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BROWNIAN MOTION—WIENER PROCESS

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0. **Prologue and summary.** Bachelier (1900), Einstein (1905) and Smoluchowski (1915) provided a theory of the peculiar erratic motion of small particles suspended in a liquid, first described in 1826 by the English botanist Brown. In a series of papers beginning in 1920 Wiener undertook a mathematical analysis of Brownian motion. In his 1956 autobiography (pp. 38, 39) Wiener writes:

"Here I had a situation in which particles describe not only curves but statistical assemblages of curves. It was an ideal proving ground for my ideas concerning the Lebesgue integral in a space of curves, and it had the abundantly physical texture of the work of Gibbs. It was to this field that I had decided to apply the work that I had already done along the lines of integration theory. I met with a considerable degree of success.

The Brownian motion was nothing new as an object of study by physicists. There were fundamental papers by Einstein and Smoluchowski that covered it, but whereas these papers concerned what was happening to any given particle at a specific time, or the long-time statistics of many particles, they did not concern themselves with the mathematical properties of the curve followed by a single particle.

Here the literature was very scant, but it did include a telling comment by the French physicist Perrin in his book Les Atomes where he said in effect that the very irregular curves followed by particles in the Brownian motion led one to think of the supposed continuous non-differentiable curves of the mathematicians. He called the motion continuous because the particles never jump over a gap and non-differentiable because at no time do they seem to have a well-defined direction of movement.

In the physical Brownian motion, it is of course true that the particle is not subject to an absolutely perpetual influence resulting from the collision of the molecules but that there are short intervals of time between one collision and the next. These, however, are far too short to be observed by any ordinary methods. It therefore becomes natural to idealize the Brownian

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motion as if the molecules were infinitesimal in size and the collisions continuously described. It was this idealized Brownian motion which I studied, and which I found to be an excellent surrogate for the cruder properties of the true Brownian motion."

A decade before Kolmogorov's axiomatization (1933) of probability Wiener (1923) constructed a mathematical model of Brownian motion in which the basic probabilities were the values of a measure defined on subsets of a space of continuous functions. This measure has since been commonly called "Wiener measure". Having constructed his measure, Wiener then proved that almost every sample path of Brownian motion is nowhere differentiable. So, indeed, Perrin was right about Brownian motion and mathematics gained a beauty, the continuous nowhere differentiable Wiener process.

The above quotation from Wiener's autobiography continues:

> To my surprise and delight I found that the Brownian motion as thus conceived had a formal theory of a high degree of perfection and elegance. Under this theory I was able to confirm the conjecture of Perrin and to show that, except for a set of cases of probability *o*, all the Brownian motions were continuous non-differentiable curves.

This exposition is also concerned with continuity and non-differentiability of Wiener process. In Section 2 we give its definition and prove its existence via direct construction. The way we go about the latter is quite similar in spirit to the approach taken by Paley and Wiener (1934), where Wiener process is represented explicitly as the sum of a Fourier series with random coefficients. As to its continuity, we prove the P. Lévy modulus of continuity theorem (cf. Theorem 1) and then explain how our construction relates to the just mentioned Paley–Wiener representation.

In Section 3 the notions of continuity and big increments are linked. We draw a parallel between the P. Lévy modulus of continuity theorem and the Erdős–Rényi (1970) Law of large numbers for the Wiener process via Theorem 2, which establishes a continuous link between Strassen's 1964 law of the iterated logarithm and the Erdős–Rényi law for the Wiener process.

Section 4 is addressed to non-differentiability of the Wiener process, and we demonstrate to what extent the latter is actually true via giving the "modulus of non-differentiability" (cf. Theorem 3) of it.

Just like the notion of its continuity can be linked to large fluctuation of the Wiener process, the notion of its non-differentiability can be coupled with its small fluctuation. This is done in Section 5 (cf. Theorem 5), where we also give a summary of further problems.

The results of Sections 2–5 are joint with Pál Révész of Hungary and are presented here the first time together. Our proof of P. Lévy's theorem (cf.

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Theorem 1) and the complete proof of our Theorem 3 are given here for the first time and are taken from the first Chapter of our forthcoming book, "Strong Approximations in Probability and Statistics".

I wish to thank the Editors of Canadian Mathematical Bulletin for inviting me to write an expository paper related to my own work in recent years.

In order to make this exposition essentially self contained and understandable by the general mathematical public, a physical description of Brownian motion is given in Section 1, while Section 2 also contains those basic notions of probability which are required and hopefully sufficient for the understanding of the rest of the material.

While the paths of Wiener process are non-differentiable curves, integration with respect to Wiener measure is still possible (cf. Paley, Wiener and Zygmund (1959)). For further reading in this direction we refer to Itô (1944) and McKean (1969).

1. **Introduction.** Let $\{W(t)=W(t,\omega);\ 0\leq t\leq\infty\}$ denote the Brownian motion of a particle ω as time t goes by. Then $W(t,\omega)$ represents the position of that particle at time t. The essential point in Einstein's 1905 modelling of $W(t)$ is that the contacts between the foreign microscopic particle and the particles of the liquid occur only at moments of collision. These collisions occur irregularly but often. Thus, if the difference $t-s$ is large in comparison with the time interval between two successive collisions, then $W(t)-W(s)$ is the sum of a large number of small increments. Now, if the liquid is in macroscopic equilibrium, we may assume that the increments depend only on the length of their time interval and hence are homogeneous, and also that in disjoint time intervals they are independent. Also, if we further assume that the Brownian motion of a particle is symmetric, then the average increment over $t-s$ is $E(W(t)-W(s))=0$. Einstein showed that under these conditions the average squared increment (variance) over $t-s$ is $E(W(t)-W(s))^2=2D^2(t-s)$, where D is the so called diffusion constant of the liquid in question. Hence $\sigma^2=2D^2$ is a constant which characterizes the liquid. We observe also that, since $W(t)-W(s)$ may be considered as the sum of a large number of independent small increments (independent random variables), the central limit theorem suggests that $W(t)-W(s)$ has a normal distribution with mean zero and variance $\sigma^2(t-s)$.

The just described assumptions which are confirmed by experimental data, are the ones one postulates also when one defines the rigorous mathematical model of Brownian motion, namely the Wiener process. As to another example in nature, the movement of a particle of a gas under low pressure may also be treated as Brownian motion. In general, we can consider as Brownian motion the movement of any body which is subject to collision with other bodies, provided the dimension of the given body is small in comparison with the

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dimensions of the other bodies and if the contacts occur only at moments of collision and these collisions are of a random character.

In general let $EW(t)=\mu t(-\infty<\mu<+\infty)$ and $E(W(t)-\mu t)^2=\sigma^2 t$. As far as mathematical considerations go, we may assume without loss of generality that $\sigma^2=1$ and $\mu=0$. Such a Brownian motion is called *normalized* (standard) *Brownian motion*, and this is the one this exposition is concerned with. We note that $(W(t)-\mu t)/\sigma$ is a normalized Brownian motion if $W(t)$ is a Brownian motion with mean $EW(t)=\mu t$ and variance $E(W(t)-\mu t)^2=\sigma^2 t$.

