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ON FINITELY GENERATED LATTICES OF FINITE WIDTH

W. POGUNTKE AND B. SANDS

1. Introduction. The *width* of a lattice L is the maximum number of pairwise noncomparable elements in L.

It has been known for some time ([**5**]; see also [**4**]) that there is just one subdirectly irreducible lattice of width two, namely the five-element non-modular lattice N_5. It follows that every lattice of width two is in the variety of N_5, and that every finitely generated lattice of width two is finite.

Beginning a study of lattices of width three, W. Poguntke [**6**] showed that there are infinitely many finite simple lattices of width three. Further studies on width three lattices were made in [**3**], where it was asked whether every finitely generated simple lattice of width three is finite. In this paper we will show that, in fact, more is true:

THEOREM 1.1. *Every finitely generated subdirectly irreducible lattice of width three is finite.*

Although we will mainly be interested in lattices of width three, a preliminary theorem, in the next section, will concern lattices of arbitrary finite width.

For each integer $n \geqq 1$, subdirect products of lattices of width at most n form a variety [**1**, **2**, **8**, **9**]: in R. Wille's terminology, the variety of all lattices of primitive width at most n. The width of any subdirectly irreducible lattice in this variety does not exceed n. From these results (or alternatively, from Jónsson's Lemma) we obtain:

COROLLARY 1.2. *The variety of lattices of primitive width at most three is generated by its finite members.*

Whether the corollary holds for lattices of primitive width n, $n > 3$, is not known. However, we now show by examples that Theorem 1.1 is best possible.

The lattice of Figure 1, which is taken from [**3**], is an example of an infinite, subdirectly irreducible (in fact simple) lattice of width three

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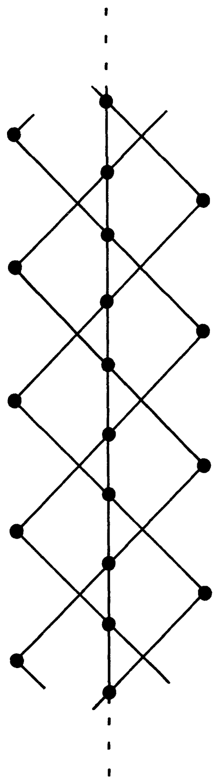

FIGURE 1. An infinite simple lattice of width three.

which is not finitely generated. On the other hand, Figure 2 illustrates a familiar finitely generated infinite lattice of width three which is not subdirectly irreducible; the structure of this lattice is of particular importance in both [**3**] and the present paper.

Finally we present the lattice L of Figure 3. L is infinite and of width four, and is finitely generated (by a_1, e_1, u, and v). Moreover, L is simple. To see this, let θ be a nontrivial congruence on L, and let $x, y \in L$ be such that $x \neq y$ and $x \equiv y(\theta)$. Observe that we may assume $x, y \in L - \{u, v, 0, 1\}$; then, by chasing transposes up and down, we get that $e_1 \equiv d_1(\theta)$. Now,

$$e_1/d_1 \nearrow 1/b_1 \searrow v/0 \nearrow 1/u \searrow a_1/0 \nearrow 1/v,$$

and so $0 \equiv v \equiv 1(\theta)$, as claimed.

This section concludes with a brief outline of the rest of the paper. In § 2 we define certain partially ordered sets called towers, and prove that a finitely generated lattice of width $n + 1$ cannot contain a tower of width n, for $n \geqq 2$. This theorem for the case $n = 2$ will be utilized in

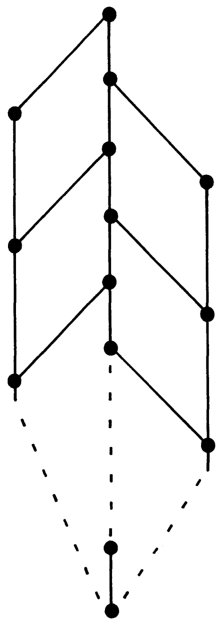

FIGURE 2

§ 3 and § 4. There, we define a lattice called the herringbone, and show that it or its dual is a sublattice of every finitely generated infinite lattice of width three (in fact we prove a little more). Finally, in § 5 we argue essentially as in [**3**] to prove our main result.

2. Towers. For $n \in \omega$, a *tower of width n* is a partially ordered set

$$\{a(i, j) | i \in \omega, 1 \leqq j \leqq n\}$$

with the ordering: $a(i, j) < a(k, l)$ if and only if $i > k$ (Figure 4 illustrates a tower of width three). Observe that if a tower of width n is embedded in a lattice, then

$$\bigvee_{j=1}^{n} a(i + 1, j) \leqq \bigwedge_{j=1}^{n} a(i, j) \quad \text{for all } i.$$

Let L be a lattice and let S be a nonempty subset of L. We shall say that S is *removable* if $L - S$ is a sublattice of L. It is clear that every set of generators of L must contain at least one element from every removable subset. This obvious, but key, concept will be needed both in this section and the next two.

We now proceed to the main theorem of this section.

THEOREM 2.1. *Let L be a finitely generated lattice of width $n + 1$, $n \geqq 2$. Then L cannot contain a tower of width n as a subset.*

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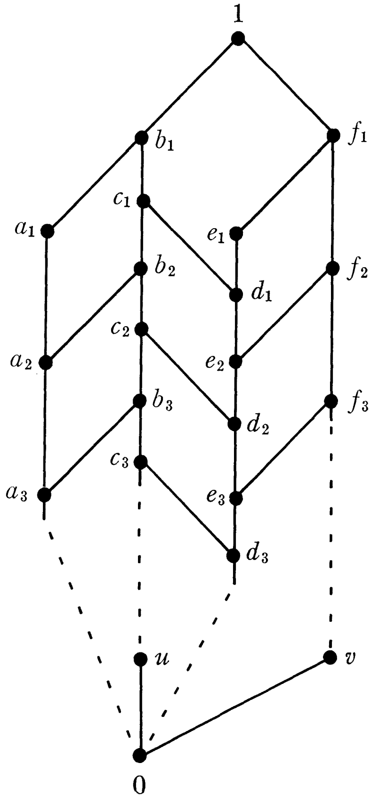

FIGURE 3. An infinite four-generated simple lattice of width four.

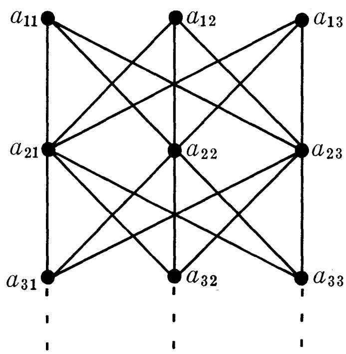

FIGURE 4. A tower of width three.

