ON THE ABSOLUTE NÖRLUND SUMMABILITY OF A FOURIER SERIES (II)[1]

FU CHENG HSIANG

(Received 21 March 1967)

1

Let $\sum_{n=0}^{\infty} a_n$ be a given series with its partial sums $\{s_n\}$ and $\{p_n\}$ a sequence of real or complex parameters. Write

$$P_n = p_0 + p_1 + p_2 + \cdots + p_n;\ p_{-k} = P_{-k} = 0 \qquad (k \geqq 1).$$

The transformation given by

$$t_n = \frac{1}{P_n} \sum_{\nu=0}^{n} p_{n-\nu} s_\nu$$

defines the Nörlund means of $\{s_n\}$ generated by $\{p_n\}$. The series $\sum a_n$ is said to be absolutely summable (N, p_n) or summable $|N, p_n|$, if $\{t_n\}$ is of bounded variation, i.e., $\sum |t_n - t_{n-1}|$ converges.

2

Let $f(t)$ be a periodic function with period 2π and integrable in the sense of Lebesgue over $(-\pi, \pi)$. Let

$$f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$
$$\approx \sum_{n=1}^{\infty} A_n(t).$$

We write

$$\varphi(t) = \tfrac{1}{2}\{f(x+t) + f(x-t)\}.$$

In this note, we prove the following theorem concerning the $|N, p_n|$ summability of the Fourier series of $f(t)$ at $t = x$.

THEOREM. *Let $\{p_n\}$ be a sequence of non-negative and non-increasing real parameters such that $\{\Delta p_n\}$ is monotonic. If* (i) *$\varphi(t)$ is of bounded variation in* $(0, \pi)$ *and* (ii) *$\{P_n \sum_{\nu=n}^{\infty} (\nu p_\nu)^{-1}\}$ is bounded, then the Fourier series of $f(t)$ is summable $|N, p_n|$ at $t = x$.*

[1] The first paper appears under the same title in this journal, vol. 7 (1967), 252–256.

3

The following lemmas are required.

LEMMA 1 (McFadden) [1]. *For* $0 \leqq a < b < \infty$, $0 \leqq t \leqq \pi$,

$$\left|\sum_{\nu=a}^{b} p_\nu e^{i(n-\nu)t}\right| \leqq AP_\tau$$

where $\tau = [t^{-1}]$.

LEMMA 2. *If* $\{p_\nu\}$ *is monotonic increasing and* $\{\Delta p_\nu\}$ *monotonic, then, for a fixed* n, $\{(P_n-P_\nu)(n-\nu)^{-1}\}$ *is non-increasing and* $\{(p_\nu-p_n)(n-\nu)^{-1}\}$ *monotonic in the same direction as* $\{\Delta p_\nu\}$.

PROOF. If $\{p_\nu\}$ is monotonic, then the sequence

$$\sigma_k = \frac{p_1+p_2+\cdots+p_k}{k}$$

is also monotonic in the same direction as $\{p_\nu\}$. Thus, we see that, for a fixed n, if $p_\nu \geqq 0$, $p_\nu \geqq p_{\nu+1}$, then

$$\frac{P_n-P_\nu}{n-\nu} = \frac{p_{\nu+1}+p_{\nu+2}+\cdots+p_n}{n-\nu}$$

is non-increasing for $\nu < n$ and since $\{\Delta p_\nu\}$ is monotonic,

$$\frac{p_\nu-p_n}{n-\nu} = \frac{(p_\nu-p_{\nu+1})+(p_{\nu+1}-p_{\nu+2})+\cdots+(p_{n-1}-p_n)}{n-\nu}$$

is also monotonic in the same direction as $\{p_\nu\}$. This proves the lemma.

LEMMA 3 (McFadden) [1].

$$\begin{aligned}
I &= \sum_{n=1}^{\infty} \frac{1}{P_nP_{n-1}} \left|\sum_{\nu=0}^{n-1} (P_np_\nu-P_\nu p_n)\frac{\sin (n-\nu)t}{n-\nu}\right| \\
&\leqq \sum_{n=1}^{\tau} \frac{1}{P_nP_{n-1}} \left|\sum_{\nu=0}^{n-1} (P_np_\nu-P_\nu p_n)\frac{\sin (n-\nu)t}{n-\nu}\right| \\
&\quad + \sum_{n=\tau+1}^{\infty} \frac{1}{P_nP_{n-1}} \left|\sum_{\nu=0}^{k-1} (P_np_\nu-P_\nu p_n)\frac{\sin (n-\nu)t}{n-\nu}\right| \\
&\quad + \sum_{n=\tau+1}^{\infty} \frac{1}{P_{n-1}} \left|\sum_{\nu=k}^{n-1} (p_\nu-p_n)\frac{\sin (n-\nu)t}{n-\nu}\right| \\
&\quad + \sum_{n=\tau+1}^{\infty} \frac{1}{P_nP_{n-1}} \left|\sum_{\nu=k}^{n-1} \left(\frac{P_n-P_\nu}{n-\nu}\right)\sin (n-\nu)t\right| \\
&= I_1+I_2+I_3+I_4,
\end{aligned}$$

say, where $\tau = [t^{-1}]$ *and* $k = [n/2]$.

