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ON MATRICES ARISING IN FINITE FIELD HYPERGEOMETRIC FUNCTIONS

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Abstract

Lehmer ['On certain character matrices', *Pacific J. Math.* **6** (1956), 491–499, and 'Power character matrices', *Pacific J. Math.* **10** (1960), 895–907] defines four classes of matrices constructed from roots of unity for which the characteristic polynomials and the kth powers can be determined explicitly. We study a class of matrices which arise naturally in transformation formulae of finite field hypergeometric functions and whose entries are roots of unity and zeroes. We determine the characteristic polynomial, eigenvalues, eigenvectors and kth powers of these matrices. The eigenvalues are natural families of products of Jacobi sums.

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1. Introduction

In [8], Lehmer remarks that the class of matrices for which one can explicitly determine the eigenvalues and the general kth power is very limited. Using the Legendre character on finite fields, Lehmer constructs two classes of matrices for which this is possible. More generally, using characters of arbitrary orders, Carlitz [2] and Lehmer [9] construct other classes of matrices for which they determine the characteristic polynomials and kth powers.

Here we consider a class of matrices, whose entries are roots of unity and zeroes, which arise in the transformation formulae for Gaussian hypergeometric functions over finite fields defined by Greene [3]. We first recall the definition of these functions. If p is a prime, $q = p^r, n \geq 1$, and $A_1, \ldots, A_n, B_2, \ldots, B_n$ are complex-valued multiplicative characters over $\mathbb{F}_q^{\times}$, then the finite field hypergeometric functions are defined by

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$$ {}_nF_{n-1}\left(\begin{matrix} A_1, & A_2, & \dots, & A_n \\ & B_2, & \dots, & B_n \end{matrix} \,\middle|\, x\right)_q := \frac{q}{q-1}\sum_{\chi}\binom{A_1\chi}{\chi}\binom{A_2\chi}{B_2\chi}\cdots\binom{A_n\chi}{B_n\chi}\chi(x), \tag{1.1} $$

where the summation is over multiplicative characters χ of $\mathbb{F}_q^{\times}$ and the binomial coefficient $\binom{A}{B}$ is a normalised Jacobi sum, given by

$$ \binom{A}{B} := \frac{B(-1)}{q}J(A,\overline{B}) := \frac{B(-1)}{q}\sum_{x\in\mathbb{F}_q} A(x)\overline{B}\,(1-x). \tag{1.2} $$

These functions have deep connections to étale cohomology [6] and often arise in geometry where they count the number of $\mathbb{F}_q$-points on various algebraic varieties (see [1, Theorem 1.5]). For example, if $\lambda \in \mathbb{F}_q \setminus \{0,1\}$ and E_λ is the Legendre normal form elliptic curve

$$ E_\lambda : y^2 = x(x-1)(x-\lambda), $$

then (see [7, Section 4] and [11, Theorem 1]),

$$ \#E_\lambda(\mathbb{F}_q) = 1 + q + q\cdot\phi_q(-1)\cdot {}_2F_1\left(\begin{matrix} \phi_q, & \phi_q \\ & \varepsilon \end{matrix} \,\middle|\, \lambda\right)_q, $$

where ϕ_q and ε are respectively the Legendre symbol and the trivial character on $\mathbb{F}_q^{\times}$.

Moreover, these functions satisfy analogues of several transformation formulae of their classical counterparts, such as the generalised Euler integral transform (see [13, (4.1.1)]). More precisely (see [3, Theorem 3.13]),

$$ \begin{aligned} &{}_{n+1}F_n\left(\begin{matrix} A_1, & A_2, & \dots, & A_n, & A_{n+1} \\ & B_2, & \dots, & B_n & B_{n+1} \end{matrix} \,\middle|\, x\right)_q \\ &\quad= \frac{A_{n+1}B_{n+1}(-1)}{q}\sum_{y\in\mathbb{F}_q} {}_nF_{n-1}\left(\begin{matrix} A_1, & A_2, & \dots, & A_n \\ & B_2, & \dots, & B_n \end{matrix} \,\middle|\, xy\right)_q \cdot A_{n+1}(y)\overline{A_{n+1}}B_{n+1}(1-y). \end{aligned} \tag{1.3} $$

Motivated by the transformation formula (1.3), Ono as well as Griffin and Rolen study the matrix corresponding to this transformation when $q = p^r$ is odd, $A_{n+1} = \phi_q$ and $B_{n+1} = \epsilon$. Consider the $(q-2)\times(q-2)$ matrix $M = (M_{ij})$ indexed by $i, j \in \mathbb{F}_q \setminus \{0,1\}$, where

$$ M_{ij} = \phi_q(1-ij)\phi_q(ij) $$

and let f_q be its characteristic polynomial. In this notation, Griffin and Rolen [4] prove a conjecture by Ono that

$$ f_q(x) = \begin{cases} (x+1)(x-1)(x+2)(x^2-q)^{(q-5)/2} & \text{if } \phi_q(-1) = 1, \\ x(x^2-3)(x^2-q)^{(q-5)/2} & \text{if } \phi_q(-1) = -1. \end{cases} $$

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The purpose of this paper is to study, à la Lehmer, a more general analogue of the matrix M that arises when the characters A_{n+1} and B_{n+1} are arbitrary. More precisely, we consider the $(q-1)\times(q-1)$ matrix $M_q = (M_q)_{ij}$ indexed by $i, j \in \mathbb{F}_q^\times$, where

$$(M_q)_{ij} := A(ij)\overline{A}B(1-ij).$$

We first determine the characteristic polynomial f_q of M_q.

