\}$ is, with these conventions, the set of natural numbers, and $\{\alpha : \alpha < \omega_1\}$ the set of countable ordinals.

COFINALITY. For any chain $\underset{\sim}{A}$, a subset B of A is said to be cofinal in $\underset{\sim}{A}$ if for no $a \in A$ is $B \subseteq (\rightarrow, a)$. The cofinality of $\underset{\sim}{A}$ is defined to be the first ordinal β for which some subset of A of order type β is cofinal in $\underset{\sim}{A}$. If $\underset{\sim}{A}$ has order type α, the cofinality of $\underset{\sim}{A}$ is denoted by $\mathrm{cf}(\alpha)$. One can show that $\mathrm{cf}(\alpha)$ is always an initial ordinal. $\underset{\sim}{A}$ has cofinality 1 if $\underset{\sim}{A}$ has last element, 0 if A is empty, ω_α for some α otherwise.

An initial ordinal α is *regular* if $\alpha = \mathrm{cf}(\alpha)$. In particular, ω and ω_1 are regular. Clearly

$$\mathrm{cf}(\alpha) = \mathrm{cf}(\mathrm{cf}(\alpha)),$$

so:–

(2.7) The cofinality of any order type is a regular initial ordinal.

Finally, it is easy to show:–

(2.8) $$\mathrm{cf}(\alpha + \beta) = \mathrm{cf}(\beta) \text{ if } \beta \neq 0$$

(2.9) $$\mathrm{cf}(\alpha \cdot \beta) = \mathrm{cf}(\beta) \text{ if } \mathrm{cf}(\beta) \neq 1.$$

3. Examples

The first example is obtained by repeatedly inserting between adjacent elements of a chain of order type ω_1 new chains each of order type ω_1. It will be shown that certain of the sets $(\rightarrow, s)$ have cofinality ω while others have cofinality ω_1, from which it is clear that the constructed chain, with its first element removed, is not uniform. On the other hand it is shown that this chain is densely subuniform.

The third example is obtained similarly from a chain of type ω_1 by inserting chains of order types ω_1^* and ω_1 at alternate steps in some, but not all, places. The chain so formed, with its first element removed, is densely subuniform, not uniform and isomorphic to its converse.

EXAMPLE A. There is a densely subuniform chain of cardinality $\aleph_1$ which is not uniform. The converse of this chain is, in fact, uniform.

CONSTRUCTION. Let X be the set of all sequences $\langle \alpha_n \rangle_{n<\omega}$ of countable ordinals for which $\alpha_n = 0$ for all but finitely many n. Order X lexicographically, that is, if $s, t \in X$, $s = \langle \alpha_n \rangle_{n<\omega}$, $t = \langle \beta_n \rangle_{n<\omega}$, $s \neq t$ then $s \leqq t$ if, and only if, $\alpha_k < \beta_k$ where k is the first n for which $\alpha_n \neq \beta_n$. $\underset{\sim}{X} = \langle X, \leqq \rangle$, with its first element removed, is then the desired chain.

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PROOF. The assertion follows from a series of observations.

A1. $\underset{\sim}{X}$ is dense with a first but no last element. Further, if $s_1, s_2 \in X$, $s_1 < s_2$ then $s \in X$ with $s_1 < s < s_2$ may be chosen to be of either form $\langle \alpha_0, \cdots, \omega, 0, 0 \cdots \rangle$ or $\langle \alpha_0, \cdots, 1, 0, 0 \cdots \rangle$.

A2. Let $\underset{\sim}{X}$ have order type ξ. $\underset{\sim}{X} = \Sigma\{\underset{\sim}{X}_\alpha : \alpha < \omega_1\}$, where $X_\alpha = \{s \in X : s = \langle \alpha, \cdots \rangle\}$. But, from the definition of $\underset{\sim}{X}$, each $\underset{\sim}{X}_\alpha \cong \underset{\sim}{X}$, showing that $\xi = \xi \cdot \omega_1$.

