Proceedings of the Edinburgh Mathematical Society (2014) **57**, 175–189
DOI:10.1017/S0013091513000874

HODGE THEORY ON GENERALIZED NORMAL CROSSING VARIETIES

YUJIRO KAWAMATA

Graduate School of Mathematical Sciences, University of Tokyo, Komaba, Meguro, Tokyo 153-8914, Japan (kawamata@ms.u-tokyo.ac.jp)

Abstract We generalize some results in Hodge theory to generalized normal crossing varieties.

Keywords: generalized normal crossing varieties; generalized normal crossing pair; mixed Hodge structure

2010 *Mathematics subject classification:* Primary 14D07
Secondary 14D06; 14E15; 14F10; 32G20

1. Introduction

A normal crossing variety is defined to be a variety that is locally isomorphic to a normal crossing divisor in a smooth variety. Similarly, we define a *generalized* normal crossing variety to be a variety that is locally isomorphic to a direct product of normal crossing divisors in smooth varieties.

A suitably defined generalized normal crossing *divisor* in a generalized normal crossing variety is again a generalized normal crossing variety. In this way, a certain induction argument on the dimension works for generalized normal crossing varieties and divisors. For example, a reducible divisor often appears and is transformed to a normal crossing divisor by resolution of singularities. Moreover, one sometimes has to consider a reducible divisor on a reducible variety during the induction argument on the dimension, as in [**11**].

A generalized normal crossing variety appears naturally in a higher-dimensional stable reduction. Indeed, a fibre of a semi-stable family is a normal crossing variety, and a fibre of a generalized semi-stable family of Abramovich and Karu [**1**] is a generalized normal crossing variety.

If each irreducible component of the intersections of some of the irreducible components of a generalized normal crossing variety is smooth, then it is called a generalized *simple* normal crossing (GSNC) variety.

The purpose of this paper is to prove that some Hodge theoretic statements for smooth varieties or pairs can be generalized to generalized simple normal crossing varieties or pairs. It is similar to the extensions in [**12**] to simple normal crossing varieties. The proofs are, therefore, similar in many places, but we need to take additional care over the details. For example, we have a cell complex instead of a simplicial complex in Proposition 2.1. We concentrate on the differences from [**12**] in this paper.

The paper has the following structure. We first prove that there exists a naturally defined cohomological mixed Hodge $\boldsymbol{Q}$-complex on a projective GSNC pair: a pair consisting of a GSNC variety and a GSNC divisor (Theorem 3.2). By the standard machinery due to Deligne (see [**3**]), we obtain degenerations of spectral sequences as corollaries. The point is that we have explicit descriptions of the graded pieces (see Lemma 3.1), so concrete calculations are possible.

We generalize the above absolute case to the relative case where every closed stratum is assumed to be smooth over the base (see Theorem 4.3). By using the theory of canonical extensions, we then prove that higher direct images of the structure sheaves are locally free (see Theorem 5.1).

As an application, we prove a generalization of Kollár's vanishing theorem (see Theorem 6.1), thereby modifying the contents of [**11**, § 4], and correcting an error in the proof of [**11**, Theorem 4.3]. Fujino [**6**] found a simpler alternative proof of its main theorem, but our construction, which shows that the original proof still works, might also be useful at some point.

In § 1, we give definitions concerning the GSNC varieties and GSNC pairs. We prove some lemmas on desingularizations and coverings. In § 2, we construct a cohomological mixed Hodge $\boldsymbol{Q}$-complex on GSNC pairs. The results in § 2 are generalized to relative settings in § 3. We extend the results in § 4 to the case where there are degenerate fibres, by using the theory of canonical extensions. We prove a vanishing theorem of Kollár type for GSNC varieties in § 5, following the original proof of [**14**]. It is generalized to the $\boldsymbol{Q}$-divisor version by using the covering lemma, thereby completing the modification of the argument in [**11**].

We work over the base field $\boldsymbol{C}$.

2. GSNC pairs

A reduced complex analytic space X is said to be a *generalized simple normal crossing variety (GSNC variety)* if the following conditions are satisfied.

(1) (Local.) At each point $x \in X$, there exists a complex analytic neighbourhood that is isomorphic to a direct product to normal crossing varieties, i.e. varieties isomorphic to normal crossing divisors on smooth varieties.

(2) (Global.) Any irreducible component of the intersection of some of the irreducible components of X is smooth.

The first condition can be put as follows: there exists a complex analytic neighbourhood X_x at each point x that is embedded into a smooth variety V with a coordinate system such that X_x is a complete intersection of divisors defined by monomials of coordinates. The *level* of X at x is the smallest number of such equations. For example, a fibre of a semi-stable family (see [**1**]) satisfies the first condition. X is a locally complete intersection, hence Gorenstein and locally equidimensional.

A *closed stratum* of X is an irreducible component of the intersection of some of the irreducible components of X. A closed stratum is smooth by assumption. In the case where X is connected, let $X^{[n]}$ denote, for an integer n, the disjoint union of all the closed strata of codimension n in X.

The combinatorics of the closed strata is described by the *dual graph*. It is a cell complex where a closed stratum of codimension n corresponds to an n-cell, and the inclusion of closed strata corresponds to the boundary relation in the opposite direction. Any cell is a direct product of simplices due to the local structure of X.

Proposition 2.1. *There exists a Mayer–Vietoris exact sequence*

$$0 \to \boldsymbol{Q}_X \to \boldsymbol{Q}_{X^{[0]}} \to \boldsymbol{Q}_{X^{[1]}} \to \cdots \to \boldsymbol{Q}_{X^{[N]}} \to 0,$$

where $N = \dim X$ and the arrows are alternating sums of restriction homomorphisms.