Proof. Suppose L contains the tower $\{a(i, j)\}$ of width n. We may clearly assume that L is linearly indecomposable (for instance, by initially inducting on the number of generators of L). Thus for each i

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there is an element, in fact a generator g_i of L, which is not comparable with $\bigwedge_{j=1}^{n} a(i, j)$. As L is finitely generated, some generator g of L must be chosen infinitely often as a g_i, and it follows that $g \| a(i, j)$ for all j and all but finitely many i. By relabeling (starting the tower further down) we may assume $g \| a(i, j)$ for all i and j.

Our plan is to define, for each $i \geqq 3$, a removable set R_i. Later we will show that the R_i's are sufficiently disjoint to contradict the fact that L is finitely generated.

Fix i. For $j \in \{1, \ldots, n\}$, set

$$J(i, j) = \bigvee \{a(i, k) | 1 \leqq k \leqq n, k \neq j\}.$$

Choose $s_i \in \{1, \ldots, n\}$ such that $a(i, s_i) \wedge g \leqq J(i, s_i)$. Such a choice is possible, since if $a(i, j) \wedge g \nleqq J(i, j)$ for each j, it is easy to check that

$$\{a(i, 1) \wedge g, \ldots, a(i, n) \wedge g, a(i + 1, 1), \ldots, a(i + 1, n)\}$$

is an antichain, which (for $n \geqq 2$) contradicts the assumption that L has width $n + 1$.

Set

$$T_i = \{x \in L | x \nleqq a(i, t_i), x \wedge g \leqq J(i, s_i)\},$$

where t_i is any fixed element of $\{1, \ldots, n\}$ other than s_i. From above, $a(i, s_i) \in T_i$; we now show T_i is removable. Let $x, y \in L$ be such that $x \vee y \in T_i$ but $x \notin T_i$, $y \notin T_i$. Then

$$(x \vee y) \wedge g \leqq J(i, s_i),$$

and it follows that $x \leqq a(i, t_i)$ and $y \leqq a(i, t_i)$. But then $x \vee y \leqq a(i, t_i)$, a contradiction. If $x \wedge y \in T_i$ but $x, y \notin T_i$, then

$$x \wedge y \nleqq a(i, t_i) \quad \text{and}$$
$$x \wedge y \wedge g \leqq J(i, s_i),$$

and we can deduce $x \nleqq a(i, t_i)$, $y \nleqq a(i, t_i)$, $x \wedge g \nleqq J(i, s_i)$, and $y \wedge g \nleqq J(i, s_i)$. Observe that

$$x \wedge g \nleqq y \wedge g,$$

for otherwise

$$x \wedge g = x \wedge y \wedge g \leqq J(i, s_i).$$

Also,

$$x \wedge g \nleqq a(i, s_i),$$

for otherwise

$$x \wedge g \leqq a(i, s_i) \wedge g \leqq J(i, s_i).$$

It follows from these and similar considerations for y that

$$\{x \wedge g, y \wedge g, a(i, 1), a(i, 2), \ldots, a(i, n)\}$$

is an antichain, which is impossible. Thus T_i is removable.

Dually, setting

$$M(i, j) = \wedge \{a(i, k) | 1 \leqq k \leqq n, k \neq j\}$$

for each $j \in \{1, \ldots, n\}$, we can find $s_i^* \in \{1, \ldots, n\}$ such that

$$a(i, s_i^*) \vee g \geqq M(i, s_i^*).$$

We then obtain the removable set

$$T_i^* = \{x \in L | x \ngeqq a(i, t_i^*), x \vee g \geqq M(i, s_i^*)\},$$

where $t_i^* \in \{1, \ldots, n\}, t_i^* \neq s_i^*$.

An observation: if our goal were merely to construct a removable subset R_i for each i, we would be done; just use the T_i's, and ignore the T_i^*'s. However, to achieve enough disjointness among the R_i's we must proceed with somewhat more care. The definition of R_i requires three cases.

Case 1. If $a(i - 2, 1) \wedge g \nleqq a(i, 1)$ set $R_i = T_i$.

Case 2. If $a(i - 2, 1) \wedge g \leqq a(i, 1)$ and $a(i + 2, 1) \vee g \ngeqq a(i, 1)$ set $R_i = T_i^*$.

Case 3. Assume $a(i - 2, 1) \wedge g \leqq a(i, 1) \leqq a(i + 2, 1) \vee g$. Let R_i be the set

$$U_i = \{x \in L | x \| g, x \| a(i, 1)\}.$$

Then U_i is nonempty, containing $a(i, 2)$ for example. We show that $L - U_i$ is closed under joins. Let $x \vee y \in U_i$ with $x, y \notin U_i$. It is easy to see that, without loss of generality, $x < g$, $y < a(i, 1)$, $y \| g$, and $x \| a(i, 1)$. Now note that $x \vee y \ngeqq a(i - 1, j)$ for any j, for otherwise $x \vee y \geqq a(i, 1)$; also, $x \vee y \nleqq a(i - 1, j)$ for any j, for otherwise $x \leqq a(i - 2, 1) \wedge g \leqq a(i, 1)$. Hence

$$\{g, a(i - 1, 1), a(i - 1, 2), \ldots, a(i - 1, n), x \vee y\}$$

is an $n + 2$-element antichain, which is impossible. Thus $L - U_i$ is closed under joins. A dual argument shows that $L - U_i$ is closed under meets, and so U_i is removable.

Having defined R_i for all $i \geqq 3$, we now prove that each element of L lies in only finitely many of the R_i. This is an immediate consequence of the next two claims.

Claim 1. If $i_1 < i_2 < i_3$ then $U_{i_1} \cap U_{i_2} \cap U_{i_3} = \emptyset$.

Suppose $t \in U_{i_1} \cap U_{i_2} \cap U_{i_3}$. Since $\{g, a(i_2, 1), \ldots, a(i_2, n), t\}$ is not an antichain, t must be comparable with some $a(i_2, j)$. However, $t < a(i_2, j)$ implies $t < a(i_1, 1)$, and $t > a(i_2, j)$ implies $t > a(i_3, 1)$, both contradictions to the definition of U_i.

Claim 2. If $2 < i_1 < i_2 < i_3 < i_4 < i_5 < i_6$ and $a(i_5 - 2, 1) \wedge g \nleqq a(i_5, 1)$, then $\bigcap_{k=1}^{6} T_{i_k} = \emptyset$. A dual property holds for the $T_i{}^*$'s.

