A CHARACTERIZATION OF ARTINIAN RINGS

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1. Introduction. Throughout this paper we consider associative rings with identity and assume that all modules are unitary. As is well known, cyclic modules play an important role in ring theory. Many nice properties of rings can be characterized by their cyclic modules, even by their simple modules. See, for example, [**2**], [**3**], [**6**], [**7**], [**13**], [**14**], [**15**], [**16**], [**18**], [**21**]. One of the most important results in this direction is the result of Osofsky [**14**, Theorem] which says: a ring R is semisimple (i.e. right artinian with zero Jacobson radical) if and only if every cyclic right R-module is injective. The other one is due to Vamos [**18**]: a ring R is right artinian if and only if every cyclic right R-module is finitely embedded.

Starting from the Osofsky's result, Boyle has introduced an interesting class of rings whose proper cyclic right modules are injective (the right PCI rings for short). Right PCI rings and related rings have drawn the attention of many authors (see, for example, [**3**], [**4**], [**7**], [**9**]). One other type of investigation in this direction is to characterize rings by means of their cyclic right modules all of which are assumed to satisfy some decomposition properties (see, for example, [**15**], [**16**], [**2**], [**6**]). Following the investigation of Smith in [**15**], Chatters [**2**, Theorem 3.1] obtained a nice characterization of right noetherian rings as rings whose cyclic right modules are direct sums of a projective module and a noetherian module.

In connection with all that above, we shall prove the following theorem.

THEOREM 1.1. *A ring R is right artinian if and only if every cyclic right R-module is a direct sum of an injective module and a finitely embedded module.*

Using this theorem, we can improve the module characterization of hereditarily artinian rings given in [**6**, Theorem 1] and prove some related results (Theorem 4.1, Propositions 4.2, 4.3).

2. Preliminaries. Let R be a ring. For a module M, M_R means that M is a right R-module, $\mathrm{Soc}(M_R)$ denotes the socle of M_R, $\dotplus$ and $\oplus$ stand for a module and ring theoretic direct sum, respectively. A submodule H of a module M is called *essential* in M if for each non-zero submodule N of M, $H \cap N \neq 0$. A module M is defined to be *finitely embedded* if $\mathrm{Soc}(M)$ is finitely generated and essential in M. Now, we say that a ring R satisfies the property (P) if every cyclic right R-module is a direct sum of an injective module and a finitely embedded module.

LEMMA 2.1. *If R is a ring satisfying (P) then every homomorphic image of R satisfies (P) too.*

Proof. Straightforward.

A ring R is called *right self-injective* if R_R is injective, and R is called *regular* (in the sense of von Neumann) if $a \in aRa$ for any $a \in R$. The following statement was proved by Osofsky [**14**, Lemma 5].

LEMMA 2.2. *Let R be a right self-injective regular ring and $\{e_i\}_{i=1}^{\infty}$ an infinite set of orthogonal idempotents of R. Then $R/(\dot{\sum}_{i=1}^{\infty} e_iR)$ is not injective over R.*

LEMMA 2.3. *Let R be a regular ring and e an idempotent of R. Then, for any $a \in R$, there exists an idempotent $f \in R$ such that $ef = fe = 0$ and $eR + aR = eR \dotplus fR$.*

Proof. See Lambek [**12**, p. 111].

A ring R is called an RM ring if, for each non-zero ideal I of R, R/I is right artinian. As a generalization of the Vamos result mentioned in the introduction, Armendariz and Hummel [**1**, Proposition 3.1] proved the following result.

LEMMA 2.4. *Let R be a ring such that, for each non-zero ideal I of R, R/I is a finitely embedded right R/I-module. Then R is an RM ring.*

3. Proof of Theorem 1.1. It is clear that every right artinian ring satisfies (P). Assume now that R is a ring satisfying (P). We first consider the case that R is semiprime. By (P) we have

$$R_R = A \dotplus B, \tag{1}$$

where A_R is a finitely embedded module and B_R is injective. Since R is semiprime, it is easy to see that $A_R = \mathrm{Soc}(A_R)$ is a direct sum of finitely many minimal idempotent right ideals of R. Without loss of generality, we can assume that A_R contains no non-zero injective submodules. Let C be the sum of all non-injective simple submodules of R_R. Then C is an ideal of R, obviously. Since $A \subseteq C$ we get, by (1), $C = A \dotplus C \cap B$. Since $\mathrm{Soc}(C_R) = C$, every minimal submodule of C_R is non-injective. On the other hand, every minimal submodule of B_R is injective by the injectivity of B_R and semiprimeness of R. Hence $C \cap B = 0$, which implies $C = A$. Then the fact that R is semiprime forces $R = A \oplus B$ at once. From this A is a semisimple ring and B is a right self-injective regular ring satisfying (P).

