Canad. Math. Bull. Vol. **18** (3), 1975

PERMANENTS OF (0, 1)-MATRICES HAVING AT MOST TWO ZEROS PER LINE

BY
J. R. HENDERSON

SUMMARY. Let U_n denote the nth ménage number. Within the class of order n matrices of zeros and ones with at most two zeros in every row and column the minimum permanent is U_n when n is even and $-1+U_n$ when n is odd.

If $A=(a_{ij})$ is a real order n matrix, the permanent of A (per A) is defined to be $\sum a_{1\pi(1)}a_{2\pi(2)}\cdots a_{n\pi(n)}$, the sum being over all permutations $\pi \in S_n$, the symmetric group on n letters. Permanents have considerable combinatorial interest, a result in part due to the fact that

$$\text{per } A = \text{per } B \tag{1}$$

when A, B are "combinatorially equivalent", i.e., when there exist permutation matrices P, Q such that $B=PAQ$. For example, the "problème des ménages" asks for the number (M_n) of ways $2n$ symbols $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ may be arranged in a circle so that the a's and b's occur alternately but a_i, b_i are not adjacent for any $i=1, 2, \ldots, n$. The answer is $M_n=2n!\, U_n$ where an explicit formula for the nth ménage number U_n is known [4] but equivalently, it may be defined as the permanent of the order n (0, 1)-matrix having exactly two zeros in every line (row or column),

$$\begin{bmatrix} 0 & 0 & & & & & & & & & \\ 0 & 1 & 0 & & & & & & & 1 & \\ & 0 & 1 & \cdot & & & & & & & \\ & & \cdot & \cdot & \cdot & & & & & & \\ & & & \cdot & \cdot & \cdot & & & & & \\ & & & & \cdot & \cdot & \cdot & & & & \\ & & & & & \cdot & \cdot & \cdot & & & \\ & & & & & & \cdot & \cdot & \cdot & & \\ & & & & & & & \cdot & 1 & 0 & \\ & 1 & & & & & & & 0 & 1 & 0 \\ & & & & & & & & & 0 & 0 \end{bmatrix} \tag{2}$$

or of any combinatorially equivalent matrix [3].

Further interest in permanents is fostered by various unresolved conjectures concerning maximum and minimum permanent values within certain classes of

Received by the editors December 31, 1973.

matrices, e.g., [2]. In particular, the van der Waerden conjecture remains unresolved [1]. Let $\vartheta(n, k)(\mathscr{B}(n, k))$ denote the class of all order n(0, 1)-matrices having exactly $n-k$ (at most $n-k$) zeros per line. It is the purpose of this paper to determine the minimum permanent within $\mathscr{B}(n, n-2)$. Specifically we shall show:

THEOREM 1. *The minimum permanent in* $\mathscr{B}(n, n-2)$ *is* U_n *for n even and* $-1+U_n$ *for n odd.*

Before turning to the proof we shall make a simplification (Lemma 1) and obtain some useful formulae (Lemma 2). Use will be made of the fact that members of $\vartheta(n, n-2)$ are combinatorially equivalent to (0, 1)-complements of the direct sum of (0, 1)-complements of matrices of type (2) of orders $p_1, p_2, \ldots, p_v$ where $n=p_1+p_2+\cdots+p_v$ is a partition of n with all $p_i \geq 2$.

LEMMA 1. *The minimum permanent in* $\mathscr{B}(n, n-2)$ *can be found in the union of the following two subclasses of* $\mathscr{B}(n, n-2)$:

(i) $\vartheta(n, n-2)$

(ii) *the class of order n matrices of the form:*

$$\begin{bmatrix} 0 & 1 \quad 1 \quad \cdot \quad \cdot \quad \cdot \quad 1 \\ 1 & \\ 1 & \\ \cdot & B \\ \cdot & \\ \cdot & \\ 1 & \end{bmatrix} \tag{3}$$

with $B \in \vartheta(n-1, n-3)$. (*Note, since the constant line sum of B is two less than its order, all combinatorially inequivalent forms for B have been implicitly described above.*)