Suppose $t \in \bigcap_{k=1}^{6} T_{i_k}$. Then in particular

$$t \wedge g \leqq J(i_2, s_{i_2}) < a(i_1, t_{i_1}),$$

implying $t \| g$. Also, $t \nleqq a(i_2, j)$ for any j, for otherwise $t \leqq a(i_2, j) < a(i_1, t_{i_1})$. Suppose $t > a(i_2, j)$ for some j. Then

$$t > a(i_3, 1) \geqq a(i_5 - 2, 1),$$

so

$$a(i_5 - 2, 1) \wedge g \leqq t \wedge g \leqq J(i_6, s_{i_6}) \leqq a(i_5, 1),$$

a contradiction. Thus $t \| a(i_2, j)$ for all j; but now

$$\{a(i_2, 1), a(i_2, 2), \ldots, a(i_2, n), g, t\}$$

is an antichain, which is impossible.

The proof of Theorem 2.1 may now be easily concluded. Since, say, R_3 is nonempty and removable, there must be a generator g_1 of L in R_3. Since g_1 is in only finitely many of the R_i's, we may choose an R_i which does not contain g_1. This R_i does contain a generator g_2 however, which is guaranteed to be distinct from g_1. Continuing in this way, we produce an infinite list of generators of L. This contradiction shows that L cannot contain a tower of width n.

Remarks. Our theorem does not allow the case $n = 1$; nevertheless it is amusing to note that when $n = 1$ the statement of the theorem is equivalent to the fact (mentioned above) that every finitely generated lattice of width two is finite.

Secondly, it is possible for a tower of width n to be a subset of a finitely generated lattice of width $n + 3$ (Figure 5 illustrates such an example for $n = 5$). This result is sharpened even further in the case $n = 2$; Figure 6 illustrates a finitely generated lattice of width four containing a tower of width two.

Problem. Can a finitely generated lattice of width $n + 2$ contain a tower of width n, for $n > 2$?

3. The herringbone as subset. We begin this section with a corollary of the last theorem.

COROLLARY 3.1. *Let L be a finitely generated lattice of width three, and*

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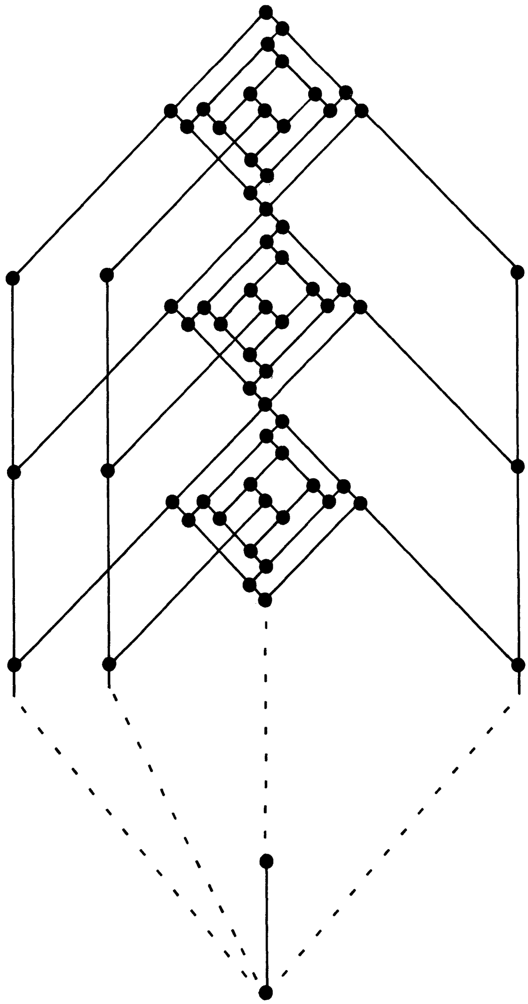

FIGURE 5

let $a_1 > a_2 > a_3 > \ldots$ *be an infinite chain in* L. *Then there is* $k < \omega$ *such that* $[a_n, a_k] \subseteq L$ *is a chain for all* $n \geqq k$.

The lattice illustrated in Figure 7 will be called the *herringbone*. Our goal for the next two sections is to prove that every finitely generated infinite lattice of width three contains the herringbone or its dual. In this section, we shall be content to prove the following.

THEOREM 3.2. *Let* L *be a finitely generated lattice of width three, and assume* L *contains an infinite descending chain. Then* L *contains a subset order-isomorphic to the herringbone.*

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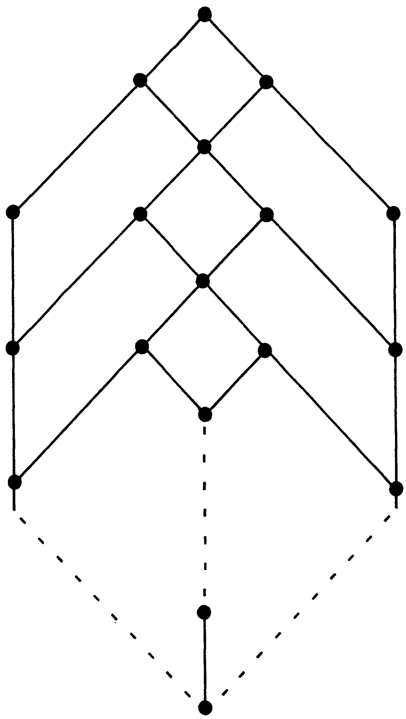


FIGURE 6

Remark. The term "herringbone" has occasionally been used informally to denote the lattice of Figure 2. As well, in [**3**] the herringbone was defined as a certain partial lattice essentially based on Figure 2.

For $n \in \omega$, let B_n denote the partially ordered set of Figure 8; that is, B_n has underlying set $\{a_1, b_1, a_2, b_2, \ldots, a_n, b_n\}$, and the ordering: $b_i < a_j$ if and only if $i \leqq j$.

LEMMA 3.3. *Let L be a lattice of width n, and let B_n be a subset of L. Then* $L = [b_1) \cup (a_n]$.

Proof. Suppose $x \in L$ with $x \ngeqq b_1$ and $x \nleqq a_n$. Since the set $\{a_1, a_2, \ldots, a_n, x\}$ cannot be an antichain, we must have $x < a_i$ for some i. Let i be minimal such that $x < a_i$ and consider the set

$$\{a_1, \ldots, a_{i-1}, a_i \wedge a_n, b_{i+1}, \ldots, b_n, x\}.$$

From $a_i \wedge a_n \geqq b_1 \vee b_i$ and the choice of i, this is an $n + 1$-element antichain, which is impossible.

For the rest of this section, we assume that L is a finitely generated lattice of width three which contains an infinite descending chain.