In order to show that B is right artinian, it is enough to show that B does not contain an infinite set of orthogonal idempotents. Assume the contrary that B contains an infinite set $\{e_i\}_{i=1}^{\infty}$ of orthogonal idempotents e_i. Put

$$D = \dot{\sum_{i=1}^{\infty}} e_iB.$$

Then the cyclic right B-module $\bar{B} = B/D$ has a direct decomposition

$$\bar{B} = \bar{K} \dotplus \bar{H},$$

where $\bar{K}_B$ is injective and $\bar{H}_B$ is finitely embedded. Let H be the inverse image of $\bar{H}$ in B. Then B/H is injective over B and H/D is finitely embedded over B. We first show that

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H_B is injective. If $B=H$, the statement is clear. We consider now the case $B\neq H$. It follows that

$$H=aB+\sum_{i=1}^{\infty} e_iB$$

for some $a\in H$. For $aB+e_1B$, Lemma 2.3 shows the existence of an idempotent f_0 of B with $f_0f_1=f_1f_0=0$ (here we set $f_1=e_1$) and $aB+e_1B=f_0B\dotplus f_1B$. Using Lemma 2.3, we can inductively show that

$$aB+e_1B+\ldots+e_nB=f_0B\dotplus f_1B\dotplus\ldots\dotplus f_nB$$

holds for each $n=1,2,\ldots$, where $\{f_j\}_{j=0}^{n}$ is a system of orthogonal idempotents. Hence

$$H=\bigcup_{n=1}^{\infty}(aB+e_1B+\ldots+e_nB)=\bigcup_{n=0}^{\infty}(f_0B\dotplus\ldots\dotplus f_nB)=\sum_{j=0}^{\dotplus} f_jB.$$

Since B/H is injective, Lemma 2.2 allows only finitely many non-zero f_j to occur in $\{f_j\}_{j=0}^{\infty}$, say $f_0,\ldots,f_m$. Let $f=f_0+\ldots+f_m$. Then $H=fB$; therefore H_B is injective.

Since the right B-module H/D has a finitely generated essential socle, H/D clearly has finite Goldie dimension, k say. Let $I_1,\ldots,I_{k+1}$ be infinite subsets of the index set $\{1,2,\ldots\}$ such that

$$I_1\cup I_2\cup\ldots\cup I_{k+1}=\{1,2,\ldots\}$$

and $I_i\cap I_j=\varnothing$ for $i\neq j$. Put

$$S_i=\sum_{j_i\in I_i}^{\dotplus} e_{j_i}B \qquad (e_{j_i}\in\{e_i\}_{i=1}^{\infty}). \tag{2}$$

Then

$$D=S_1\dotplus\ldots\dotplus S_{k+1}.$$

Let $E(S_i)$ be the injective hull of S_i in H with $E(S_i)\supseteq S_i$. Then we get

$$H=E(S_i)\dotplus\ldots\dotplus E(S_{k+1})\dotplus H'.$$

By (2) and since H is cyclic, $S_i\neq E(S_i)$ for each $i=1,\ldots,k+1$. From this, it is easy to see that the Goldie dimension of H/D is at least $k+1$, a contradiction. Hence B does not contain infinite sets of orthogonal idempotents. It follows that B and therefore R is a semisimple ring.

Now we go to the general case. Let N be the prime radical of R. Then $\bar{R}=R/N$ is a semisimple ring by Lemma 2.1 and the consideration above. As is well known, $\bar{R}_{\bar{R}}$ has the direct decomposition

$$\bar{R}=\bar{e}_1\bar{R}\dotplus\ldots\dotplus\bar{e}_m\bar{R},$$

where each $\bar{e}_i\bar{R}$ is a minimal right ideal of $\bar{R}$ and $\{\bar{e}_i\}_{i=1}^{m}$ is a set of orthogonal idempotents of $\bar{R}$. Since N is a nil ideal of R, there are orthogonal idempotents e_i of R with $e_i\in\bar{e}_i$ such that $e = e_1 + \ldots + e_m$ is the identity of R and