One of the simplest models for a normalized Brownian motion (simply Brownian motion from now on) can be given in terms of the coin tossing or random walk model. Suppose that a particle is moving on the real line, starting from the origin. In each time unit it can only move one step to the right, or to the left, with probability one half and these steps are assumed to be independent. Say the ith step of the particle is X_i; then $X_1, X_2, \ldots$ are independent identically distributed random variables (i.i.d.rv) with

$$P(X_i=1)=P(X_i=-1)=\tfrac{1}{2}, \qquad i=1,2,\ldots, \tag{1}$$

and after n steps the particle will be located at $S_n=X_1+X_2+\cdots+X_n$. The thus created path $S_1, S_2, \ldots$ imitates Brownian motion quite well if the time unit and steps are short enough. In a more realistic model of Brownian motion the particle makes instantaneous steps to the right or to the left, that is a continuous time scale is used instead of a discrete one, and the lengths X_i of steps are assumed to be normally distributed instead of the distribution (1). In the next section the proof of existence of Wiener process (the idealized version of Brownian motion) takes into account the just sketched model.

2. **Definition and existence of Wiener process.** Probability theory, in general, deals with mathematical models of situations depending on chance. We may call such a situation an *experiment*. To every experiment there corresponds a non-empty set Ω, the set of possible outcomes of an experiment, the so called *basic space*, and its elements, denoted by ω, are called the *outcomes* of an experiment. A collection of subsets $\mathscr{A}$ of Ω is called a collection of *events* if the following assumptions hold:

(i) If $A\in\mathscr{A}$, then the complement $\bar{A}$ of the event A in Ω is also an event, i.e., $\bar{A}\in\mathscr{A}$,
(ii) If $A_n\in\mathscr{A}$ $(n=1,2,\ldots)$, then $\bigcup A_n\in\mathscr{A}$
(iii) $\Omega\in\mathscr{A}$.

The event Ω occurs whatever is the outcome of an experiment. Whence Ω is called *the certain event*. Just like in other branches of mathematics, a family of subsets $\mathscr{A}$ of Ω, having the properties (i), (ii) and (iii), is called a σ-algebra (of

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events, in probability). Thus we arrive at the following notion of the *mathematical model of an experiment:*

An *experiment* $\mathscr{E}$ is a non-empty set Ω of elements ω, which are called outcomes of the experiment, and a σ-algebra $\mathscr{A}$ of subsets of Ω, which are called events of the experiment. For the sake of brevity we put $\mathscr{E}=(\Omega, \mathscr{A})$. A system $\mathscr{E}=(\Omega, \mathscr{A})$ which we call an experiment is usually called a *measurable space* in measure theory.

A non-negative set function $P(\cdot)$ defined on a σ-algebra $\mathscr{A}$ of subsets of Ω is called a *probability measure* (distribution) if

(I) (normalization) $P(\Omega)=1$,

(II) (σ-additivity) for every finite or countable collection A_k of sets (*events*) in $\mathscr{A}$ such that A_k is disjoint from A_j, $k \neq j$, (i.e., $A_k \cap A_j=\emptyset$, $k \neq j$, the *empty set* (*impossible event*)),

$$P\left(\bigcup_k A_k\right)=\sum_k P(A_k).$$

Thus a *probability measure* $P(\cdot)$ on $\mathscr{E}=(\Omega, \mathscr{A})$ is a non-negative, normed, σ-additive set function.

A triple $(\Omega, \mathscr{A}, P)$, where Ω is an arbitrary non-empty set, $\mathscr{A}$ is a σ-algebra of subsets of Ω and $P(\cdot)$ is a probability measure on $\mathscr{A}$, is called a *probability space.* The notion of a probability space is due to Kolmogorov (1933).

We note that (I) and (II) imply $P(\phi)=0$. However $P(A)=0$ does not necessarily imply that $A=\phi$, i.e., P is not a strictly positive measure.

A real valued function $X=X(\omega)$ defined on the space Ω, i.e., $X:\Omega \rightarrow R^1$, is called a *random variable* (rv) if

$$A_x=\{\omega: X(\omega) \leq x\} \in \mathscr{A} \quad \text{for every} \quad x \in R^1, \tag{2}$$

i.e. a real valued random variable is a real valued *measurable function.*

In this exposition all random variables are going to be real valued and we will simply talk about them as random variables. So, in short, we can say then that a function $X(\omega)$ defined on Ω is called a random variable if for every Borel set B of the real line R^1, the set $\{\omega: X(\omega) \in B\}$ is an event (i.e., it is in $\mathscr{A}$ of $(\Omega, \mathscr{A})$). Whether a given function is a random variable, it depends of course on the pair (experiment) $(\Omega, \mathscr{A})$. The reason underlying (2) is that we want probability assigned to all sets of the form $\{\omega: X(\omega) \in I\}$, where I is some interval of the real line, which, in turn, implies that $\{\omega: X(\omega) \in B\}$ is an event for every Borel set B of R^1.

Since, by definition, A_x is in $\mathscr{A}$ of $(\Omega, \mathscr{A})$, the probability $P(A_x)$ is always defined for any given probability space $(\Omega, \mathscr{A}, P)$ and the function

$$F(x)=F_X(x)=P(A_x)=P(\omega: X(\omega) \leq x), \qquad x \in R^1 \tag{3}$$

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is called the *distribution function* of the rv X. F, As a function of ω is a set function, and it is a monotone non-decreasing, right continuous function of x with $F(+\infty)=\lim_{x\uparrow+\infty}F(x)=1$ and $F(-\infty)=\lim_{x\downarrow-\infty}F(x)=0$.

A *stochastic process* is a collection of $\{X(t)=X(t,\omega);\ t\in T\subset R^1\}$ of rv on a probability space $(\Omega,\mathscr{A},P)$.

If T is countable, then we talk about a *discrete time parameter stochastic process* and if T is an interval, then we talk about a *continuous time parameter stochastic process*. The very essence of the definition of a stochastic process is that, for any fixed t, $X(t)=X(t,\omega)$ is a rv. Whenever convenient, the notation $\{X(t);\ t\in T\}$ or simply $\{X(t)\}$ will be used.

Since $X(t)$ is a random variable for every fixed t, the so called finite dimensional distribution functions of $X(t)$

$$F_n(x_1,\ldots,x_n)=P\left(\bigcap_{k=1}^{n}\{\omega:X(t_k,\omega)\le x_k\}\right), \tag{4}$$

where $t_1<t_2<\cdots<t_n$ are fixed values in T and $x_1,x_2,\ldots,x_n$ are in R^1 $(n=1,2,\ldots)$, are always defined. Two processes have the same probability distribution P if and only if all their finite dimensional distribution functions are equal.

Frequently the distribution of a stochastic process is defined by first giving a set of distribution functions $\{F(x_1,\ldots,x_n);\ n=1,2,\ldots\}$ which satisfy the so called Kolmogorov *consistency condition* which can be stated as follows. For each $m\ge1$ and $(x_1,\ldots,x_m)\in R^m$ we have if $n\ge m$:

$$\begin{aligned}\lim_{x_{m+1}\uparrow+\infty,\ldots,X_n\uparrow+\infty}F_n(x_1,\ldots,x_m,x_{m+1},\ldots,x_n)&=F_n(x_1,\ldots,x_m,+\infty,\ldots,+\infty)\\&=F_n(x_1,\ldots,x_m).\end{aligned} \tag{5}$$

The famous *Kolmogorov extension theorem* asserts that for any given consistent system of finite dimensional distribution functions there exists a probability space $(\Omega,\mathscr{A},P)$ and a stochastic process $\{X(t,\omega);\ T\in T\}$ on the latter such that the finite dimensional distribution functions of $X(t)$ are the a priori given ones. This means that the definition of a stochastic process via the route of (5) is feasible. Whence, and also because of our description of Brownian motion on intuitive grounds in the Introduction, we now give the mathematical