A3. For any $\alpha \leqq \omega_1, \xi \cdot \alpha$ may now be computed. In fact $\xi \cdot \alpha = \xi \cdot \mathrm{cf}(\alpha) = 0, \xi$ or $\xi\omega$. This is clear for $\alpha = 0$ and, by **A2**, for $\alpha = \omega_1$. If $\mathrm{cf}(\alpha) = 1$, we may suppose $\alpha = \beta + 1$. Then $\xi \cdot \alpha = \xi(\beta + 1) = \xi \cdot \beta + \xi = \xi \cdot \beta + \xi \cdot \omega_1 = \xi(\beta + \omega_1) = \xi \cdot \omega_1 = \xi$. In the remaining case, $\mathrm{cf}(\alpha) = \omega$, and so $\alpha = \alpha_0 + 1 + \alpha_1 + 1 + \cdots$, for some $\alpha_0, \alpha_1, \alpha_2, \cdots$, and so

$$\begin{aligned}\xi \cdot \alpha &= \xi(\alpha_0 + 1 + \alpha_1 + \cdots) = \xi \cdot \alpha_0 + \xi + \xi \cdot \alpha_1 + \xi + \cdots \quad \text{by (2.3)},\\ &= \xi \cdot \alpha_0 + \xi \cdot \omega_1 + \xi \cdot \alpha_1 + \xi \cdot \omega_1 + \cdots \\ &= \xi(\alpha_0 + \omega_1) + \xi(\alpha_1 + \omega_1) + \cdots \\ &= \xi \cdot \omega_1 + \xi \cdot \omega_1 + \cdots \\ &= \xi + \xi + \cdots \\ &= \xi \cdot \omega.\end{aligned}$$

A4. By (2.9), $\mathrm{cf}(\xi \cdot \omega_1) = \omega_1$ and $\mathrm{cf}(\xi \cdot \omega) = \omega$. But, by **A2**, $\xi\omega_1 = \xi$. Hence $\mathrm{cf}(\xi) \neq \mathrm{cf}(\xi \cdot \omega)$ and so $\xi \neq \xi \cdot \omega$.

A5. If $s \in X$, where $s = \langle \alpha_0, \cdots, \alpha_n, 0, 0, \cdots \rangle, \alpha_n \neq 0$, then

$$(\rightarrow, s) = \underset{\sim}{Y}_{\alpha_0} + \underset{\sim}{Y}_{\alpha_0, \alpha_1} + \cdots + \underset{\sim}{Y}_{\alpha_0, \cdots, \alpha_n},$$

where

$$Y_{\alpha_0, \cdots, \alpha_k} = \{s \in X : s = \langle \alpha_0, \cdots, \alpha_{k-1}, \beta, \cdots \rangle, \beta < \alpha_k\}.$$

As in **A2**, $\underset{\sim}{Y}_{\alpha_0, \cdots, \alpha_k} \cong \underset{\sim}{Y}_{\alpha_k}$ which has order type $\xi \cdot \alpha_k$, showing that $(\rightarrow, s)$ has order type

$$\begin{aligned}&\xi \cdot \alpha_0 + \xi \cdot \alpha_1 + \cdots + \xi \cdot \alpha_n \\ &= \xi(\alpha_0 + \alpha_1 + \cdots + \alpha_n) \quad \text{by (2.3)} \\ &= \xi \cdot \mathrm{cf}(\alpha_0 + \alpha_1 + \cdots + \alpha_n) \quad \text{by } \mathbf{A3}, \text{ since } \quad \alpha_0 + \alpha_1 + \cdots + \alpha_n < \omega_1, \\ &= \xi \cdot \mathrm{cf}(\alpha_n) \quad \text{by (2.8)}, \\ &= \xi \quad \text{or} \quad \xi \cdot \omega.\end{aligned}$$

A6. A moment's reflection shows that consistently adding or removing

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first or last elements does not affect isomorphism between chains. Thus **A1** and **A5** show that $\underset{\sim}{X}$ without its first element is densely subuniform. **A4** shows that this is not uniform.