Proof. If A has minimum permanent within the class $\mathscr{B}(n, n-2)$ then we can assume that every one in A is in some line with sum exactly equal to $n-2$, else such a one can be removed without change in the permanent value. For convenience we shall say such a reduced A has "property R". Now suppose $A \notin \vartheta(n, n-2)$. Clearly A cannot have a line, say a row, of sum n since by property R every column would contain two zeros and hence since the matrix is square, some row would contain more than two zeros, i.e., $A \notin \mathscr{B}(n, n-2)$. It follows that A has both a row and a column with sum $n-1$. Let the first column of A be such a column, taking its zero to be in the first row. By property R the last $n-1$ rows have row sums equal to $n-2$ so the first row must be the one with sum $n-1$. Hence A is of the form (3) with the row sums of B equal to $n-3$. Repeating the argument for the column sums of B we conclude $B \in \vartheta(n-1, n-3)$.

It is obvious that an even simpler application of property R solves the analogous problem of the minimum permanent in $\mathscr{B}(n, n-1)$. Namely, the minimum permanent in this class is D_n, the nth derangement number, for which an explicit formula is known but which can be equivalently defined as the permanent of the (0, 1)-complement of the order n identity matrix [3].

By a "list" we mean a finite unordered set of positive integers. A particular list of length μ will be denoted either as

$$[p_1, p_2, \ldots, p_\mu] \tag{4}$$

or, more commonly as

$$(p_2, p_2, \ldots, p_\mu). \tag{5}$$

The notation (5) will be used rather than (4) when some terms have possibly been suppressed, i.e.,

$$(p_1, p_2, \ldots, p_\mu) = [p_1, p_2, \ldots, p_\mu, p_{\mu+1}, \ldots, p_{\mu+\nu}]$$

for some non-negative integer ν and for some positive integers $p_{\mu+1}, \ldots, p_{\mu+\nu}$. By $U[p_1, p_2, \ldots, p_\mu]$ we shall denote the permanent of the matrix which is the (0, 1)-complement of the direct sum of (0, 1)-complements of matrices of type (2) of orders $p_1, p_2, \ldots, p_\mu$, respectively. If some $p_i=1$, the corresponding summand will be an order one zero matrix. By (1) the value $U[p_1, p_2, \ldots, p_\mu]$ is independent of the ordering of the p_i's and by Lemma 1 $U[p_1, p_2, \ldots, p_\mu]$, for some partition $n=p_1+p_2+\cdots+p_\mu$ of n, is the minimum permanent wanted. $U(p_1, p_2, \ldots, p_\mu)$ has the same definition as $U[p_1, p_2, \ldots, p_\nu]$, the notation implying that indication of some matrix summands might have been suppressed. In particular, an equation involving $U(p_1, p_2, \ldots, p_\mu)$, $U(q_1, q_2, \ldots, q_\nu)$, etc., holds when identical, arbitrary (positive integer) terms are adjoined to all lists.

LEMMA 2. *If* $k>1$, $l>1$,

$$U(k, l) = U(k+l)+2\sum_{i=1}^{k+l-1} U(i)-\sum_{i=1}^{k-1} U(i, l)-\sum_{i=1}^{l-1} U(k, i). \tag{6}$$

If $k=1$, $l>1$, (6) *has the modified form*

$$\sum_{i=1}^{l} U(1, i) = 3\sum_{i=1}^{l-1} U(i)+2U(l)+U(l+1). \tag{7}$$

If $l>2$,

$$U(1, l) = U(l-1)+U(l)+U(l+1). \tag{8}$$

Proof. If the zero in the (1, 1) position of the matrix A_n in (2) is replaced by a one, the new permanent value, per A_n^1, is

$$U_n+\text{per } A_{n-1}^1$$

so by induction,

$$\text{per } A_n^1 = \sum_{i=1}^{n} U_i. \tag{9}$$

https://doi.org/10.4153/CMB-1975-064-6 Published online by Cambridge University Press

