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ON B_4-SEQUENCES

BY
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ABSTRACT. In [2], Erdös showed that the counting function $A(n)$ of a B_2-sequence satisfies $\underline{\lim} A(N)\log^{1/2}/n^{1/2} < \infty$. Here it is shown that $A(n)$ satisfies an analogous relationship for B_4-sequences! $\underline{\lim} A(n)\log^{1/4} n/n^{1/4} < \infty$.

Notation and terminology. A denotes a set of positive integers. $nA = \{a_1 + a_2 + \cdots + a_n | a_i \in A\}$. $A(n) = |A \cap \{1, 2, , \ldots, n\}|$. A is a B_4-sequence if the equation

$$(1) \qquad n = a_1 + a_2 + \cdots + a_k, a_1 \leqq a_2 \leqq \cdots \leqq a_k, a_i \in A,$$

has at most one solution for all n.

Introduction. In [2], Erdös showed that

$$(2) \qquad \underline{\lim} A(n)\log^{1/2} n/n^{1/2} < \infty$$

for all B_2-sequences. I will show that the analogous relationship

$$(3) \qquad \underline{\lim} A(n)\log^{1/4} n/n^{1/4} < \infty$$

for all B_4-sequences.

Let A be a B_4-sequence, so that $A(N) \ll N^{1/4}$. Then A is also a B_2-sequence (as well as a B_3-sequence) and therefore, if n is large enough,

$$(2A)(n) \geqq \binom{A[n/2]}{2} \geqq A\left(\left[\frac{n}{2}\right]\right)^2.$$

Thus (3) would follow at once from

$$(4) \qquad \underline{\lim} (2A)(n)\log^{1/2}/n^{1/2} < \infty;$$

and (4) would be true if $2A$ were a B_2-sequence. While this is not the case – $(a+c)+(b+d) = (a+b)+(c+d) = (a+d)+(b+c)$ – we shall see that $2A$ is close enough in structure to a B_2-sequence for Erdös' proof of (2) to apply.

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Lemma 1 below contains the essence of Erdös' argument.

LEMMA 1. *Let C be any sequence of positive integers and let D_l denote the number of elements of C in the interval $(l-1)N < c \leqq lN$, $(l = 1, 2, \ldots, N)$. If*

$$\sum_{l=1}^{N} D_l^2 \ll N, \tag{5}$$

then

$$\underline{\lim} C(n) \log^{1/2} n / n^{1/2} < \infty. \tag{6}$$

PROOF. (See [1], pp. 89–90.)

Let $\tau_A(N) = \int_{n \geqq N} A(n)(\log n/n)^{1/2}$. We shall show that $\tau_A(N) \ll 1$, where the implied constant is absolute. By Cauchy's inequality,

$$\left(\sum_{l=1}^{N} \frac{1}{l}\right)\left(\sum_{l=1}^{N} D_l^2\right) \geqq \left(\sum_{l=1}^{N} \frac{D_l}{l^{1/2}}\right)^2. \tag{A}$$

Furthermore,

$$\begin{aligned}
\sum_{l=1}^{N} \frac{D_l}{l^{1/2}} &= \sum_{l=1}^{N} \left(A(lN) - A\left((l-1)N\right)\right) \frac{1}{l^{1/2}} \\
&= \sum_{l=1}^{N} A(lN) \left(\frac{1}{l^{1/2}} - \frac{1}{(l+1)^{1/2}}\right) + \frac{A(N^2)}{(N+1)^{1/2}} \\
&\geqq \tau_A(N) \sum_{l=1}^{N} \left(\frac{lN}{\log lN}\right)^{1/2} \left(\frac{1}{l^{1/2}} - \frac{1}{(l+1)^{1/2}}\right) \\
&\gg \tau_A(N) \left(\frac{N}{\log N}\right)^{1/2} \sum_{l=1}^{N} \frac{1}{l}
\end{aligned}$$