https://doi.org/10.1017/S1446788700005723 Published online by Cambridge University Press

4

We have

$$t_n = \frac{1}{P_n} \sum_{\nu=0}^{n} p_{n-\nu} s_\nu$$

$$= \frac{1}{P_n} \sum_{\nu=0}^{n} P_\nu a_{n-\nu}$$

$$= a_0 + \frac{1}{P_n} \sum_{\nu=0}^{n-1} P_\nu a_{n-\nu},$$

and

$$t_{n-1} = \frac{1}{P_{n-1}} \sum_{\nu=0}^{n-1} P_\nu a_{n-\nu-1}$$

$$= a_0 + \frac{1}{P_{n-1}} \sum_{\nu=0}^{n-1} P_{\nu-1} a_{n-\nu},$$

thus,

$$|t_n - t_{n-1}| = \left| \sum_{\nu=0}^{n-1} \left(\frac{P_\nu}{P_n} - \frac{P_{\nu-1}}{P_{n-1}} \right) a_{n-\nu} \right|$$

$$= \frac{1}{P_n P_{n-1}} \left| \sum_{\nu=0}^{n-1} (P_n p_\nu - P_\nu p_n) a_{n-\nu} \right|.$$

Also, for the Fourier series of $f(t)$ at $t = x$,

$$A_n(x) = \frac{1}{\pi} \int_0^\pi \varphi(t) \cos nt dt.$$

5

In order to establish the theorem, it is enough to prove that, under the conditions of the theorem,

$$\sum_{n=1}^{\infty} \left| \int_0^\pi \varphi(t) \Omega(n, t) dt \right| \leqq A,$$

where

$$\Omega(n, t) = \frac{1}{P_n P_{n-1}} \sum_{\nu=0}^{n-1} (P_n p_\nu - P_\nu p_n) \cos (n-\nu)t,$$

and here and elsewhere A is an absolute constant not necessarily the same at each occurrence. Noticing that

$$\int_0^\pi \varphi(t) \Omega(n, t) dt = - \int_0^\pi \left\{ \int_0^t \Omega(n, u) du \right\} d\varphi(t),$$

and that

https://doi.org/10.1017/S1446788700005723 Published online by Cambridge University Press

$$\sum_{n=1}^{\infty}\left|\int_0^\pi\left\{\int_0^t \Omega(n, u)du\right\} d\varphi(t)\right| \leqq \int_0^\pi |d\varphi(t)|\left\{\sum_{n=1}^{\infty}\left|\int_0^t \Omega(n, u)du\right|\right\},$$

by (i), since $\varphi(t)$ is of bounded variation in $(0, \pi)$.

$$\int_0^\pi |d\varphi| < \infty,$$

we establish the theorem if we can show that

$$\sum_{n=1}^{\infty}\left|\int_0^t \Omega(n, u)du\right| \leqq A,$$

uniformly for $0 < t < \pi$, or that is the same thing,

$$I = \sum_{n=1}^{\infty}\frac{1}{P_nP_{n-1}}\left|\sum_{\nu=0}^{n-1}(P_np_\nu - P_\nu P p_n)\frac{\sin (n-\nu)t}{n-\nu}\right| \leqq A.$$

6

We denote $\tau = [t^{-1}]$ and $k = [n/2]$ and separate I in McFaddens' way as in Lemma 3. Since $P_n/p_n \geqq P_\nu/p_\nu$ for $\nu \leqq n$ and $|\sin (n-\nu)t/(n-\nu)| \leqq At$,

$$\begin{aligned}I_1 &= \sum_{n=1}^{\tau}\frac{1}{P_nP_{n-1}}\left|\sum_{\nu=0}^{n-1}(P_np_\nu - P_\nu p_n)\frac{\sin (n-\nu)t}{n-\nu}\right| \\ &\leqq At\sum_{n=1}^{\tau}\frac{1}{P_nP_{n-1}}\sum_{\nu=0}^{n-1}P_np_\nu \\ &= At\sum_{n=1}^{\tau}\frac{1}{P_{n-1}}\sum_{\nu=0}^{n-1}p_\nu \\ &\leqq A.\end{aligned}$$