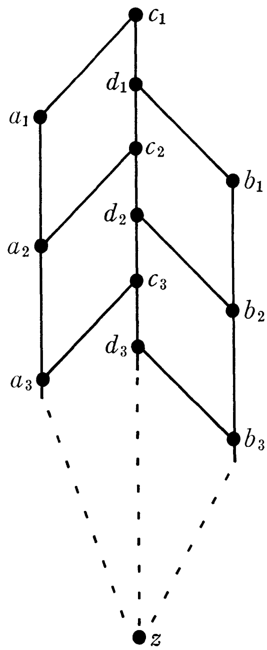
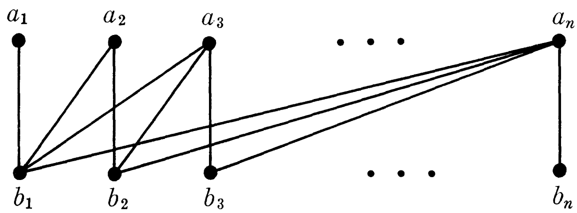


FIGURE 7. The herringbone



B_n

FIGURE 8

PROPOSITION 3.4. *L does not contain an infinite chain $c_1 > d_1 > c_2 > d_2 > c_3 > d_3 > \ldots$ such that each c_i is join-reducible and each d_i is meet-reducible.*

Proof. Assume L contains such a chain. From Corollary 3.1, we may assume (by relabeling, if necessary) that $[c_n, c_1]$ is a chain for all $n < \omega$.

Let $i < \omega$, and let $x_i, y_i \in L$ be such that $x_i \| y_i$ and $x_i \vee y_i = c_i$. Since $[c_{i+1}, c_1]$ is a chain, it follows without loss of generality that $x_i \vee d_i \geqq y_i \vee d_i$, which implies $x_i \vee d_i = c_i$. Set $a_i = x_i$. Dually, for each $i < \omega$, we can find $b_i > d_i$ such that $b_i \wedge c_i = d_i$ (see Figure 9). Note that, for all i, j with $i \leqq j$, $a_i \| d_j$ and $d_i \| b_{j+1}$; also, $a_i \nleqq b_i$ for all i.

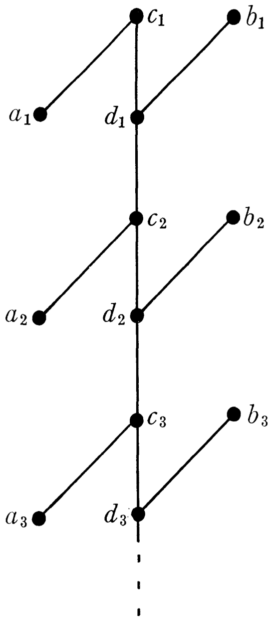


FIGURE 9

Because of the finite width of L, the set $\{a_i | i < \omega\}$ has to contain an infinite descending chain A; by relabeling the elements of the set $\bigcup_{a_i \in A} \{a_i, b_i, c_i, d_i\}$, we see that we may assume $a_1 > a_2 > a_3 > \ldots$. A further relabeling, and we may also assume $b_1 > b_2 > b_3 > \ldots$.

For each $i < \omega$, let

$$R_i = \{x \in L | x \ngeq a_{2i}, x \nleq b_{2i+1}\}.$$

Since $d_{2i} \in R_i$, we know $R_i \neq \emptyset$; also,

$$L - R_i = [a_{2i}) \cup (b_{2i+1}]$$

is a sublattice of L, so R_i is removable. We now claim that the R_i's are pairwise disjoint, which will contradict the fact that L is finitely generated. Suppose $x \in R_i \cap R_j$ where $i < j$; then $x \ngeq a_{2j}$ and $x \nleq b_{2i+1}$. But

$$\{a_{2j}, a_{2i}, d_{2j}, d_{2i}, b_{2j+1}, b_{2i+1}\} \simeq B_3,$$

which from Lemma 3.3 implies that

$$L = [a_{2j}) \cup (b_{2i+1}].$$

Thus x cannot exist, and the proof is complete.

COROLLARY 3.5. *For each infinite chain $c_1 > c_2 > c_3 > \ldots$ in L, there is a $k < \omega$ such that for all $n > k$ either*

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(i) $[c_n, c_k]$ *is a chain of join-reducible, meet-irreducible elements, or*
(ii) $[c_n, c_k]$ *is a chain of meet-reducible, join-irreducible elements.*

Proof. The only observation we need make is that L can contain only finitely many doubly irreducible elements, since each one must be a generator. The result then follows from Corollary 3.1 and Proposition 3.4.

LEMMA 3.6. *L contains infinite chains $a_0 > a_1 > a_2 > \ldots$ and $b_1 > b_2 > b_3 > \ldots$ such that*
(i) $b_i < a_j$ *for all* $i > j$, $b_i \| a_j$ *for all* $i \leqq j$;
(ii) $a_i \wedge b_i = b_{i+1}$ *for all* i;
(iii) *for each* $n < \omega$, $[a_n, a_0]$ *is a chain of meet-irreducible elements;*
(iv) *for each* $n < \omega$, $[b_n, b_1]$ *is a chain of join-irreducible elements.*

(Note: (i) and (ii) say that $S = \{a_0, a_1, b_1, a_2, b_2, \ldots\}$ is a meet-subsemilattice of L isomorphic to $\mathbf{2} \times \omega^*$.)

Proof. Let $c_1 > c_2 > c_3 > \ldots$ be an infinite chain in L. Assume (i) of Corollary 3.5 holds (the proof is similar in the other case). By relabeling, we have that $[c_n, c_1]$ is a chain of join-reducible, meet-irreducible elements for each n. For each $i < \omega$, choose $x_i, y_i \in L$ such that $x_i \| y_i$ and $x_i \vee y_i = c_i$. Arguing as in the proof of Proposition 3.4, we may assume

$$x_i \vee c_{i+1} \geqq y_i \vee c_{i+1},$$

so that $x_i \vee c_{i+1} = c_i$. Observe that $x_i \| c_j$ for all $j > i$. By choosing an infinite chain from the x_i's and relabeling them and the corresponding c_i's, we have $x_1 > x_2 > x_3 > \ldots$. Let $b_i = x_i$ and $a_{i-1} = c_i$ for each i; then (i) is established. Furthermore, it is easy to adjust the b_i's so that (ii) holds. Finally, since each b_n is meet-reducible, Corollary 3.5 shows that both (iii) and (iv) must hold under a suitable relabeling.

The next lemma and its corollary are somewhat technical.

LEMMA 3.7. *Let $a_0, a_i, b_i (i = 1, 2, \ldots)$ be as in Lemma 3.6, and let $x, y \in L$ and $k < \omega$ be such that $x \| b_i$ for all $i \geqq k$, $y < b_i$ for all i, and $x \vee y \geqq b_k$. Further suppose that $x', y' \in L$ and $l \geqq k$ are such that $x' \nleqq b_k$, $x' \vee y' \geqq b_l$, and $x' \vee y' \ngeqq a_l$. Then $x' \vee y' \geqq b_k$.*

Proof. Consider the set $\{x, a_l, b_k, x' \vee y'\}$. First, $x \ngeqq a_l$, since otherwise $x \geqq b_{l+1}$; also $x \nleqq a_l$, since otherwise $b_k \leqq x \vee y \leqq a_l$. Thus $\{x, a_l, b_k\}$ is an antichain. Now $x' \vee y' \nleqq x$ or a_l, since otherwise $b_l \leqq x$ or a_l; and since $x' \nleqq b_k$ we have $x' \vee y' \nleqq b_k$. As L has width three, either $x' \vee y' \geqq b_k$ as claimed, or $x' \vee y' \geqq x$. But in the latter case $x' \vee y' \geqq x \vee b_l \geqq x \vee y \geqq b_k$ anyway.

