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DENSITY OF NON QUASI-ANALYTIC CLASSES OF FUNCTIONS

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In the study of quasi-analytic classes (see [3], pp. 372–379), a class $C\{M_n\}$ is shown to have the following properties:

(a) If $M_0=1$ and $M_n^2 \le M_{n-1}M_{n+1}$ (i.e. $\{M_n\}$ is log convex), $C\{M_n\}$ forms an algebra.

(b) $C\{M_n\}$ is invariant under affine transformations.

(c) $C\{M_n\}$ is quasi-analytic iff it contains non non-trivial function with compact support.

We recall that, $\{M_n\}_{n=0}^{\infty}$ being a sequence of positive numbers, $C\{M_n\}=\{f\in C^\infty(R): \|f^{(j)}\|_\infty \le \alpha_f\beta_f^j M_j,\ j=0,1,2,\dots, \alpha_f>0$ and $\beta_f>0$ depending only on $f\}$. $C\{M_n\}$ is called quasi-analytic if $f^{(n)}(0)=0\,\forall n$ imply $f\equiv 0$. Otherwise $C\{M_n\}$ is non quasi-analytic. According to the Denjoy-Carleman theorem (see [3]), a necessary and sufficient condition for $C\{M_n\}$ to be quasi-analytic is that $\int_1^\infty \frac{\log t(r)}{r^2}\,dr$ diverges, where $t(r)=\sup_{n\ge 0}\left\{\frac{r^n}{M_n}\right\}$.

For complex-valued functions defined on R^m, $m>1$, we have the following

DEFINITION. $M_{(j)}$, $(j)=(j_1, j_2, \dots, j_m)$, $j_{k\ge 0}$, being a multi-sequence of positive numbers.

$C\{M_{(j)}\}=\{f\in C^\infty(R^m): \|f^{(j)}\|_\infty \le \alpha_f\beta_f^{|j|}M_{(j)}$, with $\alpha_f>0$, $\beta_f>0$ depending only on f and $|j|=\sum_{k=1}^m j_k\}$, where $f^{(j)}$ denotes $(\partial^{|j|})/(\partial x_1^{j_1}\partial x_2^{j_2}\cdots\partial x_n^{j_n})f$.

$C\{M_{(j)}\}$ is said to be quasi-analytic *I* if it contains no function $f\not\equiv 0$ s.t. $f^{(j)}(0)=0$, $\forall (j)$. Otherwise it is called non quasi-analytic I. A necessary and sufficient condition for it to be so is that each of the sequences $\{M_{j_1}, 0, \dots, 0\}$, $\{M_{0,j_2,0,\dots,0}\}, \dots, \{M_{0,0,\dots,j_m}\}$ gives rise to a quasi-analytic class (in one variable), as proved in [2].

$C\{M_{(j(}\}$ is said to be quasi-analytic II if it contains no function $f\not\equiv 0$ with compact support. Otherwise it is non quasi-analytic II. In [1], P. Lelong proved that $C\{M_{(j)}\}$ is so if and only if the integral $\int_1^\infty \log T(r)/(r^2)\,dr$ diverges where

$$T(r)=\sup_{p\ge 0}\left\{\frac{r^p}{\mu_p}\right\} \quad \text{with} \quad \mu_p=\inf_{|j|=p}\{M_{(j)}\}.$$

We let $C_0(R^m)$ denote the space of continuous functions vanishing at infinity and have the following

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THEOREM. *Let the class* $C\{M_{(j)}\}$ *be non quasi-analytic* II *with* $M_{(j)} \geq M_{j_1,0,\ldots,0}M_{0,j_2,\ldots,0}\cdots M_{0,0,\ldots,j_m}$ *for any* (j). *Then* $C\{M_{(j)}\}$ *is uniformly dense in* $C_0(R^m)$.

Proof. For $m=1$, we know that the two non quasi-analytic classes are identical.

Let $C_c\{M_n\}$ denote the subclass of $C\{M_n\}$ consisting of functions with compact support. $C_c\{M_n\}$ is obviously an algebra closed under complex conjugation and since it is also invariant under affine transformations, it follows that it separates points and vanishes identically at no point of R. $C_c\{M_n\}$ (and hence $C\{M_n\} \cap C_0(R)$) is then uniformly dense in $C_0(R)$.

For $m>1$, condition:

$$M_{(j)} \geq M_{(j_1,0,\ldots,0)}M_{(0,j_2,0,\ldots,0)}\cdots M_{(0,0,\ldots,j_m)}$$

for any (j), implies that $C\{M_{(j)}\}$ is non quasi-analytic II if and only if each of the m classes $C\{M_{(0,0,j_k,0,\ldots,0)}\}$, $1 \leq k \leq m$, is non quasi-analytic (in one variable).

Indeed, if $C\{M_{(j)}\}$ is non quasi-analytic II, then $\int_1^\infty (\log T(r))/(r^2)\,dr < \infty$. For any k, $1 \leq k \leq m$, if $j_k = p$, we have $\mu_p = \inf_{|j|=p}\{M_{(j)}\} \leq M_{0,\ldots,p,\ldots,0}$ and hence $T(r) \geq t_k(r)$ where $t_k(r) = \sup_{p\geq 0}\{r^p/M_{0,\ldots,p,\ldots,0}\}$. So $\int_1^\infty (\log t_k(r))/(r^2)\,dr < \infty$ and the class $C\{M_{(0,\ldots,j_k,\ldots,0)}\}$ is non quasi-analytic. Conversely, if the m classes are non quasi-analytic, there exists a function with compact support in each class and the product of these functions belongs to $C\{(M_{(j)}\}$ by the above condition on $M_{(j)}$.

We can now suppose, without loss of generality, that each sequence $\{M_{(0,\ldots,0,j_k,0,\ldots,0)}\}$ is log-convex, $1 \leq k \leq m$.

It remains to show that $P = C_c\{M_{(j_1,0,\ldots,0)}\} \otimes C_c\{M_{(0,j_2,\ldots,0)}\} \otimes \cdots \otimes C_c\{M_{(0,\ldots,0,j_m)}\}$ satisfies the hypotheses of the Stone-Weierstrass theorem: P is a subset of $C_c\{M_{(j)}\}$ and is a subalgebra of $C_0(R^m)$, closed under complex conjugation. We can then verify that P separates points of R^m and does not vanish identically at any point there, using the invariance under affine transformations of each class $C_c\{M_{(0,\ldots,0,j_k,0,\ldots,0)}\}$, $1 \leq k \leq m$. This completes the proof of the theorem in R^m.

REMARK. The above theorem will not hold if we suppose $C\{M_{(j)}\}$ non quasi-analytic I. Since there exist non quasi-analytic I classes which are quasi-analytic II, it is immediate that functions with compact support in $C_0(R^m)$ will not be approximated by functions of these classes.

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