11

Mesonic spectrum from current algebra

11.1 Introduction

In this chapter we study the mesonic spectrum of various QCD_2 theories. The main idea is to use the current algebra of the underlying ungauged theories. In addition we combine the bosonization techniques developed in Chapter 6 with that of a large N expansion of Chapter 7 and a light-front quantization as in Chapter 10. We will focus our attention on the massive mesonic spectrum of conformal field theories coupled to non-abelian gauge fields. In particular massless multi-flavor fundamental quarks and adjoint quarks that will be shown to correspond to the particular case of $N_f = N_c$.

First a universality theorem, that states that the massive mesonic spectrum does not depend on the representation of the matter field but rather only on its ALA level, will be derived, following Kutasov and Schwimmer [148].

We then present a detailed determination of the massive mesonic spectrum using a 't Hooft-like equation for the wave functions of "currentballs" states. We will discuss in particular the special cases of $N_f = 1, N_f = N_c$ and $N_f \gg N_c$. The last section is devoted to the spectrum of states built by the action of a single current creation operator on the adjoint vacuum. In both cases it will be shown that the bosonization approach leads to the introduction of current quanta as the basic degrees of freedom. Once the mass operator $P^+P^- = M^2$ is expressed in terms of the current quanta, the bosonization has already left the scene.

The main content of this chapter, the mesonic spectrum from current algebra, is based on [17].[1] The spectrum based on the adjoint vacuum was introduced in [3].

11.2 Universality of conformal field theories coupled to YM_2

So far we have mainly discussed the coupling of matter in the fundamental representation to the two-dimensional YM fields. Obviously one can also couple other matter fields to these non-abelian gauge fields. A natural class of matter theories that one would like to gauge are the conformal field theories which admit on top of the Virasoro algebra also an affine Lie algebra structure. These theories which are characterized by the corresponding Lie algebra G and the level k of the

[1] This was previously also discussed in [18].

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affine Lie algebra, are candidates for coupling to non-abelian gauge fields of the group G. A particular family of such theories are the WZW models, invariant under $G \times G$ of level k. We have discussed in Chapter 6 the gauging of such models. In this chapter we would like to address the issue of the spectrum of such gauged conformal field theories, and in particular the massive sector of the spectrum. In general the Lagrangian density of such a theory reads,

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_{CFT} - \frac{1}{2e^2}\mathrm{Tr}\left[F_{\mu\nu}^2\right] + \mathcal{L}_I \\ &= \mathcal{L}_{CFT} - \frac{1}{2e^2}\mathrm{Tr}\left[(\partial_- A_+)^2\right] + \mathrm{Tr}\left[A_+ J_-\right] \\ &= \mathcal{L}_{CFT} - \frac{e^2}{2}\mathrm{Tr}\left[J^+ \frac{1}{\partial_-^2} J^+\right] = \mathcal{L}_{CFT} - \frac{e^2}{2}\mathrm{Tr}\left[J \frac{1}{\partial^2} J\right],\end{aligned} \tag{11.1}$$

where we have used the light-cone gauge $A_- = 0$. We will be using the notation of J for J_- and $\bar{J}$ for J_+, and similar for other holomorphic and anti-holomorphic quantities.

A conformal field theory invariant under the symmetry generated by a G ALA has holomorphic currents J^a in the adjoint representation of G, as well as anti-holomorphic currents also in the adjoint representation of G. In general the holomorphic currents obey an ALA with level k and the anti-holomorphic currents an ALA of level $\bar{k}$. However, gauging the conformal theory requires vanishing of the chiral anomaly, namely it requires that,

$$k = \bar{k}. \tag{11.2}$$

Next we quantize the system on the light-front. This framework is very convenient since both momenta P^- and P^+, or equivalently P and $\bar{P}$, can be expressed in terms of J only (with no reference to $\bar{J}$). This decoupling of one sector (the anti-holomorphic one) can be attributed to the fact that in a frame moving to the right with the speed of light there is no way to interact with massless left-moving particles. The light-cone Hamiltonian is given by,

$$\begin{aligned}P^+ &= \frac{1}{[C(G)+k]}\int \mathrm{d}x^- \, : J^a(x^-) J^a(x^-) \\ &= \sum_{n=1}^{\infty} \frac{1}{n^2} J_{-n}^a J_n^a,\end{aligned} \tag{11.3}$$

where in the last line we have assumed that the light-cone space direction $x^- = z$ has been put on a circle. Thus the Hamiltonian acts inside current blocks, and the problem of finding the massive spectrum splits into diagonalizing the decoupled blocks of P^+ on global G singlets. We want to emphasize again that the light-front dynamics is fully independent of the anti-holomorphic sector, apart from the constraint that $k = \bar{k}$. This clearly means that we can replace the anti-holomorphic sector with another anti-holomorphic sector, provided that

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the latter has a level that equals k. Obviously we could have fixed the opposite gauge $A_+ = 0$, leaving only the anti-holomorphic sector with currents $\bar{J}$. In that gauge we could have replaced the holomorphic sector with another one, again provided that it has level k. Thus we conclude that *the massive spectrum does not depend on the representations r and $\bar{r}$, but only on the gauge group G and the level k.*

We would like to demonstrate this universality in the context of a generalization of Schwinger's model, which contains n^{R} right moving fermions ψ_i^{R} $i = 1 \ldots n^{\mathrm{R}}$ and n^{L} left-moving fermions ψ_i^{L} $i = 1 \ldots n^{\mathrm{L}}$ [120]. Both the right- and left-moving fermions are charged with respect to an abelian $U(1)$ gauge symmetry with charges q_i^{R} and q_i^{L} respectively. The system is described by the Lagrangian density,

$$\mathcal{L} = \psi_i^{\mathrm{R}\,\dagger} \bar{\partial} \psi_i^{\mathrm{R}} + \psi_i^{\mathrm{L}\,\dagger} \partial \psi_i^{\mathrm{L}} + \bar{A} J - \frac{1}{4e^2} (\partial \bar{A})^2, \tag{11.4}$$

where $J = \sum_i^{n^{\mathrm{R}}} q_i^{\mathrm{R}} \psi_i^{\mathrm{R}\,\dagger} \psi_i^{\mathrm{R}}$ and we are using the gauge $A = 0$. Upon integrating $\bar{A}$ we get,

$$\mathcal{L} = \psi_i^{\mathrm{R}\,\dagger} \bar{\partial} \psi_i^{\mathrm{R}} + \psi_i^{\mathrm{L}\,\dagger} \partial \psi_i^{\mathrm{L}} - e^2 J \frac{1}{\partial^2} J. \tag{11.5}$$