COROLLARY 3.8. *We may assume that if $x', y' \in L$ and $l < \omega$ are such that $x' \| b_i$ for all i, $y' < b_i$ for all i, $x' \vee y' \geqq b_l$, and $x' \vee y' \ngeqq a_l$, then $x' \vee y' \geqq b_1$.*

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Proof. Let K be the set of all $k < \omega$ such that there exist $x, y \in L$ satisfying $x \| b_i$ for all $i \geqq k$, $y < b_i$ for all i, $x \vee y \geqq b_k$, and $x \vee y \ngeqq a_k$. If K is empty, the corollary is true vacuously. Otherwise set $k = \min K$. By Lemma 3.7, if there exist $x', y' \in L$ and $l < \omega$ such that $x' \| b_i$ for all $i \geqq k$, $y' < b_i$ for all i, $x' \vee y' \geqq b_l$, and $x' \vee y' \ngeqq a_l$, then $x' \vee y' \geqq b_k$. The corollary now follows by renumbering the a_i's and b_i's so that $k = 1$.

Proof of Theorem 3.2. For each $i < \omega$, set

$$W_i = \{x \in L | x \ngeqq b_i, x \geqq b_j \text{ for some } j > i\}.$$

Clearly $W_i \neq \emptyset$. It is also easy to see that $L - W_i$ is closed under taking meets. On the other hand, observe that if $x \in W_i$, then there is a $j_0 > i$ with $x \notin W_j$ for any $j \geqq j_0$; since L is finitely generated, this implies that W_i cannot be removable for infinitely many i. Thus, we might as well assume that, for every i, $L - W_i$ is not closed under taking joins; that is, for each i there are $x_i, y_i \in L - W_i$ with $x_i \vee y_i \in W_i$. This means that $x_i, y_i \ngeqq b_k$ for any k, and that $x_i \vee y_i \ngeqq b_i$ but there exists an integer $j(i) > i$ such that $x_i \vee y_i \geqq b_{j(i)}$.

Observe that we cannot have both $x_i \leqq a_{j(i)}$ and $y_i \leqq a_{j(i)}$, for this would imply

$$b_{j(i)} \leqq x_i \vee y_i \leqq a_{j(i)}.$$

Thus we can assume that $x_i \nleqq a_{j(i)}$, and it follows that $x_i \| a_k$ for all $k \geqq j(i)$ and $x_i \| b_k$ for all $k \geqq j(i) + 1$. Moreover, if $x_i \leqq b_1$ then

$$b_{j(i)+1} < x_i \vee b_{j(i)+1} \leqq b_1,$$

and $[b_{j(i)+1}, b_1]$ contains a join-reducible element, contradicting Lemma 3.6 (iv); hence $x_i \| b_k$ for all k. For each $k \geqq j(i)$, the set $\{x_i, y_i, a_k, b_k\}$ is not an antichain, from which we conclude that $y_i < a_k$ for all k.

Claim. $x_i \vee y_i \geqq a_{j(i)+1}$.

To prove this, consider the set

$$\{x_i, b_i, a_{j(i)+1}, b_{j(i)+1} \vee y_i\}.$$

We already know that $\{x_i, b_i, a_{j(i)+1}\}$ is an antichain, so $b_{j(i)+1} \vee y_i$ must be comparable with one of these three elements. Obviously

$$b_{j(i)+1} \vee y_i \ngeqq x_i \text{ or } a_{j(i)+1};$$

also

$$b_{j(i)+1} \vee y_i \leqq a_{j(i)},$$

which implies that $b_{j(i)+1} \vee y_i \ngeqq x_i$ or b_i. If $b_{j(i)+1} \vee y_i \geqq a_{j(i)+1}$, then

$$x_i \vee y_i \geqq b_{j(i)} \vee y_i \geqq b_{j(i)+1} \vee y_i \geqq a_{j(i)+1}$$

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as claimed. Finally suppose $b_{j(i)+1} \vee y_i \leqq b_i$. Then $y_i \leqq b_i$, and from Lemma 3.6 (ii) it follows that $y_i \leqq b_k$ for all k. If $x_i \vee y_i \ngeqq a_{j(i)}$ then from Corollary 3.8 we would have $x_i \vee y_i \geqq b_1 \geqq b_i$, a contradiction; hence

$$x_i \vee y_i \geqq a_{j(i)} > a_{j(i)+1},$$

and the claim is established.

So $x_i \vee y_i \geqq a_{j(i)+1}$, but since $x_i \vee y_i \ngeqq b_i$ we know that $x_i \vee y_i \ngeqq a_{i-1}$. In fact, since

$$a_{j(i)+1} \leqq (x_i \vee y_i) \wedge a_{i-1} \leqq a_{i-1},$$

Lemma 3.6 (iii) implies that $x_i \vee y_i \leqq a_{i-1}$, and so $x_i < a_{i-1}$.

In summary, for each $i < \omega$ we have found an element $x_i < a_{i-1}$ and an integer $j(i) > i$ such that $x_i \| a_k$ for all $k \geqq j(i)$ and $x_i \| b_k$ for all k.

Now let $i_1 = 1$, and inductively let $i_k = j(i_{k-1}) + 1$. It follows that, for each k, $x_{i_k} < a_{i_k - 1}$, $x_{i_k} \| a_l$ for all $l \geqq i_{k+1} - 1$, and $x_{i_k} \| b_l$ for all l. Furthermore, $x_{i_j} < x_{i_k}$ for $j > k$, since L has width three. Therefore, letting 0 be the least element of L, the set

$$\{0\} \cup \bigcup_{k=1}^{\infty} \{x_{i_{2k}}, a_{i_k - 1}, b_{i_{2k}-1}\}$$

is easily seen to be order-isomorphic to the herringbone.

Note. It was proved recently in [**7**] that every finitely generated infinite lattice of finite width contains a subset isomorphic to the herringbone or its dual.