Moreover, this is the permanent of any matrix obtained by replacing a zero of A_n by a one since all such matrices are combinatorially equivalent. Let $k>1$, $l>1$, and consider the matrices which define $U[k, l]$, $U[k+l]$:

$$(10)\qquad \begin{array}{c} k \\ \\ l \end{array}\left[\begin{array}{cccccc|cccccc}
0 & 0 & & & & & & & & & & \\
0 & 1 & \cdot & & 1 & & & & & & & \\
 & \cdot & \cdot & \cdot & & & & & 1 & & & \\
 & & \cdot & \cdot & \cdot & & & & & & & \\
 & & & \cdot & \cdot & \cdot & & & & & & \\
1 & & & \cdot & 1 & 0 & & & & & & \\
 & & & & 0 & \boxed{0} & \boxed{1} & & & & & \\
\hline
 & & & & & \boxed{1} & \boxed{0} & 0 & & & & \\
 & & & & & & 0 & 1 & \cdot & & 1 & \\
 & & & & & & & \cdot & \cdot & \cdot & & \\
 & & 1 & & & & & & \cdot & \cdot & \cdot & \\
 & & & & & & & & & \cdot & \cdot & \cdot \\
 & & & & & & 1 & & & \cdot & 1 & 0 \\
 & & & & & & & & & & 0 & 0
\end{array}\right]
\begin{array}{c} k \\ \\ l \end{array}\left[\begin{array}{cccccc|cccccc}
0 & 0 & & & & & & & & & & \\
0 & 1 & \cdot & & 1 & & & & & & & \\
 & \cdot & \cdot & \cdot & & & & & 1 & & & \\
 & & \cdot & \cdot & \cdot & & & & & & & \\
 & & & \cdot & \cdot & \cdot & & & & & & \\
1 & & & \cdot & 1 & 0 & & & & & & \\
 & & & & 0 & \boxed{1} & \boxed{0} & & & & & \\
\hline
 & & & & & \boxed{0} & \boxed{1} & 0 & & & & \\
 & & & & & & 0 & 1 & \cdot & & 1 & \\
 & & & & & & & \cdot & \cdot & \cdot & & \\
 & & 1 & & & & & & \cdot & \cdot & \cdot & \\
 & & & & & & & & & \cdot & \cdot & \cdot \\
 & & & & & & 1 & & & \cdot & 1 & 0 \\
 & & & & & & & & & & 0 & 0
\end{array}\right]$$

These matrices differ only in an order 2 submatrix as indicated. By comparing the sets of $k+l$ ones, no two in a line, occurring in the two matrices (10) and by making use of (9), we obtain the identity (6). This fails if $k=1$, but a slight modification of the comparison gives

$$U(1, l)-\sum_{i=1}^{l-1} U(i)-2\sum_{i=1}^{l} U(i) = U(l+1)-\sum_{i=1}^{l-1} U(1, i)$$

which is a rearrangement of (7). If $l-1>1$ we can rewrite (7) with $l-1$ replacing l. Subtraction of these two forms of (7) then gives (8).

By (8) $U(1, l)>U(l+1)$ for $l>2$ so that except possibly for $U(1, 1)$, $U(1, 2)$, permanent values smaller than $U(1, l)$ can be found among the values $U(l+1)$. Theorem 1 will now be proven. First we show $U[p_1, p_2, \ldots, p_\mu]\geq U[q]$ if

$$q = \sum_{i=1}^{\mu} p_i,$$

unless $\mu=2$ and $|p_1-p_2|=1$. Finally we complete the proof by showing $U[k, k+1]=-1+U[2k+1]$.