We can now bosonize the system. Note that the fermions at hand are not Dirac fermions but rather n^{R} right and n^{L} left chiral fermions. The system is consistent in the sense that there is no chiral anomaly when,

$$k^{\mathrm{R}} \equiv \sum_{i=1}^{N^{\mathrm{R}}} q_i^{\mathrm{R}} \quad k^{\mathrm{L}} \equiv \sum_{i=1}^{N^{\mathrm{L}}} q_i^{\mathrm{L}} \quad k^{\mathrm{R}} = k^{\mathrm{L}} = k. \tag{11.6}$$

One can use the prescription for chiral bosonization described in Section 6.4. In fact it is enough to note that the interaction term takes the form,

$$\mathcal{L}_{\mathrm{int}} = -e^2 J \frac{1}{\partial^2} J = e^2 (\phi)^2, \tag{11.7}$$

where $\phi = \sum_i^{N^{\mathrm{R}}} q_i^{\mathrm{R}} \phi_i^{\mathrm{R}} = \sum_i^{N^{\mathrm{L}}} q_i^{\mathrm{L}} \phi_i^{\mathrm{L}}$ and ϕ^{L} and ϕ^{L} are the right and left chiral bosons that corresponds to the right and left chiral fermions. Thus we conclude that the spectrum includes one massive mode corresponding to ϕ plus $n^{\mathrm{R}} - 1$ and $n^{\mathrm{L}} - 1$ massless right- and left-moving particles, respectively. It is now evident that indeed in accordance with the universality theorem, the massive sector does not depend on the explicit sequence of charges q_i^{R} and q_i^{L} but only on the combination expressed in ϕ.

Another example of the universality theorem is the case of adjoint fermions. The ALA associated with the currents built from the adjoint fermions $J^{ab} = \psi^{ac} \psi^{cb}$ is of level N_c. The CFT based on a WZW model of $SU(N_c)$ of level $k = N_c$ is another theory with the same ALA, and hence the massive sector of the spectrum of these theories should, according to the theorem, be the same.

In the next section we describe the massive spectrum of such models based on a 't Hooft-like equation for the currents.

11.3 Mesonic spectra of two-current states

In this section we derive the massive meson spectrum built from two current creation operators acting on the vacuum. In the next section we will discuss states constructed from a single current acting on the adjoint vacuum.

The first step in the determination of the spectrum is the derivation of a 't Hooft-like equation for the wave functions of the "currentball" states, at arbitrary level N_f. This equation should interpolate between the description of a single flavor ('t Hooft model), the model $N_f = N_c$ equivalent to adjoint fermions and the large N_f limit. We will argue that the equation obtained suggests that the underlying degrees of freedom in the problem are interacting "gluons" with mass $\frac{e^2 N_f}{\pi}$. Actually, these are related to the color currents, but are color singlets.

Then we will solve the equation for the lowest massive state. Whereas the 't Hooft model $N_f = 1$ is exactly solvable, the multi-flavor case with $N_f > 1$ is not solvable even in the Veneziano limit when both N_c and N_f are taken to infinity (with a fixed ratio), since pair creation and annihilation are not suppressed.

For the case of the adjoint quarks, the results derived using the current quanta will be shown to be compatible with those computed with fermions as the basic degrees of freedom discussed in Chapter 12. For large N_f it will be shown that the exact massive spectrum is a single particle with $M^2 = \frac{e^2 N_f}{\pi}$. This phenomenon is explained by the fact that this limit can be viewed as an "abelianization" of the model.

11.3.1 The basic setup

We now establish the basic setup. We start with the fermionic formulation of the various theories, impose the light-cone gauge, introduce the bosonized version and finally write down the mass operator.

In Section 8.4 the classical theory of QCD_2 with Dirac fermions in the fundamental representation was described. Here we will address this case as well as massless Majorana fermions in the adjoint representation. Recall that these theories are described by the following classical Lagrangian:

$$\mathcal{L} = -\mathrm{Tr}\left[\frac{1}{2e^2}F_{\mu\nu}^2 + i\bar{\Psi}\not{D}\Psi\right], \tag{11.8}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$ and the trace is over the color and flavor indices. For case (i) Ψ has the group structure Ψ_{ia} where $i = 1,\ldots,N_c$ and $a = 1,\ldots,N_f$ with $D_\mu = \partial_\mu - iA_\mu$, whereas for case (ii) $\Psi \equiv \Psi^i_j$ and $D_\mu = \partial_\mu - i[A_\mu,\ ]$. In both cases Ψ is two-spinor parametrized as $\Psi = \binom{\bar\psi}{\psi}$.

As we have seen in Chapter 10 it is useful to handle these models in the framework of light-front quantization, namely, to use light-cone space-time coordinates and to choose the chiral gauge $A_- = 0$. In this scheme the Lagrangian takes the form,

$$\mathcal{L} = -\frac{1}{2e^2}(\partial_- A_+)^2 + i\psi^\dagger \partial_+ \psi + i\bar{\psi}^\dagger \partial_- \bar{\psi} + A_+ J^+, \tag{11.9}$$

where color and flavor indices were omitted and J^+ denotes the $+$ component of the color current $J^+ \equiv \psi^\dagger \psi$. This Lagrangian density is identical to (11.4) when one replaces the complex coordinates with light-cone ones.

By choosing x^+ to be the 'time' coordinate it is clear that A_+ and $\bar{\psi}$ are non-dynamical degrees of freedom. In fact, $\bar{\psi}$ are decoupled from the other fields, so in order to extract the physics of the dynamical degrees of freedom, one has to functionally integrate over A_+. The result of this integration is the following simplified Lagrangian,

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I = i\psi^\dagger \partial_+ \psi + i\bar{\psi}^\dagger \partial_- \bar{\psi} - \frac{e^2}{2} J^+ \frac{1}{\partial_-^2} J^+. \tag{11.10}$$

Since our basic idea is to solve the system in terms of the "quanta" of the colored currents, it is natural to introduce bosonization descriptions of the various fields.

(i) As was discussed in Section (9.3.2), the bosonized action of colored-flavored Dirac fermions in the fundamental representation is expressed in terms of a WZW action of a group element $u \in U(N_c \times N_f)$, with an additional mass term that couples the color, flavor and baryon number sectors. In the massless case when the latter term is missing, the action takes the form,

$$S_0^{\text{fund}} = S_{(N_f)}^{\text{WZW}}(g) + S_{(N_c)}^{\text{WZW}}(h) + \frac{1}{2}\int \mathrm{d}^2 x \partial_\mu \phi \partial^\mu \phi, \tag{11.11}$$

where $g \in SU(N_c)$, $h \in SU(N_f)$ and $\mathrm{e}^{i\sqrt{\frac{4\pi}{N_c N_f}}\phi} \in U_B(1)$, with $U_B(1)$ denoting the baryon number symmetry, and the WZW action was given in Section 4.1.