$$R = e_1R \dotplus \ldots \dotplus e_mR, \tag{3}$$

where each e_iR is an indecomposable right R-module. Then by (P), every e_iR is injective or it contains a finitely generated essential socle. Suppose for example that e_1R is injective. If $e_1R \cap N = 0$, e_1R is a minimal right ideal of R. For the case $e_1R \cap N \neq 0$, let x be a non-zero element in $e_1R \cap N$. Since xR can not contain non-zero injective submodules, xR contains a finitely generated essential socle by (P). In particular, e_1R contains a minimal submodule M. Then the injective hull of M in e_1R must coincide with e_1R. Hence $\mathrm{Soc}(e_1R) = M$. On the other hand we have, by (3),

$$\mathrm{Soc}(R_R) = \mathrm{Soc}(e_1R) \dotplus \ldots \dotplus \mathrm{Soc}(e_mR).$$

Combining these facts we get that $\mathrm{Soc}(R_R)$ is finitely generated and essential in R_R. From this and Lemma 2.1, every homomorphic image of R has also this property. Then Lemma 2.4 shows that R is an RM ring. Since $\mathrm{Soc}(R_R) \neq 0$, R is right artinian.

The proof of Theorem 1.1 is now complete.

REMARKS. A ring R is called *right PCI* if every proper (i.e. not isomorphic to R_R) cyclic right R-module is injective (cf. [**7**, p. 363]). As we have mentioned in the introduction, the right PCI rings have drawn the attention of many authors. It is remarkable to mention that every right PCI ring is right noetherian, see [**4**]. Now, similarly we could call a ring R right PCIA if every proper cyclic right R-module is a direct sum of an injective module and an artinian module. Then, in the connection with Theorem 1.1, it would be worth while to test for example whether or not a right PCIA ring is right noetherian. If it were the case then the problem raised by Camillo and Krause in [**22**, Open problems] could be answered positively.

In [**2**, Theorem 3.1], Chatters characterized right noetherian rings as those rings whose cyclic right modules are direct sums of a projective module and a noetherian module. Concerning this we consider the ring $\mathbb{Z}$ of all integers. It is clear that every cyclic $\mathbb{Z}$-module is either projective or finite. Thus, the property that every cyclic right module is a direct sum of a projective module and a module with finite length cannot characterize the class of right artinian rings.

4. Rings whose ideals are right artinian rings. A ring R is called *hereditarily artinian* if every ideal of R is a right artinian ring. Every semisimple ring is hereditarily artinian, however the converse is not true in general. The structure of hereditarily artinian rings was investigated in [**11**], [**19**], [**20**] and recently in [**6**]. Using Theorem 1.1, we now improve a result given in [**6**, Theorem 1].

THEOREM 4.1. *For a ring R the following conditions are equivalent*:

(a) *R is hereditarily artinian*;

(b) *R and each prime ideal of R are right artinian rings*;

(c) *$R = S \oplus F$, where S is semisimple and F is finite*;

(d) *R and the prime radical N of R are right artinian rings*;

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(e) *R is right artinian and N/N^2 is finite*;

(f) *every cyclic right R-module is a direct sum of an injective module whose factor modules contain no non-zero finite submodules and a finite module*;

(g) *every cyclic right R-module is a direct sum of an injective module and a finite module*;

(h) *every cyclic right R-module is a direct sum of an injective module and a module with a finite essential submodule.*

Proof. (a)$\Rightarrow$(b)$\Rightarrow$(c)$\Rightarrow$(d)$\Rightarrow$(e)$\Rightarrow$(f) are proved in [**6**, Theorem 1]; (f)$\Rightarrow$(g)$\Rightarrow$(h) are evident. Assume now that R satisfies (h). Then R is right artinian by Theorem 1.1. Let F be the largest finite ideal of R. Then R/F is a right artinian ring without non-zero finite right ideals, (cf. [**5**, Theorem 2]). By a similar argument to Lemma 2.1, R/F satisfies (h) too. Hence every principal right ideal of R/F is injective over R/F; therefore R/F is semisimple. Thus (a) holds.

The proof of Theorem 4.1 is now complete.

An element c of a ring R is called *regular* if c is not a right or left zero-divisor of R. Let $R \subseteq Q$ be rings. Then R is defined to be a *right order* in Q if every regular element of R is a unit in Q and every element q of Q has the form $q = ac^{-1}$ with a, $c \in R$ and c regular. In this case one says also that Q is the *classical right quotient ring* of R. Using Theorem 4.1(c) and some well-known facts about orders in artinian rings, we prove the following proposition.