Substituting in (A), we obtain

$$\sum_{l=1}^{N} D_l^2 \gg N \tau_A^2(N),$$

and (4) now yields the required inequality $\tau_A(N) \ll 1$. □

Thus if (5) is true when $C = 2A$, (4) holds and (3) follows. Accordingly, we study the strictly positive differences of elements from $2A$ in blocks of length N, $[(l-1), lN]$, $1 \leqq l \leqq N$, just as Erdös did when proving (5) for B_2-sequences. Since

$$4\binom{D_l}{2} \leqq D_l^2$$

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except when $D_l = 1$, we have

$$\sum_{l=1}^{N} D_l^2 \leqq 4\sum_{l=1}^{N}\binom{D_l}{2} + \sum_{\substack{l=1\\ D_l=1}}^{N} 1 \leqq 4\sum_{l=1}^{N}\binom{D_l}{2} + N,$$

and (5) will follow from

$$\sum_{l=1}^{N}\binom{D_l}{2} \ll N. \tag{7}$$

Observe that there are precisely

$$\binom{D_l}{2}$$

positive differences that can be formed from elements of $2A$ in the l-th block, and that the difference lies in $(0, N]$. Thus, if

$$S = \left\{(a_1, a_2, a_3, a_4) : a_i \in A, a_i \leqq N^2, 1 \leqq a_1 + a_2 - a_3 - a_4 \leqq N\right\},$$

then

$$\sum_{l=1}^{N}\binom{D_l}{2} \leqq |S|,$$

so that to prove (7) it suffices to show that

$$|S| \ll N. \tag{8}$$

We divide the 4-tuples in S into two classes: the first class to consist of those 4-tuples that satisfy, in addition to the conditions implicit in the definition of S,

$$a_1 \neq a_3, a_1 \neq a_4, a_2 \neq a_3, a_2 \neq a_4, \tag{9}$$

and the second class to contain the remaining 4-tuples.

Consider the 4-tuples from the first class. If $(a_1, \ldots, a_4)$ and $(a_1', \ldots, a_4')$ belong to the first class and are such that

$$a_1 + a_2 - a_3 - a_4 = a_1' + a_2' - a_3' - a_4',$$

then $a_1 + a_2 + a_3' + a_4' = a_1' + a_2' + a_3 + a_4$; by the B_4-property of A it follows that the numbers a_1', a_2', a_3, a_4 form a permutation of the numbers a_1, a_2, a_3', a_4'. In view of (9), this can only hold in the four cases $(a_1', a_2', a_3', a_4') = (a_1, a_2, a_3, a_4)$, (a_2, a_1, a_3, a_4), (a_1, a_2, a_4, a_3) or (a_2, a_1, a_4, a_3). Thus, for each n, $1 \leqq n \leqq N$, there are at most 4-tuples $(a_1, \ldots, a_4)$ in S of the first class with $a_1 + a_2 - a_3 - a_4 = n$. The contribution to $|S|$ from the first class is therefore at most $4N$.

We now turn to the 4-tuples in S of the second class, i.e., those 4-tuples $(a_1, \dots, a_4)$ for which one of the conditions in (9) is violated. Assume, for example, that the first condition fails, so that $a_1 = a_3$ and

$$a_1 + a_2 - a_3 - a_4 = a_2 - a_4.$$

The contribution of such 4-tuples to $|S|$ is equal to $A(N^2)$ – the number of choices of a_1 – *times* the cardinality $|T|$ of the set

$$T = \left\{(a_2, a_4) : a_i \in A, a_i \leqq N^2, 1 \leqq a_2 - a_4 \leqq N\right\}.$$

The same bound applies in the case of any one of the remaining three conditions in (9) being violated, so that altogether there are at most $4A(N^2)|T|$ 4-tuples in the second class. Thus

$$|S| \leqq 4N + 4A(N^2)|T|; \tag{10}$$

since

$$A(N^2) \ll N^{1/2}, \tag{11}$$

the desired bound (8) follows from

$$|T| \ll N^{1/2}. \tag{12}$$

It remains to prove (12). Observe that

$$\begin{aligned} \binom{|T|}{2} &\leqq \#\left\{(a_1, a_2, a_3, a_4) : a_i \in A, a_i \leqq N^2, 1 \leqq a_4 - a_2 < a_1 - a_3 \leqq N\right\} \\ &\leqq \#\left\{(a_1, a_2, a_3, a_4) : a_i \in A, a_i \leqq N^2, 1 \leqq (a_1 - a_3) - (a_4 - a_2) \leqq N\right\} \\ &= |S|. \end{aligned} \tag{13}$$

For $|T| \geqq 2$ we have

$$|T|^2 \ll \binom{|T|}{2}$$

and we obtain, substituting (11) and (13) into (10),

$$|T|^2 \ll N + N^{1/2}|T|.$$

This implies (12), and the proof of (8) – and therefore also of (3) – is now complete.

References

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