Proof. We assign an arbitrarily fixed orientation to each cell of the dual graph. The corresponding chain complex then has boundary maps as alternating sums. Namely, the sign of a boundary map for each pair of closed strata is positive or negative according to whether the orientations are compatible or not. The sign convention of the sequence of the constant sheaves is defined accordingly. Since each cell is contractible, the chain complex is locally exact; hence, the sequence of sheaves is exact. □

A Cartier divisor D on X is said to be *permissible* if it does not contain any stratum. We denote by $D|_Y$ the induced Cartier divisor on a closed stratum Y. A *generalized simple normal crossing divisor (GSNC divisor)* B is a permissible Cartier divisor such that $B|_Y$ is a reduced simple normal crossing divisor for each closed stratum Y. In this case, B is again a GSNC variety of codimension 1 in X. Namely, a GSNC divisor on a GSNC variety of level r is a GSNC variety of level $r + 1$. Such an inductive structure may be an advantage. If D is a permissible $\boldsymbol{Q}$-Cartier divisor whose support is a GSNC divisor, then the round up $\ulcorner D \urcorner$ is well defined. We have $\ulcorner D \urcorner|_Y = \ulcorner D|_Y \urcorner$ for any closed stratum Y.

A *generalized simple normal crossing pair (GSNC pair)* (X, B) consists of a GSNC variety X and a GSNC divisor B on X. For each closed stratum Y of X, we define $B_Y = B|_Y$. We denote by $B_{X^{[n]}}$ the union of all the simple normal crossing divisors B_Y for irreducible components Y of $X^{[n]}$. A *closed stratum* of the pair (X, B) is an irreducible component of the intersection of some of the irreducible components of X and B.

A morphism $f\colon X' \to X$ between two GSNC varieties is said to be a *permissible birational morphism* if it induces a bijection between the sets of closed strata of X and X', and birational morphisms between closed strata. A smooth subvariety C of X is said to be a *permissible centre* for a GSNC pair (X, B) if the following conditions are satisfied.

(1) C does not contain any closed stratum, and the scheme theoretic intersections C_Y of C and the closed strata Y are smooth.

(2) At each point $x \in C_Y$, there exists a coordinate system in a neighbourhood of x such that B_Y and C_Y are defined by ideals generated by monomials.

https://doi.org/10.1017/S0013091513000874 Published online by Cambridge University Press

A blow-up $f\colon X' \to X$ whose centre is permissible with respect to a GSNC pair (X, B) is called a *permissible blow-up*. It is a permissible birational morphism from another GSNC variety, and the union of the strict transform of B and the exceptional divisor is again a GSNC divisor.

The following is a corollary of a theorem of Hironaka [**8**, **9**].

Lemma 2.2. *Let (X, B) be a GSNC pair. Consider one of the following situations.*

(a) *Let $f\colon X \dashrightarrow S$ be a rational map to another variety whose domain of definition has a non-empty intersection with each closed stratum of the pair (X, B).*

(b) *Let Z be a closed subset that does not contain any closed stratum of the pair (X, B).*

There then exists a tower of permissible blow-ups

$$g\colon (X_n, B_n) \to (X_{n-1}, B_{n-1}) \to \cdots \to (X_0, B_0) = (X, B),$$

where B_{i+1} is the union of the strict transform of B_i and the exceptional divisor, satisfying the following for the situations (a) and (b), respectively.

(a) *There exists a morphism $h\colon X_n \to S$ such that $h = f \circ g$.*

(b) *The union $\bar{B}_n = B_n \cup g^{-1}(Z)$ is a GSNC divisor on X_n.*

Proof. (a) We reduce inductively the indeterminacy locus of the rational map f. For an irreducible component Y of X, the restriction of f to Y is resolved by a tower of permissible blow-ups of the pair (Y, B_Y), by the theorem of Hironaka. Since the indeterminacy locus of f does not contain any closed stratum, the centres of the blow-ups do not contain any closed stratum either. Moreover, this property is preserved in the process. Therefore, the same centres determine a tower of permissible blow-ups of (X, B). If we follow the same process for all the irreducible components of X, then we obtain the assertion.

(b) This proof is similar to that of (a). We resolve Z at each irreducible component Y of X, since Z does not contain any closed stratum of the pair (Y, B_Y). □

We generalize a covering lemma [**10**].

Lemma 2.3. *Let (X, B) be a quasi-projective GSNC pair. Let D_j $(j = 1, \dots, r)$ be permissible reduced Cartier divisors that satisfy the following conditions:*

(1) *$D_j \subset B$,*

(2) *D_j and $D_{j'}$ have no common irreducible components if $j \neq j'$,*

(3) *$D_j \cap Y$ are smooth for all D_j and closed strata Y of X.*

Let d_j be rational numbers, and let $D = \sum d_j D_j$. There then exists a finite surjective morphism from another GSNC pair $\pi\colon (X', B') \to (X, B)$ such that $B' = \pi^{-1}(B)$ and $\pi^* D$ has integral coefficients.

Proof. The proof is similar to that of [**10**, Theorem 17], where the D_j play the role of the irreducible components of D. □

https://doi.org/10.1017/S0013091513000874 Published online by Cambridge University Press

3. Hodge theory on a GSNC pair

We construct explicitly the mixed Hodge structures of Deligne [**3**] for GSNC pairs.

Let (X, B) be a GSNC pair. We do not assume that X is compact at the moment. We define a cohomological mixed Hodge complex on X when X is projective. We can extend the construction in [**12**] to a GSNC pair, but we use here the de Rham complex of Du Bois [**5**]. The underlying local system is the constant sheaf $\boldsymbol{Q}$ in our case, while it is different and more difficult in the former case.

We define a *de Rham complex* $\bar{\Omega}_X^\bullet(\log B)$ by the following Mayer–Vietoris exact sequence:

$$0 \to \bar{\Omega}_X^\bullet(\log B) \to \Omega_{X^{[0]}}^\bullet(\log B_{X^{[0]}}) \to \Omega_{X^{[1]}}^\bullet(\log B_{X^{[1]}}) \to \cdots \to \Omega_{X^{[N]}}^\bullet(\log B_{X^{[N]}}) \to 0,$$

where $N = \dim X$ and the arrows are the alternating sums of the restriction homomorphisms, with the signs as defined in Proposition 2.1. We have $\bar{\Omega}_X^0 \cong \mathcal{O}_X$, and

$$Ri_*\boldsymbol{C}_{X\setminus B} \cong \bar{\Omega}_X^\bullet(\log B)$$

for the open immersion $i\colon X \setminus B \to X$.