Definition of Wiener Process. A stochastic process $\{W(t)=W(t,\omega);\ 0\le t<\infty\}$,—where $\omega\in\Omega$ and $(\Omega,\mathscr{A},P)$ is a probability space—, is called a Wiener process if

$$P(\omega:W(t,\omega)-W(s,\omega)\le x)=\frac{1}{\sqrt{2\pi(t-s)}}\int_{-\infty}^{x}e^{-u^2/2(t-s)}\,du \tag{a}$$

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for all $0 \leq s < t < +\infty$ and $W(0) = 0$, i.e. the rv $W(t, \omega) - W(s, \omega)$ is *normally distributed with mean* 0 *and variance* $t - s$, and we start $W(t)$ at $t = 0$ with probability one,

(b) $W(t)$ is an *independent increment process*, i.e., $W(t_2) - W(t_1)$.

$$W(t_4) - W(t_3), \ldots, W(t_{2i}) - W(t_{2i-1})$$

are independent rv for all

$$0 \leq t_1 < t_2 \leq t_3 < t_4 \leq \cdots \leq t_{2i-1} < t_{2i} < \infty \quad (i = 2, 3, \ldots),$$

i.e., by (a) and the notion of independence of events,

$$P\left(\bigcap_{i=1}^{n} \{\omega : W(t_{2i}, \omega) - W(t_{2i-1}, \omega) \leq x_i\}\right)$$
$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi(t_{2i} - t_{2i-1})}} \int_{-\infty}^{x_i} e^{-u^2/2(t_{2i} - t_{2i-1})}\, du, \qquad (n = 1, 2, \ldots),$$

(c) The *sample path function* $W(t, \omega)$ (i.e., $W(t, \omega)$ as a function of t for ω fixed) *is continuous in* t *with probability one* (i.e. except on an ω-set of P-measure zero $W(t, \omega)$ is a continuous function of t).

One way of describing in short the probability space $(\Omega, \mathscr{A}, P)$ of a Wiener process defined above with the given finite dimensional distributions (cf. (a) and (b)) is to say that by Kolmogorov's extension theorem we may use the sample space $\Omega = R^T$ with $T = [0, \infty)$, which consists of all real valued functions $\{\omega = w(t),\ t \in T\}$ defined on T, and that we can uniquely define P on the σ-algebra $\mathscr{A} = \mathscr{B}^T$ generated by the cylinders of R^T in such a way that the stochastic process $W(t) = W(t, \omega) = w(t)$ has the given distributions of (a) and (b). This description is very intuitive because the outcomes ω of our experiment $\mathscr{E} = (\Omega, \mathscr{A}) = (R^T, \mathscr{B}^T)$ and the *sample path functions* $W(t)$ agree, and the Wiener probability measure P of $(\Omega, \mathscr{A}, P) = (R^T, \mathscr{B}^T, P)$ is simply a probability measure defined in the space of functions R^T, i.e., in the space of the sample path functions. Now according to (c) we are to look at the probability P of *continuous* sample functions. Unfortunately, the set of the continuous sample functions is not an event, i.e., they do not belong to $\mathscr{B}^T$ and it is, therefore, meaningless to talk about their probability. We can, however, extend the above P to a probability measure Q defined on a σ-algebra $\mathscr{F}$ which contains $\mathscr{B}^T$ and also satisfies the following requirements: the set of continuous sample functions $C = C(T)$ belongs to $\mathscr{F}$; the above stochastic process $W(t) = W(t, \omega) = w(t)$ is also a Wiener process, with (a) and (b) as above, over the probability space $(R^T, \mathscr{F}, Q)$; *every* set (event) of $\mathscr{F}$ differs from *some* set (event) of $\mathscr{B}^T$ only by a set of Q-probability zero. What we are saying then is that the Wiener process $W(t)$ in the given representation (cf. (a) and (b)) has the fundamental property that its sample path functions are almost surely (a.s.) continuous provided that

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we extend the domain of the probability P as just explained (here, and throughout this exposition, saying that an event occurs almost surely is equivalent to saying that the event in question occurs with P-measure one). Restricting now P to subsets of C, we may look at C as the sample space of Wiener process.

The just sketched measure theoretic considerations were given only for the sake of illustrating that the above given Definition of Wiener process did make sense if we were to approach the problem of its existence from a measure theoretic point of view à la Kolmogorov (1933) combined with Doob (1953).

We should note also that (a) and (b) imply that the covariance function of a Wiener process is

$$R(s, t) = EW(s)W(t) = s \wedge t \quad \text{for all} \quad s, t \in T. \tag{6}$$

Conversely, we also note without proof here, that normalized Brownian motion (Wiener process) is completely specified by stating that it is a continuous Gaussian process (i.e., all finite vectors $\{W(t_1), \ldots, W(t_n)\}$ have a joint normal distribution) with $EW(t) = 0$ and covariance function $R(s, t) = s \wedge t$.

We have now frequently used heuristically the notion of *expectation* of a rv. Let X be a rv on $(\Omega, \mathscr{A}, P)$. The expectation of X, denoted by EX, is defined by $\int X(\omega)\, dP(\omega)$. The latter is well defined if $E|X| < \infty$. The *variance of* X is defined by $E(X - EX)^2$, and the covariance of two rv X and Y on $(\Omega, \mathscr{A}, P)$ with respect to their joint probability measure P is defined by $E(X - EX)(Y - EY)$, provided the indicated integrals are finite.

Going back to the problem of definition and existence of Wiener process, Wiener (1923), of course, did not follow the above sketched measure theoretic route. He undertook a mathematical analysis of Brownian motion a decade before Kolmogorov's 1933 extension theorem and thirty years before Doob's approach (Doob 1953). Fixing an origin in time and a direction in space, let $w(t)$ be the displacement by time t of a Brownian particle in the specified direction. Then $w(0) = 0$. For technical reasons it was convenient to restrict t to the interval $[0, 1]$. Thus Wiener was led to consider the space C of continuous functions on $[0, 1]$, vanishing at 0, and to define a measure of subsets of C based on the Daniell integral. The probability of any property of the displacement function was associated with the measure of the subset of C having this property. Let $W(t) = W(t, \omega) = w(t)$, i.e. ω is a member of C and $W(t, \omega)$ is the value of ω at t. The Wiener measure P of subsets of C then has the property that $P(C) = 1$ and conditions (a) and (b) above also hold. In view of the definition of C the values of t here are restricted to the interval $[0, 1]$ but the simple transformation

$$W^*(t) = (1+t)\left(W\left(\frac{t}{t+1}\right) - \frac{t}{t+1}\, W(1)\right), \qquad 0 \le t < \infty$$

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gives us that $W^*(t)$ is a (normalized) Wiener process on $C = C[0, \infty)$. The easiest way to see this is to accept that (6) characterizes W and then our statement concerning W^* follows from the fact that

$$EW^*(s)W^*(t) = s \wedge t, \qquad s, t \in [0, \infty).$$

In 1923, and later also in his joint book with Paley (1934), Wiener also studied the regularity of Brownian paths, proving that almost no function in C is of bounded variation in any interval, and finding estimates of the modulus of continuity for $W(t)$ (cf. Theorem 1 and the Corollary of Section 4).

In 1934 Paley and Wiener define $W(t, \omega)$ in a different and very elegant way; $W(t, \omega)$ is a function on the unit interval with Lebesgue measure, and properties (a), (b), and (c) hold. The construction does not involve the Daniell integral, and $W(t, \omega)$ is represented explicitly as the sum of a Fourier series with random coefficients.

Our approach of constructing the Wiener process is similar in spirit to that of Paley and Wiener in that we also give an explicit representation of $W(t, \omega)$ satisfying (a), (b), and (c) above and then also explain how to go about relating it to the classical Paley–Wiener (1934) representation. From now on we follow the construction and proofs of Chapter I of Csörgő–Révész (1979c).