THEOREM 1.1. *If p is an odd prime, $q = p^r$ and ω is a character of order $q-1$ of $\mathbb{F}_q^\times$, then*

$$f_q(x) = (x - J(\overline{A}B, A))(x - J(\overline{A}B, \overline{A}\phi)) \prod_{l=1}^{(q-3)/2} (x^2 - J(\overline{A}B, A\omega^l)J(\overline{A}B, A\overline{\omega^l})).$$

Our proof explicitly determines the eigenvectors of M_q. Furthermore, when $B = \varepsilon$ and $k \geq 1$, we explicitly determine the entries of M_q^k.

THEOREM 1.2. *If $k \geq 1$, we write $k = 2l$ if k is even and $k = 2l + 1$ if k is odd. In this notation, if p is an odd prime, $q = p^r$ and $B = \varepsilon$, then*

$$(M_q^k)_{ij} = A^l(-1) \cdot q^{k-1} \cdot {}_kF_{k-1}\left(\begin{matrix} A_1, & A_2, & \ldots, & A_k \\ & B_2, & \ldots, & B_k \end{matrix} \,\middle|\, \frac{j^{(-1)^k}}{i}\right)_q,$$

where

$$A_n = \begin{cases} A & \text{if } 1 \leq n \leq l, \\ \varepsilon & \text{otherwise}, \end{cases} \quad \text{and} \quad B_n = \begin{cases} \varepsilon & \text{if } 2 \leq n \leq l, \\ \overline{A} & \text{otherwise}. \end{cases}$$

REMARK 1.3. If $B \neq \varepsilon$, the entries of M_q^k can be written in terms of more general finite field hypergeometric functions, such as those given by McCarthy [10, Definition 2.4] and Otsubo [12, Definition 2.7]. The proof is analogous to the proof of Theorem 1.2.

The paper is organised as follows. In Section 2, we recall facts concerning characters and finite field hypergeometric functions and determine the action of M_q on an appropriate basis. In Section 3, we prove Theorems 1.1 and 1.2.

2. Nuts and Bolts

Here we recall facts about characters on finite fields and hypergeometric functions. We also determine the behaviour of M_q on an appropriate set of vectors.

We denote by $\widehat{\mathbb{F}_q^\times}$ the group of characters on $\mathbb{F}_q^\times$. It is well known (see [5, Proposition 8.1.2]) that if $\chi \in \widehat{\mathbb{F}_q^\times}$, then

$$\sum_{x \in \mathbb{F}_q} \chi(x) = \begin{cases} q-1 & \text{if } \chi = \varepsilon, \\ 0 & \text{otherwise}, \end{cases}$$

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and that if $x \in \mathbb{F}_q$, then

$$\sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \chi(x) = \begin{cases} q-1 & \text{if } x = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

Furthermore, if $A, B \in \widehat{\mathbb{F}_q^\times}$, then the following properties of binomial coefficients are known [3, (2.6)–(2.8)]:

$$\binom{A}{B} = \binom{A}{A\overline{B}}, \tag{2.2}$$

$$\binom{A}{B} = B(-1) \cdot \binom{B\overline{A}}{B},$$

$$\binom{A}{B} = \overline{AB}(-1) \cdot \binom{\overline{B}}{\overline{A}}. \tag{2.3}$$

To state our results, we fix a generator ω of $\widehat{\mathbb{F}_q^\times}$. For $1 \le l \le q-1$, we define the vectors $\boldsymbol{w}^l$ indexed by $i \in \mathbb{F}_q^\times$, where

$$\boldsymbol{w}_i^l = \omega^l(i).$$

The following lemma determines $M_q \boldsymbol{w}^l$.