(ii) The current structure of free Majorana fermions in the adjoint representation can be recast in terms of a WZW action of level $k = N_c$, namely $S_0^{\text{adj}} = S_{(N_c)}^{\text{WZW}}(g)$, where now g is in the adjoint representation of $SU(N_c)$, so that it carries a conformal dimension of $\frac{1}{2}$. Multi-flavor adjoint fermions can be described as $S_{N_f}^{\text{WZW}}(g) + S_{N_c^2-1}^{\text{WZW}}(h)$ where $g \in SO(N_c^2-1)$ and $h \in SO(N_f)$. In the present work we discuss only gauging of $SU(N_c)$ WZW so the latter model would not be considered.

Substituting now S_0^{fund} or S_0^{adj} for S_0 the action that corresponds to 11.10 becomes,

$$S = S_0 - \frac{e^2}{2}\int \mathrm{d}^2 x J^+ \frac{1}{\partial_-^2} J^+, \tag{11.12}$$

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where the current J^+ now reads $J^+ = i\frac{k}{2\pi} g\partial_- g^\dagger$, and the level $k = N_f$ and $k = N_c$ for the multi-flavor fundamental and adjoint cases, respectively.

The light-front quantization scheme is very convenient because the corresponding momenta generators P^+ and P^- can be expressed only in terms of J^+. We would like to emphasize that this holds only for the massless case.

Using the Sugawara construction, the contribution of the colored currents to the momentum operator P^+ takes the simple form,

$$P^+ = \frac{1}{N_c + k} \int \mathrm{d}x^- : J^i_j(x^-) J^j_i(x^-) :, \tag{11.13}$$

where $J \equiv \sqrt{\pi} J^+$, N_c in the denominator is the second Casimir operator of the adjoint representation and the level k takes the values mentioned above. Note that for future purposes we have added the color indices $i, j = 1 \dots N_c$ to the currents. In the absence of the interaction with the gauge fields the second momentum operator P^- vanishes. For the various QCD_2 models it is given by,

$$P^- = -\frac{e^2}{2\pi} \int \mathrm{d}x^- : J^i_j(x^-) \frac{1}{\partial_-^2} J^j_i(x^-) : . \tag{11.14}$$

In order to find the massive spectrum of the model we should diagonalize the mass operator $M^2 = 2P^+P^-$. Our task is therefore to solve the eigenvalue equation,

$$2P^+P^-|\psi\rangle = M^2|\psi\rangle. \tag{11.15}$$

We write P^+ and P^- in term of the Fourier transform of $J(x^-)$, defined by,

$$J(p^+) = \int \frac{\mathrm{d}x^-}{\sqrt{2\pi}} \mathrm{e}^{-ip^+x^-} J(x^-).$$

Normal ordering in the expressions of P^+ and P^- are naturally with respect to p, where $p < 0$ denotes a creation operator, and to simplify the notation we will write from here on p instead of p^+. In terms of these variables the momenta generators are,

$$P^+ = \tfrac{2}{N+k} \int_0^\infty dp J^i_j(-p) J^j_i(p)$$

$$P^- = \tfrac{e^2}{\pi} \int_0^\infty dp \tfrac{1}{p^2} J^i_j(-p) J^j_i(p). \tag{11.16}$$

Recall that the light-cone currents $J^i_j(p)$ obey a level k, $SU(N_c)$ affine Lie algebra,

$$\left[J^k_i(p), J^n_l(p')\right] = \frac{1}{2}kp\left(\delta^n_i\delta^k_l - \frac{1}{N}\delta^k_i\delta^n_l\right)\delta(p+p') + \frac{1}{2}\left(J^n_i(p+p')\delta^k_l - J^k_l(p+p')\delta^n_i\right). \tag{11.17}$$

We can now construct the Hilbert space. The vacuum $|0, R\rangle$ is defined by the annihilation property,

$$\forall p > 0, \ J(p)|0, R\rangle = 0, \tag{11.18}$$

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where R is an "allowed" representation depending on the level. Thus, a physical state in Hilbert space is,

$$\mathrm{Tr}\ J(-p_1)\ldots J(-p_n)|0,R\rangle.$$

Note that this basis is not orthogonal.

11.3.2 't Hooft-like equation for the two-current wave function

We restrict ourselves to the simplest case of the two-current sector of the Hilbert space, (in Section 11.4 we will also mention the special case of one current on an adjoint vacuum),

$$|\Phi\rangle = \frac{1}{N_c N_f}\int_0^1 \mathrm{d}k\ \Phi(k) J^a(-k)J^a(k-1)|0\rangle, \tag{11.19}$$

namely to states which are color singlets of two currents with total $P^+ = 1$ momentum and a distribution of P^- momentum $\Phi(k)$. Note that Φ is a symmetric function,

$$\Phi(k) = \Phi(1-k). \tag{11.20}$$

Our task now is to find the eigenvalue (Schrödinger) equation for the wave function $\Phi(k)$. Let us start by the action of the "Hamiltonian" P^- on the state $|\Phi\rangle$.

The commutator of P^- with a current $J^b(-k)$ yields the result,

$$\begin{aligned}&\left[\int_0^\infty \frac{\mathrm{d}p}{p^2} J^a(-p)J^a(p), J^b(-k)\right]\left(\left(\frac{1}{2}N_f - N_c\right)\frac{1}{k} + N_c\frac{1}{\epsilon}\right)J^b(-k)\\ &+\int_k^\infty \mathrm{d}p\left(\frac{1}{p^2} - \frac{1}{(p-k)^2}\right) i f^{abc} J^a(-p)J^c(p-k)\\ &+\int_0^k \frac{\mathrm{d}p}{p^2} i f^{abc} J^c(p-k)J^a(-p).\end{aligned} \tag{11.21}$$

We introduced ϵ as an IR cutoff, namely, the lower limit of integration. This is the analog of λ in the derivation of the 't Hooft equation of Chapter 10. We take ϵ to go to zero at the end of the calculation.

The above expression (11.21) contains three terms on the right-hand side The first term contains a single creation operator. The second term contains an annihilation current and therefore should again be commuted with $J^b(k-1)$. The third term contains two creation currents and it would lead to a three-current state. This is a manifestation of the fact that pair creation is, generically, not suppressed in multi-flavor QCD_2.

Note that while deriving eqn (11.21) we get an "infinite" contribution $N_c\frac{1}{\epsilon}J^b(-k)$. This contribution will be cancelled by a counter contribution which comes from the regime $p \sim k$ in the first integral on the right-hand side of (11.21), as below.