The aim of this section is then to give a constructive proof for the existence of Wiener process as defined by (a), (b), and (c). Let $\{r_n\}$ be the sequence of positive dyadic rational numbers (i.e., numbers of the form $k/2^n$, $k = 1, 3, \ldots, n = 1, 2, \ldots$) and let $\{X_{r_n}\}$ be independent normal rv with mean zero and variance one ($N(0, 1)$ rv) defined on a probability space $(\Omega, \mathscr{A}, P)$. On this probability space we now proceed to *construct* a Wiener process as follows:

For any positive integer k, let

$$W(k) = X_1 + X_2 + \cdots + X_k$$

and

$$W(k + \tfrac{1}{2}) = \frac{W(k) + W(k+1)}{2} + \frac{X_k + 1/2}{\sqrt{4}}.$$

Now we wish to define $W(k/2^n)$ for $k = 1, 2, \ldots$ and $n = 1, 2, \ldots$. Assume that it is already defined for $k = 1, 2, \ldots$ and $n = 1, 2, \ldots, n_0$. Then, for $k = 1, 2, \ldots$ and $n = n_0 + 1$, let

$$W\left(\frac{2k+1}{2^n}\right) = \frac{W\left(\frac{2k}{2^n}\right) + W\left(\frac{2k+2}{2^n}\right)}{2} + \frac{X_2(k+1)2^{-n}}{\sqrt{2^{n+1}}}.$$

Whence, by induction, we have defined our Wiener process at every dyadic rational point r_n. For an arbitrary $0 < t = \sum_{k=0}^{\infty} \varepsilon_k(t)/2^k$ $(\varepsilon_0(t) = 0, 1, 2, \ldots;$

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$\varepsilon_k(t)=0,\ 1;\ k=1,\ 2,\ldots)$ we define

$$W(t) \overset{\text{a.s.}}{=} \lim_{n\to\infty} W([2^n t]/2^n) = \lim_{n\to\infty} W(t_n)$$

$$= W(\varepsilon_0(t)) + \lim_{n\to\infty} \sum_{k=1}^{n} (W(t_k) - W(t_{k-1})),$$

with $t_n = \sum_{j=0}^{n} \varepsilon_j(t)/2^j$. The existence of the above limit follows immediately from Kolmogorov's Three Series Theorem (cf., e.g., p. 112, Chung 1968) for every fixed $t>0$. However, the exceptional set of probability zero, where this latter a.s.—convergence might not hold, can depend on the particular fixed t. This, however, presents no problems, because there exists a set $\Omega_0 \subset \Omega$ of probability zero such that the series

$$\sum_{k=1}^{\infty} (W(t_k) - W(t_{k-1}))$$

converges for every t whenever $\omega \in \Omega - \Omega_0$. In fact, we are going to prove the stronger statement that the above limit representation of $W(t)$ holds uniformly in t with probability one. In order to see this, it suffices to show

$$\sum_{k=1}^{\infty} \sup_{0\le t\le 1} |W(t_k) - W(t_{k-1})| < \infty \quad \text{a.s.},$$

which, in turn, is implied by the well known estimation (cf. Feller (1968), p. 175; here $\Phi(\cdot)$ stands for the distribution function of a $N(0,1)$ rv)

$$\frac{1}{\sqrt{2\pi}}\left(\frac{1}{x} - \frac{1}{x^3}\right) e^{-x^2/2} \le 1 - \Phi(x) \le \frac{1}{\sqrt{2\pi}x} e^{-x^2/2}, \qquad x>0, \tag{7}$$

as follows. First, we have

$$P\left\{\sup_{0\le t\le 1} |W(t_k) - W(t_{k-1})| \ge u_k \frac{1}{\sqrt{K}}\right\} \le 2Ke^{-u_k^2/2},$$

where $K=2^k$ and $u_k = C\sqrt{2\log K}$, $C=\text{const.}>1$. Consequently, with

$$L = C\sum_{k=1}^{\infty} \sqrt{\left(\frac{2\log 2^k}{2^k}\right)},$$

$$P\left\{\sum_{k=1}^{\infty} \sup_{0\le t\le 1} |W(t_k) - W(t_{k-1})| \ge CL\right\} \le \sum_{k=1}^{\infty} \frac{2}{2^{k(C^2-1)}}$$

$$= \frac{2}{2^{C^2-1}} \to 0 \text{ as } \quad C\to\infty,$$

gives the desired a.s. convergence.

A little calculation now shows that the thus defined process $\{W(t);\ 0\le t<\infty\}$ satisfies conditions (a) and (b). Condition (c), however, is not immediate at all.

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The rest of this section is devoted to proving and further elaborating on condition (c) for the above constructed process $\{W(t);\ 0 \le t < \infty\}$. The following lemma plays a key role in doing this.

LEMMA 1 (Csörgő, Révész 1979c). *For any $\varepsilon > 0$ there exists a constant $C = C(\varepsilon) > 0$ such that the inequality*

$$P\left\{\sup_{0 \le s \le 1-h} \sup_{0 < t \le h} |W(s+t) - W(s)| \ge v\sqrt{h}\right\} \le \frac{C}{h} e^{-v^2/2+\varepsilon} \tag{8}$$

holds for every positive v and $h < 1$.

For a proof of Lemma 1 we refer to Section 1.1, Csörgő, Révész (1979c).

With the help of our Lemma 1 we can also prove now that the above constructed $\{W(t);\ 0 \le t < \infty\}$ is continuous in t with probability one, that is condition (c) is also satisfied. This will immediately follow from the next theorem, which also gives more, namely the modulus of continuity of the Wiener process.

THEOREM 1 (P. Lévy 1937, 1948). *We have*

$$\lim_{h \to 0} \frac{\sup_{0 \le s \le 1-h} \sup_{0 < t \le h} |W(s+t) - W(s)|}{\sqrt{(2h \log 1/h)}} \overset{\text{a.s.}}{=} 1 \tag{9}$$

and

$$\lim_{h \to 0} \frac{\sup_{0 \le s \le 1-h} |W(s+h) - W(s)|}{\sqrt{(2h \log 1/h)}} \overset{\text{a.s.}}{=} 1. \tag{10}$$

Proof. Let

$$A_h = \sup_{0 \le s \le 1-h} \sup_{0 < t \le h} |W(s+t) - W(s)|. \tag{11}$$

First we prove

$$\overline{\lim_{h \to 0}} \frac{A_h}{\sqrt{(2h \log 1/h)}} \le 1 \quad \text{a.s.} \tag{12}$$

We apply the inequality of (8) with $v = (1+\varepsilon)\sqrt{(2 \log 1/h)}$, $\varepsilon > 0$. Then

$$P\left\{\frac{A_h}{\sqrt{(2h \log 1/h)}} \ge 1 + \varepsilon\right\} \le \frac{C}{h} \exp\left\{-\frac{2\left(\log \frac{1}{h}\right)(1+\varepsilon)^2}{2+\varepsilon}\right\} \le Ch^{\varepsilon}.$$

Take $T > 1/\varepsilon$ and let $h = h_n = n^{-T}$. Then

$$\sum_{n=1}^{\infty} P\left\{\frac{A_{h_n}}{\sqrt{(2h_n \log 1/h_n)}} \ge 1 + \varepsilon\right\} \le \sum_{n=1}^{\infty} Cn^{-T\varepsilon} < \infty$$

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and the Borel–Cantelli lemma (cf. e.g. p. 72, Chung 1968) implies that

$$\overline{\lim_{n\to\infty}} \frac{A_{h_n}}{\sqrt{(2h_n \log 1/h_n)}} \le 1+\varepsilon \quad \text{a.s.}$$

As an important special case of the latter representation, we take $\{\phi_0(x)=1, \phi_k(x)=\sqrt{2}\cos \pi k x;\ 0\le x\le 1,\ k=1, 2, \ldots\}$ as our complete orthonormal system on $[0, 1]$, and get

$$W(t) = Y_0 t + \sqrt{2} \sum_{k=1}^{\infty} Y_k \int_0^t \cos k\pi x\, dx$$

$$= Y_0 t + \sqrt{2} \sum_{k=1}^{\infty} Y_k \frac{\sin k\pi t}{k\pi},$$

the classical representation of W by Paley and Wiener (1934).