LEMMA 2.1. *If* $1 \le l \le q-1$, *then*

$$M_q \boldsymbol{w}^l = J(\overline{A}B, A\omega^l)\boldsymbol{w}^{q-1-l}.$$

PROOF. Fix l. Then, for $i \in \mathbb{F}_q^\times$,

$$(M_q \boldsymbol{w}^l)_i = \sum_{j \in \mathbb{F}_q^\times} A(ij)\overline{A}B(1-ij)\omega^l(j).$$

Replacing j by j/i gives

$$\begin{aligned}(M_q \boldsymbol{w}^l)_i &= \sum_{j \in \mathbb{F}_q^\times} A(j)\overline{A}B(1-j)\omega^l\Big(\frac{j}{i}\Big) \\ &= \overline{\omega^l}(i) \sum_{j \in \mathbb{F}_q^\times} (A\omega^l)(j)\overline{A}B(1-j) \\ &= J(\overline{A}B, A\omega^l)\boldsymbol{w}_i^{q-1-l}.\end{aligned}$$

□

REMARK 2.2. Recall that the Fourier transform of $f : \mathbb{F}_q \to \mathbb{C}$ is a function $\widehat{f} : \widehat{\mathbb{F}_q^\times} \to \mathbb{C}$ defined by

$$\widehat{f}(\nu) = \sum_{\lambda \in \mathbb{F}_q} f(\lambda)\overline{\nu}(\lambda).$$

By a similar argument to the proof of Lemma 2.1, the Fourier transforms of the components of M_q^2 are products of two Jacobi sums.

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To determine the quadratic terms in Theorem 1.1, we make use of the following lemma which follows from a direct computation.

LEMMA 2.3. *If M is an $n \times n$ matrix, $\lambda_1, \lambda_2 \in \mathbb{C}$, and $v_1 \neq \pm v_2 \in \mathbb{C}^n$ such that*

$$Mv_1 = \lambda_1 v_2, \quad Mv_2 = \lambda_2 v_1,$$

then the vectors $v_1 \pm \sqrt{\lambda_1/\lambda_2} v_2$ are eigenvectors of M corresponding to the eigenvalues $\pm\sqrt{\lambda_1\lambda_2}$.

Finally, we need to determine the inverse change-of-basis matrix for the basis $\{\boldsymbol{w}^l\}_{1 \leq l \leq q-1}$.

LEMMA 2.4. *If P is the matrix given by $P_{ij} = \omega^j(i)$, where $i \in \mathbb{F}_q^\times$ and $1 \leq j \leq q-1$, then*

$$(P^{-1})_{ij} = \frac{1}{q-1}\overline{\omega^i(j)}.$$

REMARK 2.5. Note that the indices for rows and columns are inverted in P^{-1}. In other words, for P^{-1}, $1 \leq i \leq q-1$ and $j \in \mathbb{F}_q^\times$.

PROOF. Note that

$$\sum_{k \in \widehat{\mathbb{F}_q^\times}} \omega^k(i) \cdot \frac{1}{q-1}\overline{\omega^k(j)} = \frac{1}{q-1} \sum_{k \in \widehat{\mathbb{F}_q^\times}} \omega^k\Big(\frac{i}{j}\Big).$$

Since ω is a generator of $\widehat{\mathbb{F}_q^\times}$, the lemma follows by (2.1). □

3. Proofs of Theorems 1.1 and 1.2

PROOF OF THEOREM 1.1. Applying Lemma 2.1 with $l = (q-1)/2$ and $l = q-1$ shows that $x - J(\overline{A}\phi, \overline{A})$ and $x - J(\overline{A}, A)$ divide $f_q(x)$. Similarly, applying Lemma 2.1 with $1 \leq l \leq (q-3)/2$ and Lemma 2.3 to the vectors $\boldsymbol{w}^l$ and $\boldsymbol{w}^{q-1-l}$ shows that $x^2 - J(\overline{A}B, A\omega^l)J(\overline{A}B, A\overline{\omega^l})$ divides $f_q(x)$. □

PROOF OF THEOREM 1.2. We give the proof of this theorem when $k = 2l$ is even. Applying Lemma 2.1 twice shows

$$M_q^2 = PDP^{-1},$$

where

$$D_{mn} = \begin{cases} J(\overline{A}, A\omega^m)J(\overline{A}, A\overline{\omega^m}) & \text{if } m = n, \\ 0 & \text{otherwise,} \end{cases}$$

and $P_{ij} = \omega^j(i)$ for $i \in \mathbb{F}_q^\times$ and $1 \leq j \leq q-1$. By Lemma 2.4 and a direct computation,

$$(M_q^{2l})_{ij} = \frac{1}{q-1} \sum_{m=1}^{q-1} \omega^m\Big(\frac{i}{j}\Big) J(\overline{A}, A\omega^m)^l J(\overline{A}, A\overline{\omega^m})^l.$$

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By applying (1.2), (2.2) and (2.3),

$$(M_q^k)_{ij} = A^l(-1) \cdot \frac{q^m}{q-1} \sum_{m=1}^{q-1} \left(\frac{\overline{\omega^m}}{A\overline{\omega^m}}\right)^l \left(\frac{A\overline{\omega^m}}{\omega^m}\right)^l \overline{\omega^m}\binom{j}{i}.$$

Since $\overline{\omega}$ generates $\widehat{\mathbb{F}_q^\times}$, the theorem follows from (1.1).

The proof is similar when k is odd. □

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