The commutator of the second term on the right-hand side of (11.21) with $J^b(k-1)$ yields,

$$\left[\int_k^\infty \mathrm{d}p \left(\frac{1}{p^2} - \frac{1}{(p-k)^2}\right) if^{abc} J^a(-p) J^c(p-k), J^b(k-1)\right] \tag{11.22}$$
$$= N_c \int_k^\infty \mathrm{d}p \left(\frac{1}{p^2} - \frac{1}{(p-k)^2}\right) \left(J^a(-p)J^a(p-1) - J^a(p-k)J^a(k-p-1)\right).$$

Our results can be summarized by the following set of equations,

$$M^2|\Phi\rangle = \frac{1}{N_c N_f}\int_0^1 \mathrm{d}k\ \tilde{\Phi}(k) J^a(-k)J^a(k-1)|0\rangle + \frac{1}{(N_c N_f)^{\frac{3}{2}}} \tag{11.23}$$
$$\times \int_0^1 \mathrm{d}k\ \mathrm{d}p\ \mathrm{d}l\ \delta(k+p+l-1)\Psi(k,p,l) if^{abc} J^a(-k)J^b(-p)J^c(-l)|0\rangle,$$

with,

$$\Psi(k,p,l) = \frac{2e^2(N_c N_f)^{\frac{1}{2}}}{\pi}\left(\frac{\Phi(l)-\Phi(k)}{p^2}\right), \tag{11.24}$$

and,

$$\tilde{\Phi}(k) = \frac{e^2}{\pi}\Bigg((N_f - N_c)\left(\frac{1}{k} + \frac{1}{1-k}\right)\Phi(k) + \frac{2N_c}{\epsilon}\Phi(k) \tag{11.25}$$
$$- N_c \int_0^{k-\epsilon} \mathrm{d}p \frac{\Phi(p)}{(p-k)^2} - N_c \int_{k+\epsilon}^1 \mathrm{d}p \frac{\Phi(p)}{(p-k)^2} + N_c\left(\frac{1}{k^2} - \frac{1}{(1-k)^2}\right)\int_0^k \mathrm{d}p\Phi(p)\Bigg)$$

Ignoring the three-current term (see below), we get that $\Phi(k)$ obeys the eigenvalue equation,

$$\frac{M^2}{e^2/\pi}\Phi(k) = (N_f - N_c)\left(\frac{1}{k} + \frac{1}{1-k}\right)\Phi(k) \tag{11.26}$$
$$- N_c \mathcal{P}\int_0^1 \mathrm{d}p \frac{\Phi(p)}{(p-k)^2} + N_c\left(\frac{1}{k^2} - \frac{1}{(1-k)^2}\right)\int_0^k \mathrm{d}p\ \Phi(p).$$

We assumed that $\int_0^1 \mathrm{d}p\ \Phi(p) = 0$, which we will justify shortly.

For general N_c and N_f, discarding the three-current term is unjustified. However, since the length of Ψ is $|\Psi(k,p,l)| \sim e^2(N_c N_f)^{\frac{1}{2}}$, in the limit of large N_c with fixed $e^2 N_c$ and fixed N_f, or large N_f with fixed $e^2 N_f$ and fixed N_c, the three-current contribution is indeed negligible, as compared with the two-current term, the latter being of order 1.

The first integral in eqn. (11.26) should be calculated as a principal value integral (denoted by $\mathcal{P}$). The divergent part of this integral (arising from the regime $p \sim k$) cancels the previously mentioned infinity. In order to make contact with the ordinary 't Hooft equation, it is useful to integrate eqn. (11.26) with respect to k and rewrite it in terms of $\varphi(k) \equiv \int_0^k \mathrm{d}p\ \Phi(p)$, to get,

$$\frac{M^2}{e^2/\pi}\varphi(k) = (N_f - N_c)\left(\frac{1}{k} + \frac{1}{1-k}\right)\varphi(k) - N_c\mathcal{P}\int_0^1 \mathrm{d}p\frac{\varphi(p)}{(p-k)^2} + N_f\int_0^k \mathrm{d}p\frac{\varphi(p)}{p^2} + N_f\int_k^1 \mathrm{d}p\frac{\varphi(p)}{(1-p)^2}. \tag{11.27}$$

The derivation goes as follows. First, integrating eqn. (11.20) we get $\varphi(k) = -\varphi(1-k) + \text{const}$. Then taking $\varphi(1) = 0$ we get,

$$\varphi(k) = -\varphi(1-k). \tag{11.28}$$

Now $\varphi(1) = 0$ implies $\int_0^1 \mathrm{d}k\Phi(k) = 0$, which was our assumption above. Then, differentiating (11.27) we do get (11.26), and by the last equation we also get that there is no extra integration constant.

We would like to comment on the issue of the Hermiticity of the "Hamiltonian" M^2. Naively, it seems that M^2 is not Hermitian with respect to the scalar product $<\psi|\varphi> = \int_0^1 \mathrm{d}k\psi^\star(k)\varphi(k)$, since the Hermitian conjugate of (11.27) is,

$$\left(\frac{M^2}{e^2/\pi}\right)^\dagger\varphi(k) = (N_f - N_c)\left(\frac{1}{k} + \frac{1}{1-k}\right)\varphi(k) - N_c\mathcal{P}\int_0^1 \mathrm{d}p\frac{\varphi(p)}{(p-k)^2} - N_f\frac{1}{k^2}\int_0^k \mathrm{d}p\varphi(p) - N_f\frac{1}{(1-k)^2}\int_k^1 \mathrm{d}p\varphi(p). \tag{11.29}$$

However, as we shall see in the next subsection, the numerical solution yields real eigenvalues and eigenfunctions. Therefore, at least on the subspace which is spanned by the eigenfunctions, namely real functions that are zero at $k = 0, 1$ and anti-symmetric with respect to $k = \frac{1}{2}$, the operator M^2 is Hermitian. Note that (11.29) is "more regular" than (11.27), as in (11.27) it is $\varphi(p)/p^2$ that appears in the integration from zero.

Equation (11.27) is similar to the 't Hooft equation for a massive single flavor large N_c QCD_2, with $m^2 = \frac{e^2 N_f}{\pi}$. It differs from 't Hooft's equation by having two additional terms (the two last terms in (11.27)). It suggests that the dynamics that governs the lowest state of the multi-flavor model is given, approximately, by a model of a massive "glueball" with an $SU(N_c)$ gauge interaction and additional terms which are proportional to N_f.

Before we present our solution of (11.27) it is important to note that it is only an approximate solution. We neglected the three-current state with, a priori, no justification. We shall see, however, that the restriction to the truncated two-current sector is an excellent approximation for the lowest massive meson.