4. The herringbone as sublattice.

THEOREM 4.1. *Let L be a finitely generated lattice of width three, and suppose L contains an infinite descending chain. Then L contains a sublattice isomorphic to the herringbone.*

Proof. From Theorem 3.2 we may suppose that L contains the herringbone

$$\{z\} \cup \bigcup_{i=1}^{\infty} \{a_i, b_i, c_i, d_i\},$$

illustrated in Figure 7, as a subset. Let $A = \bigcup_{i=1}^{\infty} [a_i, a_1]$, $B = \bigcup_{i=1}^{\infty} [b_i, b_1]$, and $C = \bigcup_{i=1}^{\infty} [c_i, c_1]$. From Corollary 3.5, we may assume that A and B are chains of join-irreducible elements, and that C is a chain of meet-irreducible elements.

Set $a_i' = a_i$, $b_i' = b_i$, $c_i' = a_i \vee d_i$, and $d_i' = b_i \vee c_{i+1}$ for each i; then, since C is a chain, $c_i' = a_i' \vee d_j'$ and $d_i' = b_i' \vee c_{j+1}'$ for all $j \geqq i$. For each $j \geqq i > 1$, the set $\{a_i' \vee b_j', a_1', b_1', d_j'\}$ is not an antichain, which implies that $a_i' \vee b_j' > d_j'$. Hence

$$a_i' \vee b_j' \geqq a_i' \vee d_j' = c_i',$$

and so $a_i' \vee b_j' = c_i'$. Similarly, for each $j \geqq i > 1$ we get

$$b_i' \vee a_{j+1}' = d_i'.$$

Observe that these last two results can be written as the single equation

$$a_i' \vee b_j' = c_i' \vee d_j', \quad \text{for any } i, j > 1$$

(and so, by relabeling, for all i and j). Finally, from the set $\{a_i', b_i', c_{i+1}', a_1' \wedge b_1'\}$ we infer that $a_1' \wedge b_1' < c_i'$ for all i; since A and B are chains, $a_1' \wedge c_{i+1}' < a_i'$ and $b_1' \wedge c_{i+1}' < b_i'$ for all i, and hence $a_1' \wedge b_1' < a_i'$ and b_i' for all i. Setting $z' = a_1' \wedge b_1'$, we have shown that

$$\{z'\} \cup \bigcup_{i=1}^{\infty} \{a_i', b_i', c_i', d_i'\}$$

is a join-subsemilattice of L order-isomorphic to the herringbone; further, $a_1' \wedge b_1' = z'$ holds. Getting the rest of the meets to work out right will be more difficult.

An element $t \in C$ will be called a *triple point* if

$$(a_1' \wedge t) \vee c_i' = t = (b_1' \wedge t) \vee c_i',$$

where $i \geqq 1$ is such that $c_i' < t$. Notice that since C is a chain, this definition is not dependent on which $c_i'(<t)$ is chosen.

Let t be a triple point such that $c_i' < t \leqq c_{i-1}'$ for some $i > 2$, and set

$$S_t = \{x \in L | x \nleqq d_i', x \vee c_i' < t\}.$$

Then $c_i' \in S_t$ so S_t is nonempty. We claim that S_t is removable. It is easy to see that $L - S_t$ is closed under joins. Thus we suppose that $x, y \in L$ are such that $x, y \notin S_t$ but $x \wedge y \in S_t$. Then $x \wedge y \nleqq d_i'$, $(x \wedge y) \vee c_i' < t$, and it follows that $x \vee c_i' \nless t$ and $y \vee c_i' \nless t$. Since $x \wedge y < t \leqq c_{i-1}'$, and C is a chain of meet-irreducibles, we see that $x \wedge y \| d_j'$ for all $j \geqq i$. From the set $\{x \wedge y, a_{i-2}', b_{i-2}', d_i'\}$ we get that $x \wedge y < a_{i-2}'$ or $x \wedge y < b_{i-2}'$, and from the set $\{x \wedge y, a_{i+1}', b_i', d_{i+1}'\}$ we get that $x \wedge y > a_{i+1}'$ or $x \wedge y > b_i'$. Thus, either $a_{i+1}' < x \wedge y < a_{i-2}'$ or $b_i' < x \wedge y < b_{i-2}'$. Suppose the former. Since A is a chain, $a_{i-2}' \wedge t = a_1' \wedge t$, and so $x \wedge y \leqq a_1' \wedge t$. Note that if $x \wedge y = a_1' \wedge t$, then

$$(x \wedge y) \vee c_i' = (a_1' \wedge t) \vee c_i' = t,$$

contradicting $x \wedge y \in S_t$; thus $x \wedge y < a_1' \wedge t$. Again since A is a chain, we may assume $x \| a_1' \wedge t$ and $x \| a_1'$. If $x > b_1'$ then

$$x \geqq a_{i+1}' \vee b_1' = d_1' > a_1' \wedge t,$$

a contradiction; thus $x \| b_1'$. From $\{x, a_1', b_1', d_i'\}$ we get $x > d_i'$. Since C is a chain of meet-irreducibles, this implies that x is comparable to both t and c_i'. From $x \| a_1' \wedge t$ we know that $x < t$; but $x \vee c_i'$ equals x or c_i', and thus $x \vee c_i' < t$, which is a contradiction. Similarly, $b_i' < x \wedge y < b_{i-2}'$ is impossible, and S_t is removable.

If s and t are triple points such that

$$c_{j+1}' < s \leqq c_j' < d_i' < c_i' < t \leqq c_{i-1}',$$

where $2 < i < j$, then it is easy to see that $S_t \cap S_s = \emptyset$. Hence, since L is finitely generated, we may assume by relabeling that L contains no triple points.

Now set $a_1'' = a_1'$, $b_1'' = b_1'$, $c_1'' = c_1'$, and $d_1'' = d_1'$, and, for each $i > 1$, recursively define:

$$a_i'' = a_1' \wedge d_{i-1}''$$

$$c_i'' = a_i'' \vee c_i'$$

$$b_i'' = b_1' \wedge c_i''$$

$$d_i'' = b_i'' \vee d_i'.$$

We claim that

$$\{z'\} \cup \bigcup_{i=1}^{\infty} \{a_i'', b_i'', c_i'', d_i''\}$$

is a sublattice of L isomorphic to the herringbone.