We define a *weight filtration* on the complex $\bar{\Omega}_X^\bullet(\log B)$ by

$$0 \to W_q(\bar{\Omega}_X^\bullet(\log B)) \to W_q(\Omega_{X^{[0]}}^\bullet(\log B_{X^{[0]}})) \to W_{q+1}(\Omega_{X^{[1]}}^\bullet(\log B_{X^{[1]}})) \to \cdots \to W_{q+N}(\Omega_{X^{[N]}}^\bullet(\log B_{X^{[N]}})) \to 0,$$

where the W, except in the first term, denotes the filtration with respect to the order of log poles. For example, $W_q(\Omega_{X^{[0]}}^\bullet(\log B_{X^{[0]}}))$ is the subcomplex of $\Omega_{X^{[0]}}^\bullet(\log B_{X^{[0]}})$ consisting of logarithmic forms on $X^{[0]}$ whose log poles along $B_{X^{[0]}}$ have order at most q.

Before we define a weight filtration on the $\boldsymbol{Q}$-level object, we recall the definition of a convolution of a complex of objects in a triangulated category [**7**]. Let

$$a_0 \to a_1 \to \cdots \to a_{n-1} \to a_n$$

be a complex of objects. If there exists a sequence of distinguished triangles

$$b_{t+1}[-1] \to b_t \to a_t \to b_{t+1}$$

for $0 \leqslant t < n$ with an isomorphism $b_n \to a_n$, then b_0 is said to be a *convolution* of the complex. A convolution may not exist and may not be unique if it does exist.

We also need to define a *canonical filtration* of a complex. If $a_\bullet$ is a complex in an abelian category, then we define

$$\tau_{\leqslant q}(a_\bullet)_n = \begin{cases} a_n & \text{if } n < q, \\ \operatorname{Ker}(a_q \to a_{q+1}), & \\ 0 & \text{if } n > q, \end{cases}$$

so $H^n(\tau_{\leqslant q}(a_\bullet)) \cong H^n(a_\bullet)$ if $n \leqslant q$, and $H^n(\tau_{\leqslant q}(a_\bullet)) \cong 0$ otherwise.

In the same way as in [**12**], we can define a *weight filtration* $W_q(Ri_*\boldsymbol{Q}_{X\setminus B})$ as a convolution of the complex of objects

$$\tau_{\leqslant q}(R(i_0)_*\boldsymbol{Q}_{X^{[0]}\setminus B_{X^{[0]}}}) \to \tau_{\leqslant q+1}(R(i_1)_*\boldsymbol{Q}_{X^{[1]}\setminus B_{X^{[1]}}}) \to \cdots \to \tau_{\leqslant q+N}(R(i_N)_*\boldsymbol{Q}_{X^{[N]}\setminus B_{X^{[N]}}}),$$

where τ denotes the canonical filtration and $i_p\colon X^{[p]}\setminus B_{X^{[p]}} \to X^{[p]}$ are open immersions, such that there exist isomorphisms

$$W_q(Ri_*\boldsymbol{Q}_{X\setminus B}) \otimes \boldsymbol{C} \cong W_q(\bar{\Omega}^\bullet_X(\log B))$$

for all q.

We define the *Hodge filtration* as a stupid filtration

$$F^p(\bar{\Omega}^\bullet_X(\log B)) = \bar{\Omega}^{\geqslant p}_X(\log B).$$

We then have the following lemma, the proof of which can be compared with that of [**12**, Lemma 3.2].

Lemma 3.1. *We have that*

$$\mathrm{Gr}^W_q(Ri_*\boldsymbol{Q}_{X\setminus B}) \cong \bigoplus_{p\geqslant 0,\,\dim X^{[p]}-\dim Y=p+q} \boldsymbol{Q}_Y[-2p-q],$$

$$\mathrm{Gr}^W_q(\bar{\Omega}^\bullet_X(\log B)) \cong \bigoplus_{p\geqslant 0,\,\dim X^{[p]}-\dim Y=p+q} \Omega^\bullet_Y[-2p-q],$$

$$F^r(\mathrm{Gr}^W_q(\bar{\Omega}^\bullet_X(\log B))) \cong \bigoplus_{p\geqslant 0,\,\dim X^{[p]}-\dim Y=p+q} \Omega^{\geqslant r-p-q}_Y[-2p-q],$$

where the p run over all non-negative integers and the Y run over all the closed strata of the pair $(X^{[p]}, B_{X^{[p]}})$ of codimension $p+q$.

Proof. We have that

$$\begin{aligned}
\mathrm{Gr}^W_q(Ri_*\boldsymbol{Q}_{X\setminus B}) &\cong \bigoplus_p R^{p+q}(i_p)_*\boldsymbol{Q}_{X^{[p]}\setminus B_{X^{[p]}}}[-2p-q] \\
&\cong \bigoplus_{p\geqslant 0,\,\dim X^{[p]}-\dim Y=p+q} \boldsymbol{Q}_Y[-2p-q], \\
\mathrm{Gr}^W_q(\bar{\Omega}^\bullet_X(\log B)) &\cong \bigoplus_p \mathrm{Gr}^W_{p+q}(\Omega^\bullet_{X^{[p]}}(\log B_{X^{[p]}}))[-p] \\
&\cong \bigoplus_{p\geqslant 0,\,\dim X^{[p]}-\dim Y=p+q} \Omega^\bullet_Y[-2p-q], \\
F^r(\mathrm{Gr}^W_q(\bar{\Omega}^\bullet_X(\log B))) &\cong \bigoplus_p \mathrm{Gr}^W_{p+q}(F^r(\Omega^\bullet_{X^{[p]}}(\log B_{X^{[p]}})))[-p] \\
&\cong \bigoplus_{p\geqslant 0,\,\dim X^{[p]}-\dim Y=p+q} \Omega^{\geqslant r-p-q}_Y[-2p-q].
\end{aligned}$$

□

As a corollary, we have the following theorem, the proof of which is similar to that of [**12**, Theorem 3.3].

Theorem 3.2. *Let (X, B) be a GSNC pair. Assume that X is projective. Then,*

$$((Ri_*\boldsymbol{Q}_{X\setminus B}, W), (\bar{\Omega}^\bullet_X(\log B), W, F))$$

is a cohomological mixed Hodge $\boldsymbol{Q}$-complex on X.