11.3.3 The two-current mesonic spectrum

The most convenient way to solve (11.27) is to expand $\varphi(k)$ in the basis,

$$\varphi(k) = \sum_{i=0}^{\infty} A_i \left(k - \frac{1}{2}\right) [k(1-k)]^{\beta+i} . \tag{11.30}$$

The value of β is chosen so that the Hamiltonian will not be singular near $k \to 0$ or $k \to 1$. This consideration leads to the equation,

$$\left(\frac{N_f}{N_c} - 1\right) - \frac{N_f/N_c}{\beta+1} + \beta\pi \cot \beta\pi = 0, \tag{11.31}$$

as derived from eqn. (11.29). Had we started with (11.27), it would have been $-\beta$ replacing β in (11.31), and constrained to β larger than 1.

Upon truncating the infinite sum in (11.30) to a finite sum, the eigenvalue problem reduces to a diagonalization of a matrix. So, the problem can be reformulated as,

$$\lambda N_{ij} A_j = H_{ij} A_j, \tag{11.32}$$

with,

$$N_{ij} = \int_0^1 \mathrm{d}k \left(k - \frac{1}{2}\right)^2 (k(1-k))^{2\beta+i+j} , \tag{11.33}$$

and,

$$\begin{aligned}
H_{ij} &= \left(\frac{N_f}{N_c} - 1\right) \int_0^1 \mathrm{d}k \left(k - \frac{1}{2}\right)^2 (k(1-k))^{2\beta+i+j-1} \\
&- \frac{N_f}{N_c} \int_0^1 \mathrm{d}k \left(k - \frac{1}{2}\right) (k(1-k))^{\beta+i} \frac{1}{k^2} \int_0^k \left(p - \frac{1}{2}\right) (p(1-p))^{\beta+j} \\
&- \frac{N_f}{N_c} \int_0^1 \mathrm{d}k \left(k - \frac{1}{2}\right) (k(1-k))^{\beta+i} \frac{1}{(1-k)^2} \int_k^1 \left(p - \frac{1}{2}\right) (p(1-p))^{\beta+j} \\
&- \int_0^1 \mathrm{d}k\mathrm{d}p \frac{\left(k - \frac{1}{2}\right) (k(1-k))^{\beta+i} \left(p - \frac{1}{2}\right) (p(1-p))^{\beta+j}}{(k-p)^2}
\end{aligned} \tag{11.34}$$

Hence,

$$N_{ij} = \frac{B(2\beta+i+j+2, 2\beta+i+j+2)}{2(2\beta+i+j+1)}, \tag{11.35}$$

and,

$$\begin{aligned}
H_{ij} = &\left(\frac{N_f}{N_c} - 1\right) \frac{B(2\beta+i+j+1, 2\beta+i+j+1)}{2(2\beta+i+j)} \\
&- \frac{N_f}{N_c} \frac{B(2\beta+i+j+1, 2\beta+i+j+1)}{2(2\beta+i+j)(\beta+j+1)} \\
&+ \frac{(\beta+i)(\beta+j)B(\beta+i, \beta+i)B(\beta+j, \beta+j)}{8(2\beta+i+j)(2\beta+i+j+1)},
\end{aligned} \tag{11.36}$$

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Table 11.1. *The mass of the lowest massive meson, in units of* $\frac{e^2 N_c}{\pi}$, *as a function of* N_f/N_c *and* β.

β	N_f/N_c	M^2
0.0000	0	5.88
0.0573	0.2	6.91
0.1088	0.4	7.91
0.1552	0.6	8.91
0.1978	0.8	9.89
0.2366	1.0	10.86
0.2725	1.2	11.83
0.3050	1.4	12.77
0.3360	1.6	13.73
0.3645	1.8	14.67

where $B(x, y)$ is the beta function,

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{11.37}$$

In practice, the process converges rapidly and a 5×5 matrix yields the 'continuum' results.

The lowest eigenvalues of (11.27) as a function of the ratio $\frac{N_f}{N_c}$ are listed in Table 11.1 (see also Fig. 11.1). Note that by $\beta = 0, N_f/N_c = 0$ we mean the limit $\beta \to 0, N_f/N_c \to 0$.

These values are in excellent agreement with recent DLCQ calculations, as will be given in the next chapter.

The typical error is less than 0.1 %.

An interesting observation is that the eigenvalues depend linearly on N_f, Fig. 11.1. The dependence is,

$$M^2 = \frac{e^2 N_c}{\pi}\left(5.88 + 5\frac{N_f}{N_c}\right). \tag{11.38}$$

We do not have a good understanding of this observation. It is not clear why the lowest eigenvalue sits on a straight line.

In the following sections we will consider some special cases.

11.3.4 Special cases: $N_f = 1$, $N_f = N_c$ and $N_f \gg N_c$

We now discuss three special cases, the massless 't Hooft model where the fermions are in the fundamental representation with $N_f = 1$, the case of adjoint fermions namely $N_f = N_c$ and the "abelianized" model of large $N_f \gg N_c$.

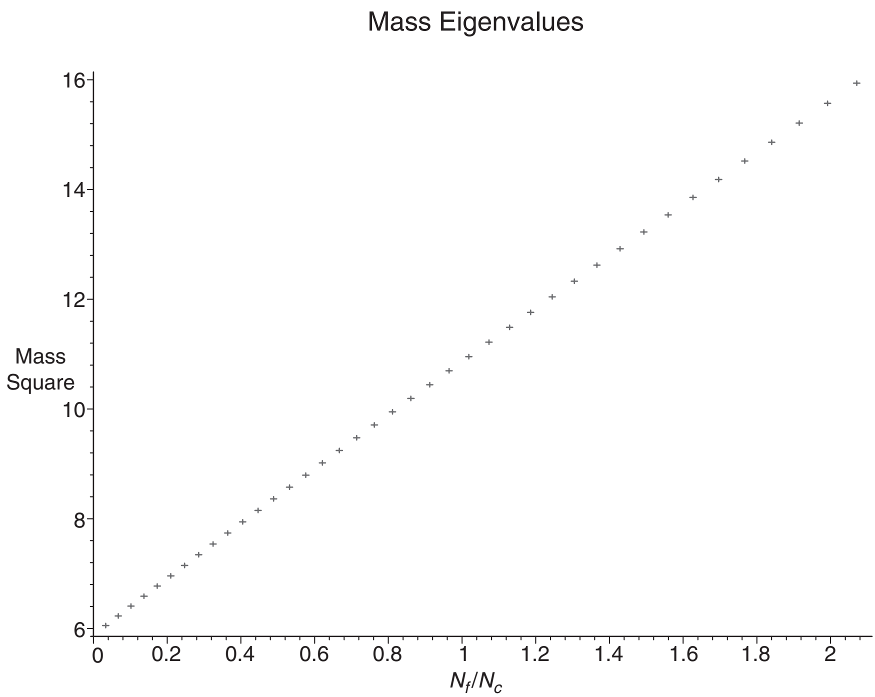

Fig. 11.1. The Green's function of the quark bilinear.