A7. As in **A5**, if $s = \langle \alpha_0, \cdots, \alpha_n, 0 \cdots \rangle \in X$, then $(s, \rightarrow)$ has order type

$$\begin{aligned} &\cdots + \xi \cdot \omega_1 + \xi \cdot \omega_1 \\ &= \cdots + \xi + \xi, \quad \text{by } \mathbf{A2}, \\ &= \xi \cdot \omega^*. \end{aligned}$$

Thus $\underset{\sim}{X}$ has its converse uniform. Incidentally, taking s to be the first element of $\underset{\sim}{X}$ shows that $\xi = 1 + \xi \cdot \omega^*$.

EXAMPLE B. There are densely subuniform chains of cardinality $\aleph_1$ which are not uniform and whose converses are not uniform.

CONSTRUCTION. It is sufficient to modify Example A and consider any chain of order type $\xi\alpha^*$ where α is an ordinal of the form ω^γ and $\gamma > 1$.

PROOF. The dense subuniformity follows from that of ξ and from the uniformity of α^*. The non-uniformity of the converse involves considering limits of uncountable ascending sequences, and is omitted. The statement is, in any case, weaker than that of the next example.

EXAMPLE C. There is a densely subuniform chain which is not uniform and which is isomorphic to its converse.

CONSTRUCTION. As in example A, but more complicated. Let Z be the set of sequences $\langle \alpha_0, \beta_0^*, \alpha_1, \beta_1^*, \cdots, \alpha_n, \beta_n^*, \cdots \rangle$ of order types for which

(i) each α_n, β_n is a countable ordinal,

(ii) for all but finitely many $n, \alpha_n = \beta_n = 0$,

(iii) if α_n is a limit ordinal or zero, then $\alpha_k = 0$ for $k > n$ and $\beta_k = 0$ for $k \geqq n$,

(iv) if β_n is a limit ordinal or zero, then $\alpha_k = \beta_k = 0$ for $k > n$.

Order Z "lexicographically" in the sense that if $s, t \in Z, s \neq t$ then $s \leqq t$ if, and only if

$$s = \langle \alpha_0, \beta_0^*, \cdots, \alpha_n, \beta_n^*, \cdots \rangle, \; t = \langle \gamma_0, \delta_0^*, \cdots, \gamma_n, \delta_n^*, \cdots \rangle,$$

where $\alpha_k = \gamma_k$ and $\beta_k = \delta_k$ for $k < n$ and either $\alpha_n < \gamma_n$ or both $\alpha_n = \gamma_n$ and $\beta_n > \delta_n$. $\underset{\sim}{Z}$, again without its first element, is the desired chain. The restrictions (iii) and (iv) will be required in **C5**, below, and without them, $\underset{\sim}{Z}$ would be uniform.

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PROOF. In sketch, corresponding to that of Example A.

C1. $\underset{\sim}{Z}$ is dense with a first but no last element. If $s_1, s_2 \in Z$, $s_1 < s_2$, then $s \in Z$ with $s_1 < s < s_2$ may be chosen to be of either of the forms

$$\langle \alpha_0, \beta_0^*, \cdots, \alpha_n, \beta_n^*, \omega, 0, \cdots \rangle \quad \text{or}$$
$$\langle \alpha_0, \beta_0^*, \cdots, \alpha_n, \beta_n^*, 1, 0, \cdots \rangle.$$