Proof. If Y is a closed stratum of the pair $(X^{[p]}, B_{X^{[p]}})$ of codimension $p+q$, then

$$(\boldsymbol{Q}_Y, \Omega^\bullet_Y, F(-p-q))$$

is a cohomological Hodge $\boldsymbol{Q}$-complex of weight $2(p+q)$, where $F(-p-q)^r = F^{r-p-q}$. Hence,

$$(\boldsymbol{Q}_Y[-2p-q], \Omega^\bullet_Y[-2p-q], F(-p-q)[-2p-q])$$

is a cohomological Hodge $\boldsymbol{Q}$-complex of weight $2(p+q) - 2p - q = q$. □

Corollary 3.3. *The weight spectral sequence*

$$_WE_1^{p,q} = H^{p+q}(\mathrm{Gr}^W_{-p}(Ri_*\boldsymbol{Q}_{X\setminus B})) \Rightarrow H^{p+q}(\boldsymbol{Q}_{X\setminus B})$$

degenerates at E_2, and the Hodge spectral sequence

$$_FE_1^{p,q} = H^q(\bar{\Omega}^p_X(\log B)) \Rightarrow H^{p+q}(\bar{\Omega}^\bullet_X(\log B))$$

degenerates at E_1.

Proof. This is as in [**4**, 8.1.9]. □

When $B = 0$, we have more degenerations, as follows.

Corollary 3.4. *The weight spectral sequence of the Hodge pieces*

$$_WE_1^{p,q} = H^{p+q}(\mathrm{Gr}^W_{-p}(\bar{\Omega}^r_X)) \Rightarrow H^{p+q}(\bar{\Omega}^r_X)$$

degenerates at E_2 for all r.

Proof. The differentials $_Wd_1^{p,q}$ of the weight spectral sequence in the previous corollary preserve the degree of the differentials. Therefore, the E_2-degeneration of the third spectral sequence is a consequence of the first two degenerations. □

4. Relativization

We can easily generalize the results in the previous section to the relative setting when all the closed strata are smooth over the base.

We consider the following situation: (X, B) is a pair of a GSNC variety and a GSNC divisor, S is a smooth algebraic variety, which is not necessarily complete, and $f\colon X \to S$ is a projective surjective morphism. We assume that, for each closed stratum Y of the pair (X, B), the restriction $f|_Y\colon Y \to S$ is smooth.

We define a *relative de Rham complex* $\bar{\Omega}^{\bullet}_{X/S}(\log B)$ by the exact sequence

$$0 \to \bar{\Omega}^{\bullet}_{X/S}(\log B) \to \bar{\Omega}^{\bullet}_{X^{[0]}/S}(\log B_{X^{[0]}}) \to \bar{\Omega}^{\bullet}_{X^{[1]}/S}(\log B_{X^{[1]}}) \to \cdots \to \bar{\Omega}^{\bullet}_{X^{[N]}/S}(\log B_{X^{[N]}}) \to 0.$$

In particular, we have that

$$\bar{\Omega}^{0}_{X/S}(\log B) \cong \mathcal{O}_X.$$

A *weight filtration* on the complex $\bar{\Omega}^{\bullet}_{X/S}(\log B)$ is defined by using the filtration with respect to the order of log poles on the closed strata as in the previous section:

$$0 \to W_q(\bar{\Omega}^{\bullet}_{X/S}(\log B)) \to W_q(\bar{\Omega}^{\bullet}_{X^{[0]}/S}(\log B_{X^{[0]}})) \to W_{q+1}(\bar{\Omega}^{\bullet}_{X^{[1]}/S}(\log B_{X^{[1]}})) \to \cdots \to W_{q+N}(\bar{\Omega}^{\bullet}_{X^{[N]}/S}(\log B_{X^{[N]}})) \to 0.$$

We define a *Hodge filtration* by

$$F^p(\bar{\Omega}^{\bullet}_{X}(\log B)) = \bar{\Omega}^{\geqslant p}_{X}(\log B).$$

Lemma 4.1. *There exists an isomorphism*

$$Ri_* \boldsymbol{C}_{X\setminus B} \otimes f^{-1}\mathcal{O}_S \cong \bar{\Omega}^{\bullet}_{X/S}(\log B)$$

such that

$$W_q(Ri_* \boldsymbol{C}_{X\setminus B}) \otimes f^{-1}\mathcal{O}_S \cong W_q(\bar{\Omega}^{\bullet}_{X/S}(\log B)).$$

Lemma 4.2. *We have again that*

$$\operatorname{Gr}^W_q(\bar{\Omega}^{\bullet}_{X/S}(\log B)) \cong \bigoplus_{p\geqslant 0,\, \dim X^{[p]} - \dim Y = p+q} \Omega^{\bullet}_{Y/S}[-2p-q],$$

$$F^r(\operatorname{Gr}^W_q(\bar{\Omega}^{\bullet}_{X/S}(\log B))) \cong \bigoplus_{p\geqslant 0,\, \dim X^{[p]} - \dim Y = p+q} \Omega^{\geqslant r-p-q}_{Y/S}[-2p-q],$$

where the p run over all non-negative integers and the Y run over all the closed strata of the pair $(X^{[p]}, B_{X^{[p]}})$ of codimension $p+q$.

The following theorem and corollaries are similar to Theorem 4.1 and Corollary 4.2 in [**12**].

https://doi.org/10.1017/S0013091513000874 Published online by Cambridge University Press

Theorem 4.3.

$$((R^n(f\circ i)_*\boldsymbol{Q}_{X\setminus B}, W(-n)), (R^n f_*\bar{\Omega}^{\bullet}_{X/S}(\log B), W(-n), F))$$

is a variation of rational mixed Hodge structures on S*.*

Corollary 4.4. *The relative weight spectral sequence*

$$_W E_1^{p,q} = R^{p+q} f_* \operatorname{Gr}^W_{-p}(Ri_*\boldsymbol{Q}_{X\setminus B}) \Rightarrow R^{p+q}(f\circ i)_*\boldsymbol{Q}_{X\setminus B}$$

degenerates at E_2*, and the relative Hodge spectral sequence*

$$_F E_1^{p,q} = R^q f_* \bar{\Omega}^p_{X/S}(\log B) \Rightarrow R^{p+q} f_* \bar{\Omega}^{\bullet}_{X/S}(\log B)$$

degenerates at E_1*.*

If $B=0$, then, again, we have the following.