3. **How big are the increments of a Wiener process?** In Theorem 1 we saw how large the increments of a Wiener process over subintervals of length h of the unit interval can be when h is small. In this section we are going to study the similar problem of how large the increments of a Wiener process over subintervals of length a_T of the interval $[0, T]$ can be when $T\to\infty$ and a_T is a non-decreasing function of T. These two problems are closely related to each other and can be studied from the same source of information, namely from Lemma 1. Towards this end we first extend the statement of the latter from the unit interval to any finite interval of the positive half-line (Lemma 2). From this latter lemma the main result (Theorem 2) of this section follows just like Theorem 1 did from Lemma 1. This then shows that Theorems 1 and 2 are closely linked. They do not seem to follow directly from each other though (cf., however, Theorem S.1.2.1 in Csörgő, Révész 1979c).

The above mentioned immediate analogue of Lemma 1 is

LEMMA 2. *For any $\varepsilon>0$ there exists a constant $C=C(\varepsilon)>0$ such that the inequality*

$$P\left\{\sup_{0\le s\le T-h} \sup_{0\le t\le h} |W(s+t)-W(s)| \ge v\sqrt{h}\right\} \le \frac{CT}{h} e^{-v^2/2+\varepsilon} \tag{16}$$

holds for every positive v, T and $0<h<T$.

Proof. This lemma follows from (8) and from the following

OBSERVATION. *For any fixed $T>0$ we have*

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where the latter r.v. is $\overset{\text{a.s.}}{=} o(1)$ by (12), and the first one is a.s. ≥ 1 by (15). Hence we get (13). The latter combined with (12) also completes the proof of (9) and (10).

REMARK 1. The following trivial generalizations of Theorem 1 are easily obtained:

$$\lim_{h\to 0} \frac{\sup_{a\leq s\leq b} \sup_{0<t\leq h} (W(s+t)-W(s))}{(2h\log 1/h)^{1/2}} \overset{\text{a.s.}}{=} \lim_{h\to 0} \frac{\sup_{a\leq s\leq b} \sup_{0<t\leq h} |W(s+t)-W(s)|}{(2h\log 1/h)^{1/2}}$$

$$\overset{\text{a.s.}}{=} \lim_{h\to 0} \frac{\sup_{a\leq s\leq b} (W(s+h)-W(s))}{(2h\log 1/h)^{1/2}}$$

$$\overset{\text{a.s.}}{=} \lim_{h\to 0} \frac{\sup_{a\leq s\leq b} |W(s+h)-W(s)|}{(2h\log 1/h)^{1/2}} \overset{\text{a.s.}}{=} 1$$

for any $0\leq a<b<\infty$.

Our construction of Wiener process can be slightly modified so that it also gives a random infinite series representation of W. Namely we can show (for details we refer to Section 1.8 of Csörgő, Révész 1979c) that, restricting ourselves to the unit interval $[0, 1]$,

$$W(t)=\sum_{k=0}^{\infty} Y_k \int_0^t w_k(x)\,dx, \qquad 0\leq t\leq 1,$$

where $Y_0, Y_1, \ldots$ is a sequence of independent $N(0, 1)$ rv and w_k is the kth Walsh function.

Since the Walsh functions $\{w_k\}$ form a complete orthonormal system, it is only natural to ask whether $\{w_k\}$ in the above representation could be replaced by any other complete orthonormal system $\{\phi_k\}$. Indeed, it is clear that, for any such system of functions $\{\phi_k\}$, the series $\sum_{k=0}^{\infty} Y_k \int_0^t \phi_k(x)\,dx$ converges with probability one for each fixed $t\in[0, 1]$. It is also clear (direct calculations) that the covariance function of the latter series is that of a Wiener process. On the other hand, it is not clear at all that the latter convergence should hold uniformly in t. However, Itô and Nisio (1968) showed that it is so for any complete orthonormal system $\{\phi_k\}$ and also that the thus defined limit is a Wiener process, i.e., we have with probability one and uniformly in $t\in[0, 1]$ that

$$W(t)=\sum_{k=0}^{\infty} Y_k \int_0^t \phi_k(x)\,dx,$$

for any sequence $\{Y_i\}$ of independent $N(0, 1)$ rv.

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As an important special case of the latter representation, we take $\{\phi_0(x)=1, \phi_k(x)=\sqrt{2}\cos \pi kx;\ 0\le x\le 1,\ k=1, 2, \ldots\}$ as our complete orthonormal system on $[0, 1]$, and get

$$W(t)=Y_0 t+\sqrt{2}\sum_{k=1}^{\infty} Y_k \int_0^t \cos k\pi x\, dx$$

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holds for every positive v, T and $0<h<T$.

Proof. This lemma follows from (8) and from the following

OBSERVATION. *For any fixed $T>0$ we have*

$$\{W(s);\ 0\le s\le T\}\overset{\mathcal{D}}{=}\left\{\sqrt{T}\,W\left(\frac{s}{T}\right);\ 0\le s\le T\right\},$$

where $\overset{\mathcal{D}}{=}$ stands for equality in distribution.

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THEOREM 2 (Csörgő, Révész 1979a). *Let $a_T (T \geq 0)$ be a monotonically non-decreasing function of T for which*

(i) $0 < a_T \leq T$,

(ii) *T/a_T is monotonically non-decreasing. Then*

$$\overline{\lim_{T\to\infty}} \sup_{0\leq t\leq T-a_T} \beta_T |W(t+a_T)-W(t)| \overset{\text{a.s.}}{=} 1, \tag{17}$$

$$\overline{\lim_{T\to\infty}} \beta_T |W(T+a_T)-W(T)| \overset{\text{a.s.}}{=} \overline{\lim_{T\to\infty}} \sup_{0\leq s\leq a_T} \beta_T |W(T+s)-W(T)| \overset{\text{a.s.}}{=} 1 \tag{18}$$

and

$$\overline{\lim_{T\to\infty}} \sup_{0\leq t\leq T-a_T} \sup_{0\leq s\leq a_T} \beta_T |W(t+s)-W(t)| \overset{\text{a.s.}}{=} 1, \tag{19}$$

where

$$\beta_T = \left(2a_T\left[\log\frac{T}{a_T}+\log\log T\right]\right)^{-1/2}.$$

If we have also

(iii) $$\lim_{T\to\infty} (\log T/a_T)(\log\log T)^{-1} = \infty,$$

then

$$\lim_{T\to\infty} \sup_{0\leq t\leq T-a_T} \beta_T |W(t+a_T)-W(t)| \overset{\text{a.s.}}{=} 1 \tag{20}$$

and

$$\lim_{T\to\infty} \sup_{0\leq t\leq T-a_T} \sup_{0\leq s\leq a_T} \beta_T |W(t+s)-W(t)| \overset{\text{a.s.}}{=} 1. \tag{21}$$

REMARK 2. The proof of this Theorem (cf. Csörgő Révész 1979a) shows that statements (17)–(21) remain true if any one, two or all of T, t and s are running over all the integers, or if we omit the absolute value signs of these statements. Also, because of the symmetry of W, if we replace the lim sup by lim inf and sup by inf in (17)–(21), then the above results will be true with -1 instead of $+1$, when also omitting the absolute value signs. For example

$$\underline{\lim_{T\to\infty}} \inf_{0\leq t\leq T-a_T} \beta_T(W(t+a_T)-W(t)) \overset{\text{a.s.}}{=} -1 \tag{22}$$

if conditions (i)–(ii) hold true and

$$\lim_{T\to\infty} \inf_{0\leq t\leq T-a_T} \beta_T(W(t+a_T)-W(t)) \overset{\text{a.s.}}{=} -1 \tag{23}$$

if conditions (i)–(iii) hold true.