Proof of Theorem 1. Our first reduction is to show:

$$(11)\qquad U(k, l) = U(l+k)+U(l-k) \quad \text{if} \quad k > 1 \quad \text{and} \quad l > k+1.$$

Subtracting expressions for $U(2, 4)$, $U(2, 3)$ as obtained from (6) we have

$$U(2, 4)-U(2, 3) = U(6)+U(5)+U(1, 3)-U(1, 4)-U(2, 3)$$

and using (8) this simplifies to (11) when $k=2$ and $l=4$. The same approach works

https://doi.org/10.4153/CMB-1975-064-6 Published online by Cambridge University Press

for $U(k, l)$ by induction. Assume the result for $U(k', l')$ for all pairs $k', l'(k'>1$ and $l'>k'+1)$ satisfying $k'<k$ or $l'<l$ if $k'=k$. Again by (6),

$$\begin{aligned} U(k, l)-U(k, l-1) &= U(k+l)+U(k+l-1)-U(1, l) \\ &\quad -\sum_{i=2}^{k-1} U(i, l)+U(1, l-1)+\sum_{i=2}^{k-1} U(i, l-1)-U(k, l-1). \end{aligned} \tag{12}$$

Using (8) for the terms $U(1, l)$, $U(1, l-1)$ and applying the induction hypothesis to the two summations in (12) we obtain the result wanted.

The proof of Theorem 1 has now been reduced to a consideration of those lists of the form

$$[k, k, \ldots, k, k+1, k+1, \ldots, k+1] \quad \text{with} \quad k \geq 1.$$

We next note that:

$$U(k, k) \geq U(2k) \quad \text{if} \quad k \geq 1. \tag{13}$$

Applying (11) to (6) and making use of (8) (assuming $k\geq 3$), we have

$$U(k, k) = U(2k)+2U(1)+2U(2k-1)-2U(k-1, k) \tag{14}$$

$$U(k, k+1) = U(2k+1)+2U(1)+U(2k-1)+U(2k)-U(k-1, k)-U(k, k). \tag{15}$$

Equating the two expressions for $U(k, k)-U(2k)-2U(1)$ as obtained from (14), (15) we get

$$U(k, k+1)-U(2k+1) = U(k-1, k)-U(2k-1). \tag{16}$$

So (14) becomes

$$U(k, k)-U(2k) = 2\{U(1)+U(5)-U(2, 3)\}, \qquad k \geq 3 \tag{17}$$

after successive applications of (16). For $k=2$ (14), (15) must be modified, but similarly equating the analogous expressions for $U(2, 2)-U(4)-2U(1)$ we obtain

$$U(2, 3)-U(5) = U(1, 2)-U(3)-U(2).$$

Consequently, for $k>1$, (17) becomes

$$U(k, k)-U(2k) = 2\{U(1)+U(2)+U(3)-U(1, 2)\}$$

and using (7) for $l=2$,

$$U(k, k)-U(2k) = 2\{U(1, 1)-2U(1)-U(2)\}.$$

If we consider the terms in the permanent value $U(1, 1)$ we have

$$U(1, 1) = U(2)+2U(1)+U(\ ) \tag{18}$$

and therefore (17), for $k>1$ can be written

$$U(k, k)-U(2k) = 2U(\ )$$

so that $U(k, k)-U(2k)\geq 0$ if $k>1$ and the same result follows from (18) for $k=1$.

To complete the proof of Theorem 1 we note that $U[k, k+1]=-1+U[2k+1]$ for $k\geq 1$. This is readily checked for $k=1, 2$ and for $k\geq 3$ (16) gives us

$$U[k, k+1]-U[2k+1] = U[2, 3]-U[5] = -1.$$

A final point should be noted: For all $n\neq 3$, the minimum permanent in $\mathscr{B}(n, n-2)$ is attained in $\vartheta(n, n-2)$.

References

1. Andrew M. Gleason, *Remarks on the van der Waerden permanent conjecture*, J. Combinatorial Theory, **8** (1970), 54–64.

2. W. B. Jurkat and H. J. Ryser, *Matrix factorizations of determinants and permanents*, J. Algebra, **3** (1966), 1–27.

3. H. J. Ryser, *Combinatorial mathematics*, Wiley, New York, 1963.

4. J. Touchard, *Sur un problème de permutations*, C. R. Acad. Sci., Paris, **198** (1934), 631–633.

California State University, Northridge
Northridge, California 91324
U.S.A.

https://doi.org/10.4153/CMB-1975-064-6 Published online by Cambridge University Press