Corollary 4.5. *The weight spectral sequence of the Hodge pieces*

$$_{W,r} E_1^{p,q} = R^{p+q} f_* \operatorname{Gr}^W_{-p}(\bar{\Omega}^r_{X/S}) \Rightarrow R^{p+q} f_*(\bar{\Omega}^r_{X/S})$$

degenerates at E_2 *for all* r*.*

5. Canonical extension

We prove the local freeness theorem by using the theory of canonical extensions when the degeneration locus is a simple normal crossing divisor.

Let (S, B_S) be a pair of a smooth projective variety and a simple normal crossing divisor. We define $S^o = S \setminus B_S$. Let $H_{\boldsymbol{Q}}$ be a local system on S^o that underlies a variation of mixed Hodge $\boldsymbol{Q}$-structures. Assuming that all the local monodromies around the branches of B_S are quasi-unipotent, we define the *lower canonical extension* $\tilde{\mathcal{H}}$ of $\mathcal{H} = H_{\boldsymbol{Q}} \otimes \mathcal{O}_{S^o}$ as follows.

We take an arbitrary point $s \in B_S$ at the boundary. Let $N_i = \log T_i$ be the logarithm of the local monodromies T_i around the branches of B_S around s, and let t_i be the local coordinates corresponding to the branches. Here we select a branch of the logarithmic function such that the eigenvalues of N_i belong to the interval $2\pi\sqrt{-1}(-1,0]$. The lower canonical extension $\tilde{\mathcal{H}}$ is then defined as a locally free sheaf on S that is generated by local sections of the form $\tilde{v} = \exp(-\sum_i N_i \log t_i / 2\pi\sqrt{-1})(v)$ near s, where the v are flat sections of $H_{\boldsymbol{Q}}$. We note that the monodromy actions on v and the logarithmic functions are canceled and the $\tilde{v}$ are single-valued holomorphic sections of $\mathcal{H}$ outside the boundary divisors.

By [**17**], the Hodge filtration of $\mathcal{H}$ extends to a filtration by locally free subsheaves, which we denote again by F.

Let $f\colon X \to S$ be a surjective morphism between smooth projective varieties that is smooth over S^o. We define $X^o = f^{-1}(S^o)$ and $f^o = f|_{X^o}$. Let $H^q_{\boldsymbol{Q}} = R^q f^o_* \boldsymbol{Q}_{X^o}$ for an

https://doi.org/10.1017/S0013091513000874 Published online by Cambridge University Press

integer q, let $\mathcal{H}^q = H^q_{\boldsymbol{Q}} \otimes \mathcal{O}_{S^o}$, and let $\tilde{\mathcal{H}}^q$ be the canonical extension. Then, by [**15**, **16**], there exists an isomorphism

$$R^q f_* \mathcal{O}_X \cong \mathrm{Gr}^0_F(\tilde{\mathcal{H}}^q)$$

that extends the natural isomorphism

$$R^q f^o_* \mathcal{O}_{X^o} \cong \mathrm{Gr}^0_F(\mathcal{H}^q).$$

The following theorem will be used in the next section.

Theorem 5.1. *Let X be a projective GSNC variety, let (S, B_S) be a pair of a smooth projective variety and a simple normal crossing divisor, and let $f\colon X \to S$ be a projective surjective morphism. Assume that, for each closed stratum Y of X, the restriction $f|_Y\colon Y \to S$ is surjective and smooth over $S^o = S \setminus B_S$. Define $X^o = f^{-1}(S^o)$ and $f^o = f|_{X^o}$. For integers q, let $H^q_{\boldsymbol{Q}} = R^q f^o_* \boldsymbol{Q}_{X^o}$ be the local system on S^o that underlies a variation of mixed Hodge $\boldsymbol{Q}$-structures defined in the preceding section. Let $\mathcal{H}^q = H^q_{\boldsymbol{Q}} \otimes \mathcal{O}_{S^o}$, and let $\tilde{\mathcal{H}}^q$ be their canonical extensions. The following then hold.*

(1) *The weight spectral sequence of a Hodge piece*

$$_{W,0}E_1^{p,q} = R^{p+q} f_* \,\mathrm{Gr}^W_{-p}(\mathcal{O}_X) \Rightarrow R^{p+q} f_* \mathcal{O}_X$$

degenerates at E_2.

(2) *There exist isomorphisms*

$$R^q f_* \mathcal{O}_X \cong \mathrm{Gr}^0_F(\tilde{\mathcal{H}}^q),$$

for all integers q, that extend the natural isomorphisms

$$R^q f^o_* \mathcal{O}_{X^o} \cong \mathrm{Gr}^0_F(\mathcal{H}^q)$$

in Corollary 4.4.

(3) *$R^q f_* \mathcal{O}_X$ are locally free sheaves.*

Proof. We extend the definition of the weight filtration on $\mathcal{O}_{X^o} = \mathrm{Gr}^0_F(\bar{\Omega}^\bullet_{X^o/S^o})$ to $\mathcal{O}_X$ by the exact sequence

$$0 \to W_q(\mathcal{O}_X) \to W_q(\mathcal{O}_{X^{[0]}}) \to W_{q+1}(\mathcal{O}_{X^{[1]}}) \to \cdots \to W_{q+N}(\mathcal{O}_{X^{[N]}}) \to 0,$$

where $W_q(\mathcal{O}_{X^{[p]}}) = \mathcal{O}_{X^{[p]}}$ for $q \geqslant 0$, and $W_q(\mathcal{O}_{X^{[p]}}) = 0$ otherwise, for any p. By the canonical extension, we extend the E_2-degeneration of the weight spectral sequence from S^o to S as in [**12**, Theorem 5.1]. We then reduce the assertion to the theorem of Kollár and Nakayama. ☐

https://doi.org/10.1017/S0013091513000874 Published online by Cambridge University Press

6. Kollár-type vanishing

We generalize the vanishing theorem of Kollár [**14**] to GSNC varieties.