$N_f = 1$, currentized massless 't Hooft model

The limit $N_c \to \infty$ with $e^2 N_c$ fixed and $N_f \ll N_c$ corresponds to the well-known 't Hooft model. In this limit QCD_2 was solved exactly by 't Hooft [124] (see Chapter 10), using the fermionic basis. Let us see how our approach looks in the fermionic basis in this case. In the limit $N_f \ll N_c$ we can neglect terms which are proportional to N_f. Equation (11.27) takes the form,

$$\frac{M^2}{e^2/\pi}\varphi(k) - N_c\left(\frac{1}{k} + \frac{1}{1-k}\right)\varphi(k) - N_c\mathcal{P}\int_0^1 \mathrm{d}p \frac{\varphi(p)}{(p-k)^2}, \tag{11.39}$$

which is just the 't Hooft equation for the massless case. Note that (11.39) is *exact*, since in the small N_f limit the three-current state is suppressed by $N_c^{-\frac{1}{2}}$ with respect to the two-current state and therefore we can neglect it. Note also that in this eqn. (11.29) looks the same too.

Since the wave function $\varphi(k)$ is anti-symmetric, we will recover only the odd states in the spectrum of QCD_2 (the even states can be recovered by considering other sectors of the Hilbert space which decouple from the two-current state).

Though eqn. (11.39) is formally the same as the 't Hooft equation, the interpretation of $\varphi(k)$ should be different. It is the integral of the function $\Phi(k)$ which corresponds to the two-current state, namely to a mixture of 4-fermions and

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2-fermions. What is the relation between the states that we find here and the mesons in 't Hooft's model?

In order to answer this question let us expand the currents in terms of fermions. It is useful to denote the current in double index notation

$$J^a(k) \to J^i_j(k) = \int_{-\infty}^{\infty} \mathrm{d}q \left(\bar{\Psi}^i(q)\Psi_j(k-q) - \frac{1}{N_c}\delta^i_j \bar{\Psi}^k(q)\Psi_k(k-q) \right). \quad (11.40)$$

We do not bother about normal ordering, as no problem for k non zero, and we have to treat the $k=0$ part in a limiting way. The state $|\Phi\rangle$ can be written as,

$$\begin{aligned} |\Phi\rangle &= \frac{1}{2N_c}\int_0^1 \mathrm{d}k\ \Phi(k) J^i_j(-k) J^j_i(k-1)|0\rangle \qquad (11.41)\\ &= \frac{1}{2N_c}\int_0^1 \mathrm{d}k\ \Phi(k)\int_{-\infty}^{\infty} \mathrm{d}q\ \mathrm{d}p \left(\bar{\Psi}^i(-q)\Psi_j(-k+q) - \frac{1}{N_c}\delta^i_j\bar{\Psi}^k(-q)\Psi_k(-k+q)\right)\\ &\quad \times \left(\bar{\Psi}^j(-p)\Psi_i(k+p-1) - \frac{1}{N_c}\delta^j_i\bar{\Psi}^k(-p)\Psi_k(k+p-1)\right)|0\rangle. \end{aligned}$$

Note that the above expression (11.41) contains creation and annihilation fermionic operators. Written in terms of creation operators only; (11.41) reads,

$$\begin{aligned} |\Phi\rangle &= \frac{1}{2N_c}\int_0^1 \mathrm{d}k \int_0^k \mathrm{d}q \int_0^{1-k} \mathrm{d}p\ \Phi(k)\bar{\Psi}^i(-q)\Psi_j(-k+q)\bar{\Psi}^j(-p)\Psi_i(k+p-1)|0\rangle\\ &-\frac{1}{2N_c^2}\int_0^1 \mathrm{d}k \int_0^k \mathrm{d}q \int_0^{1-k} \mathrm{d}p\ \Phi(k)\bar{\Psi}^i(-q)\Psi_i(-k+q)\bar{\Psi}^j(-p)\Psi_j(k+p-1)|0\rangle\\ &-\left(1-\frac{1}{N_c^2}\right)\int_0^1 \mathrm{d}k\int_0^k \mathrm{d}q\ \Phi(k)\bar{\Psi}^i(-q)\Psi_i(q-1)|0\rangle. \qquad (11.42) \end{aligned}$$

The last term in (11.42) corresponds to a meson. It can be written also as,

$$\int_0^1 \mathrm{d}q \int_q^1 \mathrm{d}k\ \Phi(k)\bar{\Psi}^i(-q)\Psi_i(q-1)|0\rangle = -\int_0^1 \mathrm{d}q\ \varphi(q)\bar{\Psi}^i(-q)\Psi_i(q-1)|0\rangle, \quad (11.43)$$

which is exactly the 't Hooft meson. We conclude that the two-current state has an overlap with the 't Hooft meson and this is why (11.27) reproduces exactly the (odd part of the) spectrum of the 't Hooft model.

Large $N_f \gg N_c$ limit

In the limit $N_f \gg N_c$, with $e^2 N_f$ fixed, the truncation to two-current state should again predict exact results. The reason is that the three-current state is suppressed by $N_f^{-\frac{1}{2}}$ with respect to the two-current state.

In this limit eqn. (11.26) takes the form,

$$M^2 = \frac{e^2 N_f}{\pi}\left(\frac{1}{k} + \frac{1}{1-k}\right). \quad (11.44)$$

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It describes a continuum of states with masses above $2m$, where $m^2 = \frac{e^2 N_f}{\pi}$. The interpretation is clear: in this limit the spectrum of the theory reduces to a single non-interacting meson (or "currentball") with mass m.

$N_f = N_c$, *The Adjoint Fermions Model*

The case $N_f = N_c$ is the most interesting one. It was shown that the massive spectrum of this model is equivalent to the massive spectrum of a model with a single adjoint fermion, due to 'universality' [148]. Since this model is not exactly solvable, it is interesting to see how our approach reproduces, almost accurately, previous numerical results.

The mass of the lowest massive meson, predicted by (11.27), is $M^2 = 10.86 \times \frac{e^2 N_c}{\pi}$. The values reported from DLCQ calculations are $M^2 = 10.8$ and $M^2 = 10.84$, in units of $\frac{e^2 N_c}{\pi}$, as will be detailed in the next chapter.

This agreement is very surprising. In the regime $N_f \sim N_c$, the three-current state is not suppressed by factors of color or flavor with respect to the two-current state. Why, therefore, is our approach so successful? The reason seems to be that as in the fermionic basis [38], the lowest massive state is an almost pure two-current state. However, the present approach is much more successful than the fermionic basis, where the prediction for the mass of the lowest massive boson of the adjoint model is twice as much as the lowest massive boson of the 't Hooft model. It seems that the "correct" underlying degrees of freedom are currents and not fermions, as predicted by the authors of [148].

To summarize, we have used a description of massless QCD_2 in terms of currents. With this basis we wrote down a 't Hooft-like equation (11.27) for the wave function of the two-current states.