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Choosing a_T as $c \log T$, cT and 1 respectively, the following corollaries are immediate.

COROLLARY 1. *For any* $c > 0$ *we have*

$$\lim_{T\to\infty} \sup_{0\le t\le T-c\log T} \frac{|W(t+c\log T)-W(t)|}{c\log T} \overset{\text{a.s.}}{=} \sqrt{\left(\frac{2}{c}\right)}. \tag{24}$$

This latter statement is the Erdős–Rényi (1970) law of large numbers for the Wiener process.

COROLLARY 2. *For* $0 < c \le 1$ *we have*

$$\overline{\lim_{T\to\infty}} \sup_{0\le t\le T-cT} \frac{|W(t+cT)-W(t)|}{\sqrt{(2cT\log\log T)}} \overset{\text{a.s.}}{=} 1, \tag{25}$$

$$\overline{\lim_{T\to\infty}} \sup_{0\le t\le T-cT} \sup_{0\le s\le cT} \frac{|W(t+s)-W(t)|}{(2cT\log\log T)^{1/2}} \overset{\text{a.s.}}{=} 1. \tag{26}$$

With $c = 1$, (25) and (26) reduce to the classical law of iterated logarithm for the Wiener process (cf. P. Lévy 1937, 1948). In their present form (25) and (26) also follow from Strassen's law of iterated logarithm (1964). The latter will not be covered here. As to a step towards a Strassen type law of iterated logarithm concerning Theorem 2, we refer to Chan, Csörgő, and Révész (1978).

COROLLARY 3. *We have*

$$\lim_{T\to\infty} \sup_{0\le t\le T-1} \frac{|W(t+1)-W(t)|}{\sqrt{(2\log T)}} \overset{\text{a.s.}}{=} 1. \tag{27}$$

This is a well-known result which (when T and t run over the integers; cf. Remark 2) in terms of the order statistics $X_{i:n}$ (increasing in $i = 1, 2, \ldots, n$) of n independent $N(0, 1)$ rv reads

$$\lim_{n\to\infty} \frac{X_{n:n}}{\sqrt{(2\log n)}} \overset{\text{a.s.}}{=} 1. \tag{28}$$

For a proof of Theorem 2 we refer to Csörgő, Révész (1979a).

REMARK 3. It is possible to prove that in (17) and (19) the $\overline{\lim}$ cannot be changed to a lim, if condition (iii) fails, that is to say in the latter case (20) and (21) cannot be true. In fact, Deo (1977) has shown that

$$\underline{\lim}_{T\to\infty} \sup_{0\le t\le T-a_T} \sup_{0\le s\le a_T} \beta_T\,|W(t+s)-W(t)| < 1 \quad \text{a.s.} \tag{29}$$

as well as

$$\underline{\lim}_{T\to\infty} \sup_{0\le t\le T-a_T} \beta_T\,|W(t+a_T)-W(t)| < 1 \quad \text{a.s.} \tag{30}$$

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provided

$$\overline{\lim_{T\to\infty}}\,(\log T/a_T)(\log\log T)^{-1}<\infty.$$

This result suggests the following problem: find the normalizing factor $\delta_T=\delta_T(a_T)$ such that the left hand side rv of (29), resp. that of (30), should be equal to one almost surely, with δ_T replacing β_T in them. A partial answer concerning (30) was given by Book and Shore (1978), who showed that

$$\underline{\lim_{T\to\infty}}\sup_{0\le t\le T-a_T}\beta_T\,|W(t+a_T)-W(t)|\overset{\text{a.s.}}{=}\left(\frac{r}{r+1}\right)^{1/2},$$

provided $\lim_{T\to\infty}\log(Ta_T^{-1})/\log\log T=r$, $0\le r\le\infty$.

The similar question in connection with (29) was studied by Csáki and Révész (1979), who proved that

$$18^{-1}\le\underline{\lim_{T\to\infty}}\sup_{0\le t\le T-a_T}\sup_{0\le s\le a_T}\delta_T\,|W(t+s)-W(t)|\le 46\quad\text{a.s.},$$

where

$$\delta_T=\left(2a_T\log\left(1+\frac{\pi^2}{16}[Ta_T^{-1}]/\log\log T\right)\right)^{-1/2}.$$

The general question of finding the exact value of the above $\underline{\lim}_{T\to\infty}$ statement appears to be a difficult one. However, if one also has $\lim_{T\to\infty}\log(Ta_T^{-1})/\log\log\log T=\infty$, then the just mentioned $\underline{\lim}_{T\to\infty}$ is equal to one.

The special case of $a_T=T$ of these questions was studied by Chung (1948) and Hirsch (1965) who evaluated the normalizing factor μ_T resp. ν_T for which

$$\underline{\lim_{T\to\infty}}\sup_{0\le t\le T}\mu_T W(t)\overset{\text{a.s.}}{=}\underline{\lim_{T\to\infty}}\sup_{0\le t\le T}\nu_T\,|W(t)|\overset{\text{a.s.}}{=}1.$$

It should be emphasized that μ_T and ν_T are very different, which is not the case when studying the $\overline{\lim}$ instead of the $\underline{\lim}$ of these functionals.

4. **The modulus of non-differentiability of the Wiener process.** In this section we intend to prove the following analogue of Theorem 1.

THEOREM 3. (Csörgő, Révész 1979b).

$$\lim_{h\to 0}\inf_{0\le s\le 1-h}\sup_{0<t\le h}\sqrt{\left(\frac{8\log h^{-1}}{\pi^2 h}\right)}\,|W(s+t)-W(s)|\overset{\text{a.s.}}{=}1.$$

This theorem implies

THEOREM 4. (Wiener, 1923). *Almost all sample functions of a Wiener process are nowhere differentiable.*

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Theorem 3 actually gives the exact "modulus of non-differentiability" of a Wiener process.

The proof of Theorem 3 is based on the following simple lemma (cf. Lemma 1.6.1, Csörgő, Révész 1979c).

LEMMA 3.

$$\frac{4}{\pi}\left(e^{-\pi^2/8x^2}-\tfrac{1}{3}e^{-9\pi^2/8x^2}\right)\leq P\left\{\sup_{0\leq t\leq T} T^{-1/2}\,|W(t)|\leq x\right\}$$
$$\leq\frac{4}{\pi}\,e^{-\pi^2/8x^2}$$

and, if x is small enough,

$$\frac{2}{\pi}\,e^{-\pi^2/8x^2}\leq\frac{4}{\pi}\left(e^{-\pi^2/8x^2}-\tfrac{1}{3}e^{-9\pi^2/8x^2}\right).$$

The proof of Theorem 3 will be presented in two steps and it is taken from Section 1.6 of Csörgő, Révész (1979c).