Theorem 6.1. *Let X be a projective GSNC variety, let S be a normal projective variety, let $f\colon X \to S$ be a projective surjective morphism, and let L be a permissible Cartier divisor on X such that $\mathcal{O}_X(m_1 L)$ is generated by global sections for a positive integer m_1. Assume that $\mathcal{O}_X(m_2 L) \cong f^*\mathcal{O}_Z(L_S)$ for a positive integer m_2 and an ample Cartier divisor L_S on S, and, for each closed stratum Y of X, the restricted morphism $f|_Y\colon Y \to S$ is surjective. Then the following hold.*

(1) *Let $s \in H^0(X, \mathcal{O}_X(nL))$ be a non-zero section for some positive integer n such that the corresponding Cartier divisor D is permissible. The natural homomorphisms given by s,*

$$H^p(X, \mathcal{O}_X(K_X + L)) \to H^p(X, \mathcal{O}_X(K_X + L + D)),$$

are then injective for all p.

(2) *$H^q(S, R^p f_* \mathcal{O}_X(K_X + L)) = 0$ for $p \geqslant 0$ and $q > 0$.*

(3) *$R^p f_* \mathcal{O}_X(K_X)$ are torsion free for all p.*

Proof. We follow closely the proofs of [**14**, Theorems 2.1 and 2.2]. We number our steps as in that paper.

Step 1. We may assume, and will assume from now on, that $\mathcal{O}_X(L)$ is generated by global sections.

We achieve this reduction by constructing a Kummer covering and taking the Galois invariant part as in [**14**].

Step 2. We prove the dual statement of (1) in the case where $n = 1$ and D is generic. Let $s, s' \in H^0(X, L)$ be general members, and let D, D' be the corresponding permissible Cartier divisors. The natural homomorphisms given by s,

$$H^p(X, \mathcal{O}_X(-D - D')) \to H^p(X, \mathcal{O}_X(-D')),$$

are then surjective for all p.

We go into detail in this step in order to show how to generalize the argument in [**14**] to the GSNC case. Since s and s' are general, D, D', $D + D'$ and $D \cap D'$ are also GSNC varieties. We consider the following commutative diagram:

$$\begin{array}{ccccccc}
H^{p-1}(\mathcal{O}_{D+D'}) & \longrightarrow & H^p(\mathcal{O}_X(-D-D')) & \longrightarrow & H^p(\mathcal{O}_X) & \xrightarrow{d_p} & H^p(\mathcal{O}_{D+D'}) \\
{\scriptstyle b'_{p-1}}\downarrow & & \downarrow & & \downarrow{\scriptstyle =} & & {\scriptstyle b'_p}\downarrow \\
H^{p-1}(\mathcal{O}_{D'}) & \longrightarrow & H^p(\mathcal{O}_X(-D')) & \longrightarrow & H^p(\mathcal{O}_X) & \xrightarrow{e'_p} & H^p(\mathcal{O}_{D'})
\end{array}$$

In order to prove our assertion, we prove that (a) b'_{p-1} is surjective and (b) $\mathrm{Ker}(d_p) = \mathrm{Ker}(e'_p)$.

https://doi.org/10.1017/S0013091513000874 Published online by Cambridge University Press

(a) We consider the following Mayer–Vietoris exact sequence:

$$H^{p-1}(\boldsymbol{C}_{D\cap D'}) \xrightarrow{\bar{a}_p} H^p(\boldsymbol{C}_{D+D'}) \xrightarrow{\bar{b}_p+\bar{b}'_p} H^p(\boldsymbol{C}_D) \oplus H^p(\boldsymbol{C}_{D'}) \xrightarrow{\bar{c}_p-\bar{c}'_p} H^p(\boldsymbol{C}_{D\cap D'})$$

whose Gr_F^0 is

$$H^{p-1}(\mathcal{O}_{D\cap D'}) \xrightarrow{a_p} H^p(\mathcal{O}_{D+D'}) \xrightarrow{b_p+b'_p} H^p(\mathcal{O}_D) \oplus H^p(\mathcal{O}_{D'}) \xrightarrow{c_p-c'_p} H^p(\mathcal{O}_{D\cap D'}).$$

There exists a path connecting D and D' in the linear system $|L|$ on X that gives a diffeomorphism of pairs $(D, D\cap D') \to (D', D\cap D')$ fixing $D\cap D'$. Therefore, we have an equality $\mathrm{Im}(\bar{c}_p) = \mathrm{Im}(\bar{c}'_p)$, which implies that $\bar{b}'_p$ is surjective. Hence, so is b'_p.

(b) It is sufficient to prove that $\mathrm{Im}(d_p) \cap \mathrm{Ker}(b'_p) = 0$. Using a path connecting D and D', we deduce that $\mathrm{Ker}(e_p) = \mathrm{Ker}(e'_p)$ for $e_p\colon H^p(\mathcal{O}_X) \to H^p(\mathcal{O}_D)$. Thus,

$$\mathrm{Im}(d_p) \cap \mathrm{Ker}(b'_p) = \mathrm{Im}(d_p) \cap \mathrm{Ker}(b_p).$$

Therefore, it is sufficient to prove that $\mathrm{Im}(a_p) \cap \mathrm{Im}(d_p) = 0$. We prove that

$$\mathrm{Im}(a_p) \cap \mathrm{Im}(d_p) \cap W_q(H^p(\mathcal{O}_{D+D'})) = 0$$

by induction on q.