The equation interpolates smoothly between the description of a single flavor model with large N_c ('t Hooft model), the adjoint fermions model $N_f = N_c$ and the large N_f model. The equation is derived by using an a-priori unjustified suppression of the three-current coupling. Nevertheless, we observe an excellent agreement with the DLCQ results for the first excited state. For higher excited states the agreement deteriorates and it is of the order of 20%.

The accuracy of the results for the first excited state, which implies that for this state the truncation of the "pair creation terms" is harmless, deserves further investigation.

11.4 The adjoint vacuum and its one-current state

Next we construct the spectrum of states, which is obtained by the action of a current on the "adjoint vacuum", in the color singlet combination. This way we get physical states, which are in a sense "one-current" states.

The "adjoint vacuum" is created from the singlet vacuum by applying the adjoint zero mode, which is taken as the limit $\epsilon \to 0$ of the product of quark and anti-quark creation operators, each one at momentum ϵ. Hence in our case,

$$|0,\ R\rangle = \lim_{\epsilon\to 0} \psi^i_{-1}(\epsilon)\psi^\dagger_{-1,j}(\epsilon)\,|0\rangle\,, \tag{11.45}$$

where ψ^i_{-1} and $\psi^\dagger_{-1,j}$ are the creation operators of a quark and anti-quark respectively. We can represent the action of the above adjoint zero mode on the vacuum by the derivative of a creation current taken at zero momentum. Differentiating the current with respect to k, and acting on the vacuum we get,

$$\begin{aligned} J'^{\,i}_j(k)\,|0\rangle_{k=0^-} &= \sqrt{\frac{\pi}{2}}\frac{\mathrm{d}}{\mathrm{d}k}\int_0^\infty \mathrm{d}p\int_0^\infty \mathrm{d}q\delta(k+p+q)\psi^i_{-1}(p)\psi^\dagger_{-1,j}(q)\,|0\rangle_{k=0^-} \\ &= -\sqrt{\frac{\pi}{2}}\psi^i_{-1}(\epsilon)\psi^\dagger_j(\epsilon)\,|0\rangle_{\epsilon\to 0}\,. \end{aligned} \tag{11.46}$$

As the currents are traceless, we have to subtract the trace part for $i = j$. The latter can be neglected for large N_c. For any given N_c, results that follow are also the same after the trace is subtracted.

The adjoint vacuum we have is a bosonic one, constructed from fermion-antifermion zero modes, and as we show it can be written as the derivative of the current acting on the singlet vacuum. In the case of adjoint fermions there is another adjoint vacuum, a fermionic one, obtained by applying the adjoint fermion zero mode on the singlet vacuum.

As we showed already, $(J^a)'(0)\,|0\rangle$ represents the adjoint zero mode $b^\dagger(0)d^\dagger(0)\,|0\rangle$ (indices suppressed), for any N_f and N_c, so in particular also for $N_f = N_c$. But in the latter case the theory is equivalent to that of adjoint fermions, as follows from the equivalence theorem discussed in Section 11.2. As also stated there, states built on the adjoint vacuum above, cannot be distinguished from those built on the fermionic adjoint vacuum, the latter obtained by applying the adjoint fermions on the singlet vacuum.

The adjoint bosonic vacuum can also have flavor quantum numbers, when the fermion has flavor. This does not change our results about the mass of the new state we have. Our "currentball" will have flavor too in such a case. In our scheme of bosonization, which is the "product scheme", especially convenient when the quarks are massless, the flavor sector is decoupled, and so the flavor multiplets are given by the action of flavor zero modes, not changing the mass values.

Let us introduce the notation,

$$Z^a \equiv -\sqrt{\frac{2}{\pi}}(J^a)'(0).$$

The state we have in mind is,

$$|k\rangle = J^b(-k)Z^b|0\rangle.$$

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This state is obviously a global color singlet, but in our light-cone gauge $A_- = 0$ it is also a local color singlet, as the appropriate line integral vanishes.

Now,

$$\sqrt{\frac{\pi}{2}}\left[J^a(p), Z^b\right] = \frac{1}{2}N_f\delta^{ab}\delta(p) - if^{abc}(J^c)'(p), \tag{11.47}$$

and thus, for $p > 0$,

$$J^a(p)Z^b|0\rangle = Z^bJ^a(p)|0\rangle - i\sqrt{\frac{2}{\pi}}f^{abc}(J^c)'(p)|0\rangle = 0.$$

Hence the state $Z^b|0\rangle$ is annihilated by all the annihilation currents, and so it is indeed a colored vacuum.

Using,

$$\left[P^+, J^b(-k)\right] = kJ^b(-k), \tag{11.48}$$

we get that our state $|k\rangle$ is indeed of momentum k.

Note that when quantizing on a circle of radius R, the adjoint vacuum would be an eigenstate of P^+ with eigenvalue N_c/R. As we work in the continuum limit, we get zero.

11.4.1 The action of M^2 on the one-current states

First, we evaluate the commutator of P^- with a creation current,

$$\begin{aligned}
&\left[\int_0^\infty \mathrm{d}p\phi(p)J^a(-p)J^a(p), J^b(-k)\right] \\
&\quad = \frac{1}{2}N_f\frac{1}{k}J^b(-k) + if^{abc}\int_0^k \mathrm{d}p\phi(p)J^a(-p)J^c(p-k) \\
&\quad\quad + if^{abc}\int_k^\infty \mathrm{d}p\left(\phi(p) - \phi(p-k)\right)J^a(-p)J^c(p-k),
\end{aligned}$$

note that in P^- (and in P^+) we ignore contributions from zero-mode states, that is, we cut the integrals at ϵ, and then take the limit.

As P^+ and P^- act on a singlet state, and as $J^a(0)$, being the color charge, annihilates this state, the contribution from the zero modes in both P^+ and P^- is zero. Therefore it is legitimate to cut the integration limit above the zero mode and then take the cutoff to zero, as we have done. Note also that the integral of $\phi(p)$ around $p = 0$ is finite, and in fact zero when integrating over the whole line, therefore there are no divergences when we take the limit.

It is important, however, to remember that the zero mode does contribute when we act upon non singlet states, like the adjoint vacuum $Z^b|0>$ itself. When quantizing on a circle of radius R one gets that P^+ is of order $1/R$. And then, with P^- of order e^2R, M^2 is R independent, and so remains finite in the continuum limit. However, this is subtle, as P^- becomes IR divergent in

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the continuum and needs to be regularized. This subtlety does not affect our calculation as we work in the singlet sector only.