Step 1. for any $\varepsilon>0$ we have

$$\text{(31)}\qquad \varliminf_{h\to 0}\ \inf_{0\leq s\leq 1-h}\ \sup_{0<t\leq h}\sqrt{\left(\frac{8\log h^{-1}}{\pi^2 h}\right)}\,|W(s+t)-W(s)|\geq 1-\varepsilon\quad\text{a.s.}$$

Proof. Put

$$s_i=ih(\log h^{-1})^{-3}\qquad(i=0,1,2,\ldots,\rho_h),$$

where $\rho_h=[h^{-1}(\log h^{-1})^3]$. Then, by Lemma 3, we have

$$P\left\{\min_{0\leq i\leq\rho_h}\ \sup_{0<t\leq h}\left(\frac{8\log h^{-1}}{\pi^2 h}\right)^{1/2}|W(s_i+t)-W(s_i)|<1-\varepsilon\right\}$$
$$\leq(\rho_h+1)\frac{4}{\pi}\exp\left\{-\frac{1}{(1-\varepsilon)^2}\log h^{-1}\right\}=O(h^{\delta}(\log h^{-1})^3),$$

where $\delta=(1-\varepsilon)^{-2}-1>0$.

Now let $h_n=n^{-T}$ where $T>\delta^{-1}$. Then the above inequality implies:

$$\text{(32)}\qquad \varliminf_{n\to\infty}\ \min_{0\leq i\leq\rho_{h_n}}\ \sup_{0<t<h_n}\left(\frac{8\log h_n^{-1}}{\pi^2 h_n}\right)^{1/2}|W(s_i+t)-W(s_i)|\geq 1-\varepsilon\quad\text{a.s.}$$

where

$$s_i=ih_n(\log h_n^{-1})^{-3}.$$

Consider the interval $s_i\leq s\leq s_{i+1}$. Then applying Theorem 1 with $h_n/[\log(1/h_n)]^3$ instead of h, we get

$$\varlimsup_{n\to\infty}\ \max_{0\leq i\leq\rho_{h_n}}\ \sup_{s_i\leq s<s_{i+1}}\left(\frac{(\log h_n^{-1})^2}{2h_n}\right)^{1/2}|W(s)-W(s_i)|\leq 1\quad\text{a.s.}$$

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which, together with (32), implies:

$$(33)\qquad \varliminf_{n\to\infty} \inf_{0<s<1-h_n} \sup_{0<t\le h_n} \left(\frac{8\log h_n^{-1}}{\pi^2 h_n}\right)^{1/2} |W(s+t)-W(s)| \ge 1-\varepsilon. \quad \text{a.s.}$$

Finally, choosing $h_{n+1}\le h<h_n$ and taking into account that $h_n/h_{n+1}\to 1$ $(n\to\infty)$ and that

$$\inf_{0<s<1-h_{n+1}} \sup_{0\le t\le h_{n+1}} \left(\frac{8\log h^{-1}}{\pi^2 h}\right)^{1/2} |W(s+t)-W(s)|$$
$$\le \inf_{0\le s\le 1-h} \sup_{0\le t\le h} \left(\frac{8\log h^{-1}}{\pi^2 h}\right)^{1/2} |W(s+t)-W(s)|$$

we get (31).

Step 2. *For any* $\varepsilon>0$ *we have*

$$(34)\qquad \varlimsup_{h\to 0} \inf_{0\le s\le 1-h} \sup_{0<t\le h} \left(\frac{8\log h^{-1}}{\pi^2 h}\right)^{1/2} |W(s+t)-W(s)| \le 1+\varepsilon \quad \text{a.s.}$$

Proof. Put

$$s_i = ih \quad (i=0,1,2,\ldots[h^{-1}]).$$

Then, by Lemma 3, we have

$$P\left\{\min_{0\le i\le[h^{-1}]} \sup_{0<t\le h} \left(\frac{8\log h^{-1}}{\pi^2 h}\right)^{1/2} |W(s_i+t)-W(s_i)| > 1+\varepsilon\right\}$$
$$\le \left[P\left\{\sup_{0<t\le h} \left(\frac{8\log h^{-1}}{\pi^2 h}\right)^{1/2} |W(t)| > 1+\varepsilon\right\}\right]^{[1/h]+1}$$
$$\le \left[1-\frac{2}{\pi}\exp\left(-\frac{1}{(1+\varepsilon)^2}\log\frac{1}{h}\right)\right]^{[1/h]} = \left(1-\frac{2}{\pi}h^{\delta}\right)^{[h^{-1}]}$$
$$\le \exp\left\{-\frac{2}{\pi}h^{\delta}\left[\frac{1}{h}\right]\right\},$$

where $\delta=(1+\varepsilon)^{-2}<1$. Now let $h_n=n^{-1}$. Then the above inequality implies:

$$(35)\qquad \varlimsup_{n\to\infty} \inf_{0\le s\le 1-h_n} \sup_{0<t\le h_n} \left(\frac{8\log h_n^{-1}}{\pi^2 h_n}\right)^{1/2} |W(s+t)-W(s)|$$
$$\le \varlimsup_{n\to\infty} \min_{0\le i\le[h_n^{-1}]} \sup_{0<t\le h_n} \left(\frac{8\log h_n^{-1}}{\pi^2 h_n}\right)^{1/2} |W(s_i+t)-W(s_i)| \le 1+\varepsilon.$$

Finally, choosing $h_{n+1}\le h<h_n$ and taking into account that $h_n/h_{n+1}\to 1$

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$(n \to \infty)$ and

$$\inf_{0 \le s \le 1-h} \sup_{0<t\le h} \left(\frac{8 \log h^{-1}}{\pi^2 h}\right)^{1/2} |W(s+t)-W(s)|$$
$$\le \inf_{0 \le s \le 1-h} \sup_{0<t\le h_n} \left(\frac{8 \log h^{-1}}{\pi^2 h}\right)^{1/2} |W(s+t)-W(s)|,$$

we get (34). The latter combined with (31) proves Theorem 3.

COROLLARY. Almost every sample path of $W(t)$ has infinite variation on every finite interval I.

Proof. If a sample function $W(t, \omega)$ has bounded variation on I, then it has a derivative existing almost everywhere on I.

5. **How small are the increments of a Wiener process?** The connection between the results of this Section and those of Section 3 is similar to that between Theorem 3 and Theorem 1.

Let

$$\mathscr{I}_1 = \mathscr{I}_1(t) = |W(t+a_T) - W(t)|$$

and

$$\mathscr{I}_2 = \mathscr{I}_2(t) = \sup_{0 \le s \le a_T} |W(t+s) - W(t)|.$$

Now the increment $\mathscr{I}_1$ can be much smaller than the increment $\mathscr{I}_2$. In this section we investigate only the question "How small are the increments $\mathscr{I}_2(t)$ $(0 \le t \le T - a_T)$?" and, as an answer to it, we prove

THEOREM 5 (Csörgő, Révész 1979b). *Let a_T be a non-decreasing function of T for which*

(i) $0 < a_T \le T$ $(T \ge o)$,
(ii) *a_T/T is non-increasing.*

Then

$$\varliminf_{T\to\infty} \gamma_T I(T) \overset{\text{a.s.}}{=} 1, \tag{36}$$

where

$$I(T) = \inf_{0 \le t \le T - a_T} \mathscr{I}_2(t)$$

and

$$\gamma_T = \left(\frac{8(\log T a_T^{-1} + \log\log T)}{\pi^2 a_T}\right)^{1/2}.$$

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If we also have

(iii) $$\frac{\log T/a_T}{\log\log T} \nearrow +\infty$$

then

(37) $$\lim_{T\to\infty} \gamma_T I(T) \overset{\text{a.s.}}{=} 1.$$

The following examples illustrate what the meaning of this theorem is all about.