Let

$$a_{p,q}\colon\ \mathrm{Gr}_q^W(H^{p-1}(\mathcal{O}_{D\cap D'})) \to \mathrm{Gr}_q^W(H^p(\mathcal{O}_{D+D'})),$$
$$d_{p,q}\colon\ \mathrm{Gr}_q^W(H^p(\mathcal{O}_X)) \to \mathrm{Gr}_q^W(H^p(\mathcal{O}_{D+D'}))$$

be the natural homomorphisms. It is then sufficient to prove that

$$\mathrm{Im}(a_{p,q}) \cap \mathrm{Im}(d_{p,q}) = 0.$$

For $A = X$, $D\cap D'$ or $D+D'$, we have the spectral sequences

$${}_{W,A}E_1^{r,s} = H^{r+s}(\mathrm{Gr}_{-r}^W(\mathcal{O}_A)) = \bigoplus_{\dim A - \dim Y = r} H^s(\mathcal{O}_Y) \Rightarrow H^{r+s}(\mathcal{O}_A).$$

Therefore, the boundary homomorphism $d_{p,q}$ is induced from the sum of homomorphisms $H^s(\mathcal{O}_Y) \to H^s(\mathcal{O}_{Y'})$ such that $s = q$, $Y \subset X$, $Y' \subset D + D'$, $Y' \subset Y$ and

$$r = \dim X - \dim Y = \dim(D+D') - \dim Y' = p - q;$$

hence, $Y' = Y \cap D$ or $Y' = Y \cap D'$.

On the other hand, $a_{p,q}$ is induced from the sum of homomorphisms $H^s(\mathcal{O}_{Y''}) \to H^s(\mathcal{O}_{Y'})$ such that $s = q$, $Y'' \subset D \cap D'$, $Y' \subset D + D'$, $Y'' = Y'$ and

$$r = \dim(D\cap D') - \dim Y'' + 1 = \dim(D+D') - \dim Y' = p - q.$$

Therefore, there is no closed stratum Y' of $D + D'$ that receives non-trivial images from both Y and Y''; hence, we have our assertion.

https://doi.org/10.1017/S0013091513000874 Published online by Cambridge University Press

Step 3. In the case where $n = 2^d - 1$ for a positive integer d and D generic, Theorem 6.1 (1) is an immediate corollary of Step 2.

Step 4. Proof of (2) is the same as in [**14**].

Step 5. Proof of (3).

This is a generalization of Step 2; we use the same notation. D' is again generic, but D is special. More precisely, D is special along a fibre $f^{-1}(s)$ over a point $s \in S$, and generic otherwise. Therefore, the intersection $D \cap D'$ is still generic, hence a GSNC variety.

Let $\mu\colon \tilde{X} \to X$ be the blow-up along the centre $D \cap D'$. $\tilde{X}$ is again a GSNC variety. We denote by $\tilde{Y}$ the closed stratum of $\tilde{X}$ above a closed stratum Y of X. Let $g\colon \tilde{X} \to \boldsymbol{P}^1$ be the morphism induced from the linear system spanned by D and D'. Let $U \subset \boldsymbol{P}^1$ be an open dense subset such that the restricted morphisms $g|_{\tilde{Y}}\colon \tilde{Y} \to \boldsymbol{P}^1$ are smooth over U for all the closed strata Y.

(a) In order to prove that b'_p is surjective, we need to prove an inclusion $\mathrm{Im}(c'_p) \subset \mathrm{Im}(c_p)$. For this purpose, we prove that

$$\mathrm{Im}(c'_p) = \mathrm{Im}(c_p \circ e_p).$$

Since $c_p \circ e_p = c'_p \circ e'_p$, the right-hand side is contained in the left-hand side. The other direction is the essential part. We note that both c'_p and $c_p \circ e_p$ are parts of homomorphisms of mixed Hodge structures.

Let $\tilde{X}_U = g^{-1}(U)$ and $\tilde{Y}_U = \tilde{Y} \cap \tilde{X}_U$. Let $H^p_{\mathrm{prim}}(\boldsymbol{C}_{\tilde{Y}\cap D'}) \subset H^p(\boldsymbol{C}_{\tilde{Y}\cap D'})$ be the subspace of the primitive cohomologies, and let $R^p g_{*,\mathrm{prim}} \boldsymbol{C}_{\tilde{Y}_U} \subset R^p g_* \boldsymbol{C}_{\tilde{Y}_U}$ be the corresponding local subsystem. Natural homomorphisms

$$R^p g_* \boldsymbol{C}_{\tilde{X}_U} \to H^p(\boldsymbol{C}_{D\cap D'}) \times U,$$
$$R^p g_{*,\mathrm{prim}} \boldsymbol{C}_{\tilde{Y}_U} \to H^p(\boldsymbol{C}_{Y\cap D\cap D'}) \times U$$

then underlie, respectively, morphisms of variations of mixed and pure Hodge structures over U, where the targets are constant variations.

By the semi-simplicity of the category of variations of pure Hodge structures of fixed weight (see [**3**]), we deduce that the local system $R^p g_{*,\mathrm{prim}} \boldsymbol{C}_{\tilde{Y}_U}$ has a subsystem that is isomorphic to a constant local system with fibre $\mathrm{Im}(\bar{c}'_{p,Y})$ for $\bar{c}'_{p,Y}\colon H^p_{\mathrm{prim}}(\boldsymbol{C}_{Y\cap D'}) \to H^p(\boldsymbol{C}_{Y\cap D\cap D'})$. By the invariant cycle theorem (see [**3**]), we then deduce that

$$\begin{aligned}\mathrm{Im}(\bar{c}'_{p,Y}) &\subset \mathrm{Im}(H^0(U, R^p g_{*,\mathrm{prim}} \boldsymbol{C}_{\tilde{Y}_U}) \to H^p(\boldsymbol{C}_{Y\cap D\cap D'})) \\ &\subset \mathrm{Im}(H^p(\boldsymbol{C}_{\tilde{Y}}) \to H^p(\boldsymbol{C}_{Y\cap D\cap D'})),\end{aligned}$$

where the second and third homomorphisms are derived from the restrictions to the fibre D' of g. Since

$$H^p(\mathcal{O}_{Y\cap D'}) = \mathrm{Gr}^0_F(H^p_{\mathrm{prim}}(\boldsymbol{C}_{Y\cap D'})),$$

we conclude that

$$\mathrm{Im}(c'_{p,Y}\colon H^p(\mathcal{O}_{Y\cap D'}) \to H^p(\mathcal{O}_{Y\cap D\cap D'})) \subset \mathrm{Im}(H^p(\mathcal{O}_{\tilde{Y}}) \to H^p(\mathcal{O}_{Y\cap D\cap D'})).$$

https://doi.org/10.1017/S0013091513000874 Published online by Cambridge University Press

Since the combinatorics of the closed strata are the same for $\tilde{X}$ and $D \cap D'$, we have that

$$\mathrm{Im}(c'_p\colon H^p(\mathcal{O}_{D'}) \to H^p(\mathcal{O}_{D\cap D'})) \subset \mathrm{Im}(H^p(\mathcal{O}_{\tilde{X}}) \to H^p(\mathcal{O}_{D\cap D'})).$$

Since $H^p(\mathcal{O}_{\tilde{X}}) \cong H^p(\mathcal{O}_X)$, we have our assertion.