C2. Let $\underset{\sim}{Z}$ have order type ζ. As in **A2**, but allowing for the complexities of the definition of Z, one has $\zeta = (1 + \zeta^* \cdot \omega)\omega_1$. This follows from the observations that if $\underset{\sim}{Z}_\alpha$ is defined as in **A2**, then $\underset{\sim}{Z}_\alpha$ has order type 1 if α is a limit ordinal or zero, order type ζ^* otherwise, and $\underset{\sim}{Z} \cong \Sigma\{\Sigma\{\underset{\sim}{Z}_\alpha : \omega\beta \leqq \alpha < (\omega + 1)\beta\} : \beta < \omega_1\}$. Now put $\zeta = 1 + \sigma$, since $\underset{\sim}{Z}$ has a first element. Then $\zeta^* = \sigma^* + 1$, and

$$\begin{aligned} 1 + \sigma = \zeta &= (1 + \zeta^* \cdot \omega) + (1 + \zeta^* \cdot \omega)\omega_1 \\ &= 1 + (\sigma^* + 1)\omega + (1 + \zeta^* \cdot \omega)\omega_1. \end{aligned}$$

Thus $\sigma = (\sigma^* + 1) + (\sigma^* + 1)\omega + (1 + \zeta^* \cdot \omega)\omega_1$, showing that $\sigma^* \leqq \sigma$ and so by (2.5), $\sigma^* = \sigma$. Now

$$1 + \zeta^* \cdot \omega = 1 + (\sigma^* + 1)\omega = (1 + \sigma^*)\omega = (1 + \sigma)\omega = \zeta \cdot \omega.$$

So $\zeta = (1 + \zeta^* \cdot \omega)\omega_1 = (\zeta \cdot \omega)\omega_1 = \zeta(\omega \cdot \omega_1) = \zeta \cdot \omega_1$ by (2.6).

C3. As in **A3**, $\zeta \cdot \alpha = \zeta \cdot \mathrm{cf}(\alpha)$ for $\alpha \leqq \omega_1$, since only the fact that $\zeta \cdot \omega_1 = \zeta$ is used.

C4. As in **A4**, $\zeta \neq \zeta \cdot \omega$.

C5. Further calculation along the lines of **C2**, **C3** and **A5** shows that for $s \in Z$, $(\rightarrow, s)$ has order type ζ if

$$s = \langle \alpha_0, \beta_0^*, \cdots, \alpha_n, \beta_n^*, 0, \cdots \rangle, \beta_n \neq 0$$

or

$$s = \langle \alpha_0, \beta_0, \cdots, \alpha_n, 0, \cdots \rangle,$$

where α_n is a successor ordinal, while $(\rightarrow, s)$ has order type $\zeta \cdot \omega$ if

$$s = \langle \alpha_0, \beta_0^*, \cdots, \alpha_n, 0, \cdots \rangle$$

where $\alpha_n \neq 0$ and α_n is a limit ordinal.

C6. As in Example A, **C1** and **C5** show that $\underset{\sim}{Z}$ without its first element is densely subuniform, and **C4** shows that it is not uniform.

C7. $\underset{\sim}{Z}$ without first element has order type σ. Also $\sigma = \sigma^*$ from **C2**.

4. Conclusion

Other examples of densely subuniform chains may be constructed after this fashion. It would be of interest to know if any general results are possible on the structures of uniform and densely subuniform chains.

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References

H. Bachmann (1967), *Transfiniten Zahlen* (Springer, Berlin, 2nd edition).

W. D. Munn (1966), 'Uniform semilattices and bisimple inverse semigroups', *Quart. J. Math. Oxford* (2), **17**, 151–159.

W. D. Munn (1970), 'Fundamental inverse semigroups', *Quart. J. Math. Oxford* (2), **21**, 157–170.

W. D. Munn (1974) (CFIS), 'Congruence-free inverse semigroups', *Quart. J. Math. Oxford* (2), **25**, 463–484.

W. Sierpinski (1965), *Cardinal and ordinal numbers* (2nd rev. edition, Panstowe Wyndawnictwo Naukowe, Warsaw).

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