Evidently $a_i'' \leqq c_i''$ and $b_i'' \leqq d_i''$. Since $b_i'' \leqq c_i''$ and $d_i' < c_i' \leqq c_i''$, we have $d_i'' \leqq c_i''$. Similarly, $c_i'' \leqq d_{i-1}''$. Thus $d_i'' \leqq d_{i-1}''$, which implies $a_{i+1}'' \leqq a_i''$, and $c_{i+1}'' \leqq c_i''$, which implies $b_{i+1}'' \leqq b_i''$, for each i. Since $a_i' < d_{i-1}' \leqq d_{i-1}''$ and $a_i' \leqq a_1'$, we have $a_i' \leqq a_i'' \leqq a_1'$; thus $a_i'' \in A$ for each i, and similarly $b_i'' \in B$. Hence $a_i'' \| b_j''$ for all i and j. Furthermore,

$$z' = a_1'' \wedge b_1'' < a_i'', b_i'' \quad \text{for each } i, \quad \text{and}$$

$$a_i'' \vee b_j'' = a_i'' \vee b_j'' \vee (a_i' \vee b_j') = a_i'' \vee b_j'' \vee (c_i' \vee d_j')$$
$$= (a_i'' \vee c_i') \vee (b_j'' \vee d_j') = c_i'' \vee d_j'' \quad \text{for each } i, j.$$

To complete the verification, we need only show that $c_i'' > d_i''$ and $d_i'' > c_{i+1}''$ for each i; for instance, $a_i'' \| d_i''$ then follows from the fact that $c_i'' = a_i'' \vee b_i''$, and similarly we conclude $b_i'' \| c_{i+1}''$. Observe that

$$c_i'' = a_i'' \vee c_{i+1}' = (a_1' \wedge c_i'') \vee c_{i+1}',$$

since C is a chain. But c_i'' cannot be a triple point, and (noting $c_{i+1}' < c_i' \leqq c_i''$) it follows that

$$c_i'' > (b_1' \wedge c_i'') \vee c_{i+1}' = b_i'' \vee c_{i+1}' = d_i'',$$

as desired. A similar argument establishes $d_i'' > c_{i+1}''$, and we have our herringbone.

In [**8**] is exhibited a finitely generated infinite lattice of width four which does not contain a sublattice isomorphic to the herringbone or its dual.

The next result, stronger than Theorem 4.1, answers question 2 of [**3**] in the affirmative. It may also be of interest in that the lattices involved (see Figure 10) are all three-generated, whereas the herringbone is not even finitely generated.

THEOREM 4.2. *Let L be a finitely generated infinite lattice of width three, and suppose L contains an infinite descending chain. Then L contains one of the lattices of Figure* 10 *as a sublattice.*

Proof. By the previous theorem, we may assume that L contains the herringbone illustrated in Figure 7 as a sublattice. As usual, we also assume that $A = \bigcup_{i=1}^{\infty} [a_i, a_1]$ and $B = \bigcup_{i=1}^{\infty} [b_i, b_1]$ are chains of join-irreducible elements, and that $C = \bigcup_{i=1}^{\infty} [c_i, c_1]$ is a chain of meet-irreducible elements.

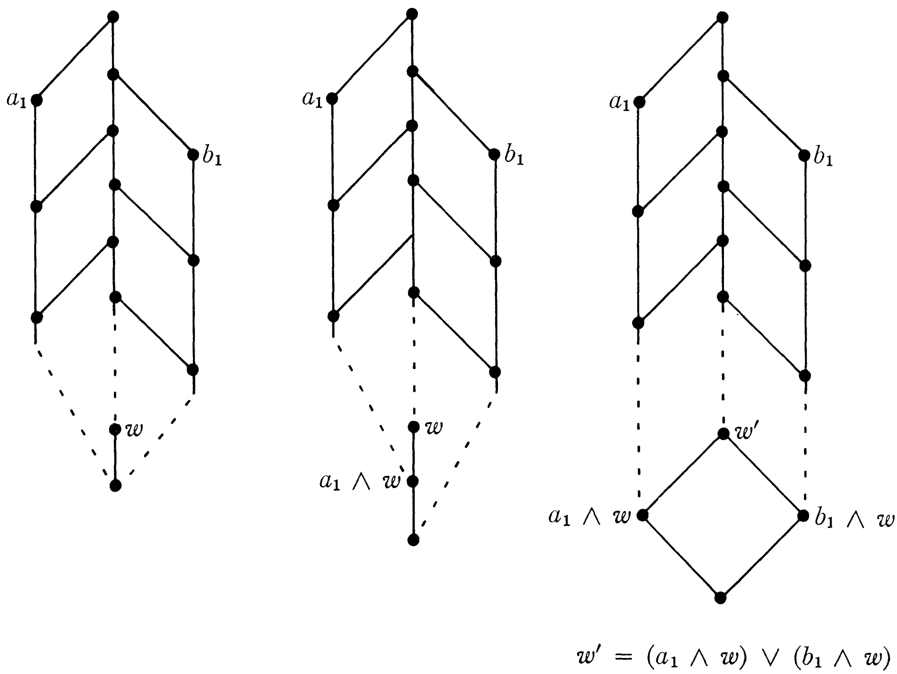

FIGURE 10

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For each $n \geqq 1$, set

$$C_n = \bigcup_{i=n}^{\infty} ([a_i, c_n] \cup [b_i, c_n]) - \{a_n, b_n, c_n\}.$$

Observe that $C_1 \supset C_2 \supset C_3 \supset \ldots$ and $\bigcap_{n=1}^{\infty} C_n = \emptyset$. Since L is finitely generated, it follows that there cannot be infinitely many n such that C_n is removable. Thus, by relabeling, we may assume that no C_n is removable.

However, $L - C_n$ is closed under meets. To see this, let $x, y \in L$ be such that $x, y \notin C_n$, and suppose $x \wedge y \in C_n$. Then $x \wedge y \notin \{a_n, b_n, c_n\}$ and there is $i \geqq n$ such that either $a_i \leqq x \wedge y < c_n$ or $b_i \leqq x \wedge y < c_n$. Note that, since L has width three, c_n covers a_n and d_n covers b_n (for $n > 1$, and so, by relabeling, for all n); therefore $x \wedge y \ngeqq a_n$ and, since $x \wedge y > b_n$ would imply $x \wedge y \in C$, $x \wedge y \ngeqq b_n$ either. Now look at $\{x, a_n, b_n, c_{n+1}\}$. Since $x \notin C_n$ we know x is not less than any of the other three elements of this set, and therefore $x \geqq a_n$, b_n, or c_{n+1}. If $x \geqq c_{n+1}$ then $c_{n+1} \leqq x \wedge c_n \leqq c_n$, and so x and c_n are comparable; however, since $x \notin C_n$, $x < c_n$ is impossible, and of course $x \geqq c_n$ implies $x > a_n$. Hence $x \geqq a_n$ or b_n. Similarly, $y \geqq a_n$ or b_n. Suppose $b_i \leqq x \wedge y < c_n$. Then $x \geqq a_n$ implies $x \geqq a_n \vee b_i = c_n > b_n$, and similarly for y; thus in any case x and y are both above b_n, which means $x \wedge y \geqq b_n$, a contradiction. Hence we assume $a_i \leqq x \wedge y < c_n$. Then $x \geqq b_n$ implies $x \geqq a_i \vee b_n = d_n$, and for the usual reasons x and c_n are comparable, in fact $x \geqq c_n > a_n$. This goes for y as well, so $x \wedge y \geqq a_n$, another contradiction. We conclude that $L - C_n$ is closed under meets.