(b) There is an obvious inclusion $\mathrm{Ker}(d_p) \subset \mathrm{Ker}(e'_p)$. We know already that

$$H^p(\mathcal{O}_X(-D-D')) \to H^p(\mathcal{O}_X(-D))$$

is surjective, because it is a statement that is independent of D. Thus, we have that $\mathrm{Ker}(d_p) = \mathrm{Ker}(e_p)$. Therefore, it is sufficient to prove that

$$\mathrm{rank}(e_p) = \mathrm{rank}(e'_p).$$

As explained in Step 6, the sheaves $R^p g_* \mathcal{O}_{\tilde{X}}$ are locally free for all p. Therefore, we have that $h^p(\mathcal{O}_D) = h^p(\mathcal{O}_{D'})$, where we define $h^p = \dim H^p$. We compare two exact sequences

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0,$$
$$0 \to \mathcal{O}_X(-D') \to \mathcal{O}_X \to \mathcal{O}_{D'} \to 0.$$

Since the corresponding h^p are equal, we deduce that the ranks of the corresponding homomorphisms of the long exact sequences are equal, and we are done.

Step 6. This is Theorem 5.1.

Step 7. Proof of (1) is the same as in [**14**]. □

Finally, we can easily generalize the vanishing theorem to the case of $\boldsymbol{Q}$-divisors by using the covering lemma (see Lemma 2.3).

Theorem 6.2. *Let X be a projective GSNC variety, let S be a normal projective variety, let $f\colon X \to S$ be a projective surjective morphism, and let L be a permissible $\boldsymbol{Q}$-Cartier divisor on X such that $m_1 L$ is a Cartier divisor and $\mathcal{O}_X(m_1 L)$ is generated by global sections for a positive integer m_1. Assume that the support of L is a GSNC divisor, $\mathcal{O}_X(m_2 L) \cong f^*\mathcal{O}_Z(L_S)$ for a positive integer m_2 and an ample Cartier divisor L_S on S, and, for each closed stratum Y of X, the restricted morphism $f|_Y\colon Y \to S$ is surjective. Then the following hold.*

(1) *Let $s \in H^0(X, \mathcal{O}_X(nL))$ be a non-zero section for some positive integer n such that the corresponding Cartier divisor D is permissible. The natural homomorphisms given by s,*

$$H^p(X, \mathcal{O}_X(K_X + \ulcorner L \urcorner)) \to H^p(X, \mathcal{O}_X(K_X + \ulcorner L \urcorner + D)),$$

are then injective for all p.

(2) *$H^q(S, R^p f_* \mathcal{O}_X(K_X + \ulcorner L \urcorner)) = 0$ for $p \geqslant 0$ and $q > 0$.*

Proof. We take a finite Kummer covering $\pi\colon X' \to X$ from another GSNC variety such that $\pi^* L$ becomes a Cartier divisor. Let G be the Galois group. We then have that

$$(\pi_* \mathcal{O}_X(K_X + L))^G \cong \mathcal{O}_X(K_X + \ulcorner L \urcorner).$$

Therefore, our assertion is reduced to the case where L is a Cartier divisor. □

https://doi.org/10.1017/S0013091513000874 Published online by Cambridge University Press

References

1. D. Abramovich and K. Karu, Weak semistable reduction in characteristic 0, *Invent. Math.* **139**(2) (2000), 241–273.
2. P. Deligne, Theoreme de Lefschetz et criteres de degenerescence de suites spectrales, *Publ. Math. IHES* **35** (1968), 259–278.
3. P. Deligne, Théorie de Hodge, II, *Publ. Math. IHES* **40** (1971), 5–57.
4. P. Deligne, Théorie de Hodge, III, *Publ. Math. IHES* **44** (1974), 5–77.
5. P. Du Bois, Complexe de de Rham filtre d'une variete singuliere, *Bull. Soc. Math. France* **109**(1) (1981), 41–81.
6. O. Fujino, On Kawamata's theorem, in *Classification of algebraic varieties*, European Mathematical Society Series of Congress Reports, pp. 305–315 (European Mathematical Society, Zürich, 2011).
7. S. I. Gelfand and Y. I. Manin, *Methods of homological algebra*, Springer Monographs in Mathematics (Springer, 2003).
8. H. Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic zero, I, *Annals Math.* **79** (1964), 109–203.
9. H. Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic zero, II, *Annals Math.* **79** (1964), 205–326.
10. Y. Kawamata, Characterization of abelian varieties, *Compositio Math.* **43** (1981), 253–276.
11. Y. Kawamata, Pluricanonical systems on minimal algebraic varieties, *Invent. Math.* **79** (1985), 567–588.
12. Y. Kawamata, Semipositivity theorem for reducible algebraic fiber spaces, *Q. J. Pure Appl. Math.* **7**(4) (2011), 1427–1447.
13. Y. Kawamata, Variation of mixed Hodge structures and the positivity for algebraic fiber spaces, *Adv. Stud. Pure Math.* (in press).
14. J. Kollár, Higher direct images of dualizing sheaves, I, *Annals Math.* **123**(1) (1986), 11–42.
15. J. Kollár, Higher direct images of dualizing sheaves, II, *Annals Math.* **124**(1) (1986), 171–202.
16. N. Nakayama, Hodge filtrations and the higher direct images of canonical sheaves, *Invent. Math.* **85**(1) (1986), 217–221.
17. W. Schmid, Variation of Hodge structure: the singularities of the period mapping, *Invent. Math.* **22** (1973), 211–319.