By Abel's transformation and Lemma 1,

$$\begin{aligned}I_2 &= \sum_{n=\tau+1}^{\infty}\frac{1}{P_nP_{n-1}}\left|\sum_{\nu=0}^{k-1}\frac{P_n - P_\nu p_n/p_\nu}{n-\nu}p_\nu \sin (n-\nu)t\right| \\ &\leqq AP_\tau\sum_{n=\tau+1}^{\infty}\frac{1}{P_nP_{n-1}}\left(\frac{P_n - P_{k-1}p_n/p_{k-1}}{n-k+1}\right) \\ &\quad + AP_\tau\sum_{n=\tau+1}^{\infty}\frac{1}{P_nP_{n-1}}\sum_{\nu=0}^{k-2}\left|\Delta_\nu\left(\frac{P_n - P_\nu p_n/p_\nu}{n-\nu}\right)\right| \\ &\leqq AP_\tau\sum_{n=\tau+1}^{\infty}\frac{1}{P_nP_{n-1}}\cdot\frac{P_n}{n-k+1}\end{aligned}$$

$$+AP_\tau \sum_{n=\tau+1}^{\infty} \frac{1}{P_n P_{n-1}} \sum_{\nu=0}^{k-2} \frac{P_n}{(n-\nu)(n-\nu-1)}$$

$$+AP_\tau \sum_{n=\tau+1}^{\infty} \frac{1}{P_n P_{n-1}} \sum_{\nu=0}^{k-2} \frac{p_n}{n-\nu} \left(\frac{P_{\nu+1}}{p_{\nu+1}} - \frac{P_\nu}{p_\nu}\right)$$

$$\leqq AP_\tau \sum_{n=\tau+1}^{\infty} \frac{1}{nP_{n-1}} + AP_\tau \sum_{n=\tau+1}^{\infty} \frac{1}{P_{n-1}} \sum_{\nu=0}^{k-2} \frac{1}{(n-\nu)(n-\nu-1)}$$

$$+AP_\tau \sum_{n=\tau+1}^{\infty} \frac{p_n}{P_n P_{n-1}} \cdot \frac{1}{n-k+2} \left(\frac{P_{k-1}}{p_{k-1}} - \frac{P_0}{p_0}\right)$$

$$\leqq A + AP_\tau \sum_{n=\tau+1}^{\infty} \frac{1}{nP_{n-1}}$$

$$\leqq A + AP_\tau \sum_{n=\tau}^{\infty} \frac{1}{nP_n}$$

$$< A,$$

by (ii). By Lemma 2, since $\{(p_\nu - p_n)(n-\nu)^{-1}\}$ is monotonic, Abel's transformation gives

$$\left|\sum_{\nu=k}^{n-1} \frac{p_\nu - p_n}{n-\nu} \sin (n-\nu)t\right|$$

$$\leqq \frac{A}{t} \frac{p_k - p_n}{n-k} + \frac{A}{t} (p_{n-1} - p_n) + \frac{A}{t} \sum_{\nu=k}^{n-2} \left|\Delta_\nu \left(\frac{p_\nu - p_n}{n-\nu}\right)\right|$$

$$\leqq \frac{A}{t} \cdot \frac{p_k}{k} + \frac{A}{t} (p_{n-1} - p_n) + \frac{A}{t} \left|\frac{p_k - p_n}{n-k} - (p_{n-1} - p_n)\right|$$

$$\leqq \frac{A}{t} \cdot \frac{p_k}{k} + \frac{A}{t} (p_{n-1} - p_n).$$

Hence,

$$I_3 \leqq \frac{A}{t} \sum_{n=\tau+1}^{\infty} \frac{p_k}{kP_{n-1}} + \frac{A}{t} \sum_{n=\tau+1}^{\infty} \frac{p_{n-1} - p_n}{P_{n-1}}$$

$$\leqq \frac{A}{t} \sum_{n=\tau+1}^{\infty} \frac{1}{k(k-1)} + \frac{A}{t} \cdot \frac{p_\tau}{P_\tau}$$

$$\leqq A + A\tau \frac{p_\tau}{P_\tau}$$

$$< A.$$

Moreover, by Lemma 2, since $\{(P_n - P_\nu)(n-\nu)^{-1}\}$ is non-increasing,

https://doi.org/10.1017/S1446788700005723 Published online by Cambridge University Press

$$\left|\sum_{\nu=k}^{n-1}\frac{P_n-P_\nu}{n-\nu}\sin(n-\nu)t\right|$$
$$\leqq \frac{A}{t}\sum_{\nu=k}^{n-2}\left|\Delta_\nu\left(\frac{P_n-P_\nu}{n-\nu}\right)\right|+\frac{A}{t}\cdot\frac{P_n-P_k}{n-k}+\frac{A}{t}p_n$$
$$\leqq \frac{A}{t}\cdot\frac{P_n-P_k}{n-k}+\frac{A}{t}p_n+\frac{A}{t}\cdot\frac{P_n-P_k}{n-k}+\frac{A}{t}p_n$$
$$\leqq \frac{AP_n}{nt}.$$

Thus, we obtain

$$I_4 \leqq \frac{A}{t}\sum_{n=\tau+1}^{\infty}\frac{p_n}{nP_n}$$
$$\leqq \frac{A}{t}\sum_{n=\tau+1}^{\infty}\frac{1}{n(n-1)}$$
$$< A.$$

This completes the proof of the theorem.

Reference

[1] L. McFadden, 'Absolute Nörlund summability', *Duke Math. Jour.*, 9 (1942), 168—207.

National Taiwan University
Formosa, China