Actually, the argument connected with P^- acting on singlets should be somewhat sharpened. Let us put the lower limit at ϵ, and let it go to zero at the end. This IR cutoff is similar to the one introduced in the derivation of 't Hooft's model discussed in Section 10. Then $J(\epsilon)$, when acting on a singlet, would go like ϵ. We have two currents in the integral, so we get ϵ^2. But then we have $1/\epsilon^2$ from the denominator, so a finite integrant. But the region for integration is of order ϵ, so indeed the total contribution goes to zero.

Now apply P^- on our state,

$$P^- J^b(-k)Z^b|0\rangle = \left[P^-, J^b(-k)\right] Z^b|0\rangle, \tag{11.49}$$

as the Hamiltonian annihilates the color vacuum as well.

Using the commutator of the Hamiltonian with a current, we get,

$$\begin{aligned}\frac{\pi}{e^2}P^- J^b(-k)Z^b|0\rangle &= \frac{1}{2}N_f\frac{1}{k}J^b(-k)Z^b|0\rangle \\ &\quad + if^{abc}\int_0^k \mathrm{d}p\phi(p)J^a(-p)J^c(p-k)Z^b|0\rangle.\end{aligned}$$

Note that we use the fact that annihilation currents do annihilate the colored vacuum also.

Let us apply the operator M^2 to our one-current state,

$$\begin{aligned}M^2 J^b(-k)Z^b|0\rangle &= 2P^-P^+J^b(-k)Z^b|0\rangle 2kP^-J^b(-k)Z^b|0\rangle \\ &= \left(\frac{e^2N_f}{\pi}\right)J^b(-k)Z^b|0\rangle + \left(\frac{2e^2}{\pi}k\right)if^{abc}\int_0^k \mathrm{d}p\phi(p)J^a(-p)J^c(p-k)Z^b|0\rangle.\end{aligned} \tag{11.50}$$

So it seems that, in the large N_f limit, the state $J^b(-k)Z^b|0\rangle$ is an (approximate) eigenstate, with eigenvalue $\frac{e^2N_f}{\pi}$.

To see the exact dependence of the two terms in the equation above (the one- and two-current states) on N_f and N_c, we should normalize them. The normalization of $J^b(-k)Z^b|0\rangle$ is,

$$\begin{aligned}\langle 0|\, Z^a J^a(k)J^b(-k)Z^b\,|0\rangle &= \langle 0|\, Z^a \left[J^a(k), J^b(-k)\right] Z^b\,|0\rangle \\ &= \frac{1}{2}N_f k\delta(0)\,\langle 0|\, Z^bZ^b\,|0\rangle + if^{abc}\,\langle 0|\, Z^aJ^c(0)Z^b\,|0\rangle \\ &= \frac{1}{2}N_f k\delta(0)\,\langle 0|\, Z^bZ^b\,|0\rangle + N_c\,\langle 0|\, Z^bZ^b\,|0\rangle\,.\end{aligned} \tag{11.51}$$

The second term in the last line can be neglected compared with the first, as it is a constant to be compared with $\delta(0)$ [the space volume divided by 2π].

Now,

$$\langle 0|Z^bZ^b|0\rangle = (N_c^2-1)\langle 0|Z^1Z^1|0\rangle,$$

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and the factor $k\delta(0)$ is the normalization of a plane wave of momentum k. So the normalized state is, for $N_c \gg 1$,

$$\frac{1}{N_c\sqrt{\frac{1}{2}N_f}} J^b(-k)Z^b|0\rangle, \tag{11.52}$$

relative to $\langle 0|Z^1 Z^1|0\rangle$.

The normalization of the second term is more complicated. A lengthy but straightforward calculation gives,

$$\begin{aligned}
&\left\| \left(i f^{\mathrm{def}} k \int_0^k \mathrm{d}q \Phi(q) J^d(-q) J^f(q-k) \right) Z^e \,|0\rangle \right\|^2 \\
&= N_c \left(N_c^2 - 1\right) \left(\frac{1}{2}N_f\right)^2 k\delta(0) \left\langle 0 \left| Z^1 Z^1 \right| 0 \right\rangle \\
&\quad \times k \left(\int_0^k \mathrm{d}p p(k-p)\Phi(p)\left(\Phi(p) - \Phi(k-p)\right) - \frac{N_c}{N_f}\int_0^k \mathrm{d}p\Phi(p)\int_0^{k-p}\mathrm{d}q q\Phi(q) \right).
\end{aligned} \tag{11.53}$$

Using the following relations,

$$\begin{aligned}
f^{abc}f^{abd} &= \mathrm{Tr}(T^cT^d) = N\delta^{cd} \\
f^{abc}f^{a'bc'}f^{aa'd}f^{cc'd} &= \mathrm{Tr}(T^bT^dT^bT^d) \\
&= i f^{bde}\mathrm{Tr}(T^eT^bT^d) + \mathrm{Tr}(T^bT^bT^dT^d) = \frac{1}{2}N^2(N^2-1),
\end{aligned}$$

we have evaluated only the terms proportional to $\delta(0)$ as they are the dominant ones.

The various momentum integrals (including the ones for the non dominant terms) are divergent for $\epsilon \to 0$, thus they should be regulated. We leave this problem for now, and assume henceforth that they are regulated and finite. For simplicity the integrals (including the factor k) appearing in the two dominant terms will be denoted R_1 and $-R_2$ in the following expressions. Note that we have $\frac{1}{\epsilon^2}$ and $\frac{1}{\epsilon}$ divergences and also $\ln(\frac{k^2}{\epsilon^2})$. It seems that these are cancelled in R_2.

Define now the normalized states,

$$|S_1\rangle = C_1\left(J^b(-k)Z^b\,|0\rangle\right) \tag{11.54}$$

$$|S_2\rangle = iC_2 k f^{abc}\int_0^k \mathrm{d}p\Phi(p)J^a(-p)J^c(p-k)Z^b\,|0\rangle\,, \tag{11.55}$$

where,

$$C_1\frac{1}{N_c\sqrt{\frac{1}{2}N_f}}, \quad C_2 = \frac{\frac{2}{N_f\sqrt{N_c^3}}}{\sqrt{R_1 + R_2\frac{N_c}{N_f}}}. \tag{11.56}$$

The mass eigenvalue equation of the normalized states is,

$$M^2|S_1\rangle = \frac{e^2}{\pi}N_f|S_1\rangle + \frac{e^2 N_c}{\pi}\sqrt{2}\sqrt{R_1\frac{N_f}{N_c} + R_2}|S_2\rangle, \tag{11.57}$$

thus, we see that in the large flavor limit, our state $|S_1\rangle$ is an eigenstate with mass

$$M = \sqrt{\frac{e^2 N_f}{2\pi}}. \tag{11.58}$$

In the large color limit, however, we actually get that the second term dominates by a factor of N_c. Moreover, while the first term goes to zero in the large N_c limit, due to the factor of e^2, the second term survives in that limit.

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