PROPOSITION 4.2. *A ring R is a right order in a hereditarily artinian ring if and only if R is a direct sum of a semiprime right Goldie ring and a finite ring.*

Proof. Let R be a right order in a hereditarily artinian ring Q, and let F be the largest finite ideal of Q. Then, by Theorem 4.1, we have

$$Q = Q_1 \oplus F, \tag{1}$$

where Q_1 is a semisimple ring. Now, $F = (F \cap R)Q$. Since $F \cap R$ is finite, $c(F \cap R) = (F \cap R)c = F \cap R$ for all regular elements c of R; therefore $c^{-1}(F \cap R) = (F \cap R)c^{-1} = F \cap R$. Hence $F = (F \cap R)Q = Q(F \cap R) = F \cap R \subseteq R$. In particular, F is an ideal of R containing an identity by (1). It follows that $R = S \oplus F$. Comparing with (1), we get that S is a right order in Q_1. Hence S is a semiprime right Goldie ring.

The converse is clear.

REMARKS. In Proposition 4.2 one can prove more: if R is a right order in a hereditarily artinian ring then R is a direct sum of a hereditarily artinian ring and a semiprime right Goldie ring with a zero socle.

In [**5**, Theorem 4(a)] it was proved that the largest finite ideal of a right and left artinian ring R is a direct summand of R. From this we can, by the same as above, show that if R is a right order in a right and left artinian ring then R contains a largest finite ideal which is a direct summand of R. Now let R be a right and left order in a right and left artinian ring. Then we can show that R_R contains a largest artinian submodule A and

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${}_RR$ contains a largest artinian submodule B. In general $A \neq B$ and they are not direct summands of R. This shows, for example, in the matrix ring

$$R = \begin{bmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{bmatrix},$$

where $\mathbb{Z}$ is the ring of integers and $\mathbb{Q}$ is the field of rational numbers. We do not know whether in case $A = B$, A is a direct summand of R. Assume in addition that R is right and left noetherian, then A $(=B)$ is a direct summand of R (see [**8**]).

As is well known, for a ring R with the classical right quotient ring Q, Q is semisimple if and only if every right Q-module is injective over R (cf. [**17**, p. 58]). For the hereditarily artinian case we can prove the following result.

Proposition 4.3. *For a ring R with the classical right quotient ring Q the following conditions are equivalent*:

(i) *Q is hereditarily artinian*;

(ii) *every cyclic right Q-module is a direct sum of an injective right R-module and a finite right R-module.*

Proof. (i)$\Rightarrow$(ii). By Theorem 4.1 and Proposition 4.2, we have

$$Q = Q_1 \oplus E, \qquad R = R_1 \oplus E,$$

where E is finite, Q_1 is semisimple and R_1 is a right order in Q_1. Let M be a cyclic right Q-module. Then $M_Q = MQ_1 \dotplus ME$. Clearly, ME is finite. By [**17**, Proposition 3.8, p. 58], MQ_1 is injective over R_1, and so also over R, which proves (ii).

(ii)$\Rightarrow$(i). Let M be a cyclic right Q-module. By (ii),

$$M_R = I \dotplus F, \tag{1}$$

where I_R is injective and F_R is finite. Let $x \in I$. Then for each regular element c of R, $xc^{-1} \in M$. By (1), $xc^{-1} = x_1 + x_2$ $(x_1 \in I, x_2 \in F)$; hence $x = (xc^{-1})c = x_1c + x_2c$; therefore $x - x_1c = x_2c \in I \cap F = 0$, i.e. $xc^{-1} = x_1 \in I$. This shows that I is a right Q-module. Similarly, F is a right Q-module too. Now, let φ be a Q-homomorphism of a right ideal H of Q into I_Q. Then φ can be considered as an R-homomorphism of H_R into I_R. Since I_R is injective, there is an R-homomorphism ψ of Q_R into I_R such that $\varphi = \psi\tau$, where τ is the inclusion map of H into Q. Let x and q be elements of Q, $q = ac^{-1}$, a, $c \in R$, c regular. Then $\psi(xq)c = \psi(xac^{-1})c = \psi(xa) = \psi(x)a$; therefore $\psi(xq) = \psi(x)ac^{-1} = \psi(x)q$. This shows that ψ is a Q-homomorphism of Q into I_Q, hence I_Q is injective. By Theorem 4.1, Q is then hereditarily artinian. This completes the proof of Proposition 4.3.

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