Therefore there must exist $x, y \in L$ such that $x \vee y \in C_n$ while $x, y \notin C_n$. This means that $a_i \leqq x \vee y < c_n$ or $b_i \leqq x \vee y < c_n$ for some $i \geqq n$. Also, $x \ngeqq a_i$ or b_i for any i unless $x = a_n$ or b_n, and similarly for y. It is obvious that $\{x, y\} \neq \{a_n, b_n\}$, so without loss of generality we assume $x \notin \{a_n, b_n\}$. For each $i \geqq n$, $\{x, a_i, b_i, c_{i+1}\}$ is not an antichain, and we infer that $x < c_i$ for all i. If $y \notin \{a_n, b_n\}$, then also $y < c_i$ and hence $x \vee y \leqq c_i$ for all i, which is impossible. If $y = a_n$ then obviously $x \| a_i$ for all i; but from $\{x \vee y, a_1, b_1, d_n\}$ we must have $x \vee y > d_n$ and hence $x \vee y \geqq a_n \vee d_n = c_n$, a contradiction. Therefore $y = b_n$, and $x \| b_i$ for all i.

By symmetric arguments, we may assume that for all n the set

$$D_n = \bigcup_{i=n}^{\infty} ([b_i, d_n] \cup [a_{i+1}, d_n]) - \{b_n, a_{n+1}, d_n\}$$

is not removable, and we can find an element $x' \in L$ such that $x' < d_i$ and $x' \| a_i$ for all i. Let $w = x \vee x' \vee z$; then $w < c_i$ and $w \| a_i, b_i$ for all i. Since c_i covers a_i and d_i covers b_i, we see that $w \vee a_i = c_i$ and $w \vee b_i = d_i$ for all i. Finally, the reader can check that the locations of $w \wedge a_1$ and $w \wedge b_1$ will determine one of the lattices of Figure 10, and the theorem is proved.

Figure 11 illustrates a finitely generated infinite lattice of width five which does not contain a sublattice isomorphic to a lattice of Figure 10, or to any other infinite three-generated lattice, for that matter (our thanks to R. Wille for this example).

Problem. Does every finitely generated infinite lattice contain an infinite sublattice generated by an antichain?

5. The proof of theorem 1.1. Let L be a finitely generated lattice of width three, and suppose L contains the herringbone of Figure 7 as a sublattice; in this section we shall show that L cannot be subdirectly irreducible, thus establishing Theorem 1.1. The argument is much the same as that in the earlier paper of H. Bauer and the first author [**3**].

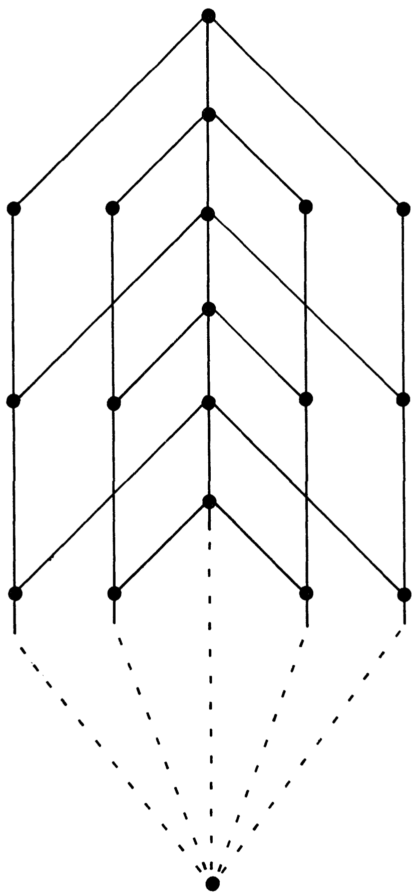

FIGURE 11

For each $n \geqq 1$, define

$$A_n = \bigcup_{i=n}^{\infty} [a_i, a_n]$$

$$B_n = \bigcup_{i=n}^{\infty} [b_i, b_n]$$

$$C_n = \bigcup_{i=n}^{\infty} [c_i, c_n].$$

It is immediate that for each n, A_n, B_n, and C_n are pairwise disjoint convex sublattices of L, and that as usual we may assume A_n and B_n are chains of join-irreducible elements and C_n is a chain of meet-irreducible elements. Thus, for every n we may define an equivalence relation θ_n on L having A_n, B_n, C_n as nontrivial equivalence classes, all other classes being singletons.

We claim that θ_n is a congruence relation on L for each n. Since all the equivalence classes are convex sublattices, it is enough to show that for every proper quotient x/y identified by θ_n, each transpose ${}^u/v$ of ${}^x/y$ is also identified by θ_n.

Let $\{x, y\} \subseteq B_n$; that is, $b_k \leqq y < x \leqq b_n$ for some $k \geqq n$. If ${}^u/v \nearrow {}^x/y$ then $u \vee y = x$ is join-reducible, which is a contradiction; hence we let ${}^u/v \searrow {}^x/y$, where $v \| x$, $v \vee x = u$, and $v \wedge x = y$. Since $v \| x$ and $v > b_k$, we see that $v \ngeqq b_1$, $v \ngeqq c_n$, and $v \nleqq a_1$. Also, $v \nleqq b_1$, for otherwise $b_k \leqq u = v \vee x \leqq b_1$, and $v \ngeqq a_1$, for otherwise $v \geqq a_1 \vee b_k = c_1 > x$. From $\{a_1, c_n, b_1, v\}$ we know that $v < c_n$ must hold, and so $u = v \vee x \leqq c_n$. Moreover, $v > b_k$ implies that $v \nleqq c_{k+1}$, and from $\{a_1, c_{k+1}, b_1, v\}$ we see that $v \geqq c_{k+1}$. Thus $\{v, u\} \subseteq C_n$, so ${}^u/v$ is identified by θ_n. A similar argument handles the case $\{x, y\} \subseteq A_n$.

Finally let $\{x, y\} \subseteq C_n$, so that $c_k \leqq y < x \leqq c_n$ for some $k > n$. ${}^u/v \searrow {}^x/y$ is impossible, so let ${}^u/v \nearrow {}^x/y$, where $u \| y$, $u \vee y = x$, and $u \wedge y = v$. From $\{a_1, b_1, c_k, u\}$ it follows that $u \leqq a_1$ or $u \leqq b_1$, so $u \leqq a_1 \wedge c_n = a_n$ or $u \leqq b_n$. From $\{a_k, b_k, c_{k+1}, u\}$ we have $u \geqq a_k$ or $u \geqq b_k$. Thus $\{u, v\} \subseteq A_n$ or B_n, which completes the proof of the claim.

Theorem 1.1 now follows from the observation that

$$\bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} B_n = \bigcap_{n=1}^{\infty} C_n = \emptyset$$

and thus

$$\bigwedge_{n=1}^{\infty} \theta_n = \omega.$$

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