EXAMPLE 1. $a_T = (8/\pi^2)\log T$. Then $\gamma_T \approx 1$ and our Theorem 5 says that for all T big enough, for any $\varepsilon > 0$ and for almost all ω there exists a $0 \le t = t(T, \varepsilon, \omega) \le T - a_T$ such that

$$\sup_{0\le s\le 8/\pi^2 \log T} |W(t+s) - W(t)| \le 1 + \varepsilon,$$

but, for all $t \in [0, T - a_T]$, with probability 1,

$$\sup_{0\le s\le 8/\pi^2 \log T} |W(t+s) - W(t)| \ge 1 - \varepsilon.$$

At the same time our Theorem 2 stated the existence of a $t \in [0, T - a_T]$ such that, with probability 1,

$$\left| W\left(t + \frac{8}{\pi^2}\log T\right) - W(t) \right| \ge \left(\frac{4}{\pi} - \varepsilon\right)\log T,$$

and hence

$$\sup_{0\le s\le 8/\pi^2 \log T} |W(t+s) - W(t)| \left(\frac{4}{\pi} - \varepsilon\right)\log T$$

but, for all $t \in [0, T - a_T]$,

$$\sup_{0< s\le 8/\pi^2 \log T} |W(t+s) - W(t)| \le \left(\frac{4}{\pi} + \varepsilon\right)\log T.$$

EXAMPLE 2. Let $a_T = T$. Then our Theorem 2 says

$$\varliminf_{T\to\infty} \left(\frac{8\log\log T}{\pi^2 T}\right)^{1/2} \sup_{0\le t\le T} |W(t)| = 1 \quad \text{a.s.},$$

which is the law of iterated logarithm of Chung (1948) when it is applied to the Wiener process.

EXAMPLE 3. Let $a_T = (\log T)^{1/2}$. Then $\gamma_T \approx [(8/\pi^2)\sqrt{(\log T)}]^{1/2}$, and our Theorem 5 says that for all T big enough, for any $\varepsilon > 0$ and for almost all ω

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there exists a $t=t(T,\varepsilon,\omega)\in[0,T-a_T]$ such that

$$\sup_{0\le s\le(\log T)^{1/2}}|W(t+s)-W(t)|\le(1+\varepsilon)\frac{\pi}{\sqrt{8}}(\log T)^{-1/4}.$$

That is to say the interval $[0,T-a_T]$ has a subinterval of length $(\log T)^{1/2}$ where the sample function of the Wiener process is nearly constant; more precisely, the fluctuation from a constant is so small as $(1+\varepsilon)\pi 8^{-1/2}(\log T)^{-1/4}$.

This result is sharp in the sense that for all T big enough, and all $t\in[0,T-a_T]$, we have with probability 1

$$\sup_{0<s<(\log T)^{1/2}}|W(t+s)-W(t)|\ge(1-\varepsilon)\frac{\pi}{\sqrt{8}}(\log T)^{-1/4}.$$

Just like that of Theorem 3, the proof of Theorem 5 is also based on Lemma 3. For details we refer to Csörgő, Révész (1979b and c).

In Section 3 and in the present Section we studied the properties of some increments of a Wiener process. In order to present some further problems, let:

$$\mathscr{I}_1^{(1)}(t)=|W(t+a_T)-W(t)|,$$

$$\mathscr{I}_2^{(1)}(t)=\sup_{0<s\le a_T}|W(t+s)-W(t)|,$$

$$\mathscr{I}_1^{(2)}(t)=W(t+a_T)-W(t),$$

$$\mathscr{I}_2^{(2)}(t)=\sup_{0<s\le a_T}(W(t+s)-W(t)),$$

$$I_i^{(j,1)}(T)=\sup_{0<t\le T-a_T}\mathscr{I}_i^{(j)}(t)\qquad(i=1,2;\ j=1,2),$$

$$I_i^{(j,2)}(T)=\inf_{0<t\le T-a_T}\mathscr{I}_i^{(j)}(t)\qquad(i=1,2;\ j=1,2).$$

Now, our question is to find the normalizing factors $\mu_T(i,j,k)$ and $\nu_T(i,j,k)$ $(i=1,2;\ j=1,2;\ k=1,2)$ for which

$$\overline{\lim_{T\to\infty}}\ \mu_T(i,j,k)I_i^{(j,k)}(T)=1\quad\text{a.s.}$$

and

$$\underline{\lim_{T\to\infty}}\ \nu_T(i,j,k)I_i^{(j,k)}(T)=1\quad\text{a.s.}$$

Of the here mentioned eight lim sup problems four were solved in Section 3 (cf. Theorem 2), namely the cases: $k=1$, $i=1$, 2, $j=1$, 2. One of the eight mentioned lim inf problems is solved in the present section, namely the case of $(k=2,\ i=2,\ j=1)$ (cf. (36) of Theorem 5). For a partial solution of the case $(i=2,\ j=1,\ k=1)$ we refer to Remark 3. Also, for small a_T i.e. when condition (iii) of Theorem 5 holds, the lim sup = the lim inf in the just mentioned completely solved five cases (cf. (37)). Thus, for a_T satisfying (i) and (ii), five and, for a_T satisfying also (iii), ten of the above problems are completely solved.

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References

1. Bachelier, L., *Théorie de la spéculation*, Ann. Sci. Ecole Norm. Sup. **17** (1900), 21–86.

2. Book, S. A. and Shore, T. R., *On large intervals in the Csörgő–Révész theorem on increments of a Wiener process*, Z. Wahrscheinlichkeitstheorie verw. Gebiete, **46** (1978), 1–11.

3. Chan, A. H. C., Csörgő, M. and Révész, P., *Strassen type limit points for moving averages of a Wiener process*, Can. J. Statist. **6** (1978), 57–75.

4. Chung, K. L., *On the maximum partial sums of sequences of independent random variables*, Trans. Amer. Math. Soc. **64** (1948), 205–233.

5. Chung, K. L., *A Course in Probability Theory*, Harcourt, Brace & World, New York, 1968.

6. Csáki, E. and Révész, P., *How big must be the increments of a Wiener process*? (1979), to appear.

7. Csörgő, M. and Révész, P., *How big are the increments of a Wiener process*? Ann. Probability, (1979a), to appear.

8. Csörgő, M. and Révész, P., *How small are the increments of a Wiener process*? Stochastic Processes and their Applications, **8** (1979b), 119–129.

9. Csörgő, M. and Révész, P., *Strong Approximations in Probability and Statistics*, (1979c), Book manuscript in progress.

10. Deo, C. M., *A note on increments of a Wiener process*, Technical Report, University of Ottawa, 1977.

11. Doob, J. L., *Stochastic Processes*, John Wiley, New York, 1953.

12. Einstein, A., *On the movement of small particles suspended in a stationary liquid demanded by the molecular-kinetic theory of heat*, Ann. Physik, **17** (1905), 549–560.

13. Erdős, P. and Rényi, A., *On a new law of large numbers*, J. Analyse Math., **13** (1970), 103–111.

14. Feller, W., *An Introduction to Probability Theory and its Applications I*, John Wiley, New York, 1968.

15. Hirsh, W. M., *A strong law for the maximum cumulative sum of independent random variables*, Comm. Pure Appl. Math., **18** (1965), 109–127.

16. Itô, K., *Stochastic integral*, Proc. Imperial Acad., Tokyo, **20** (1944), 519–524.

17. Itô, K. and Nisio, M., *On the oscillation functions of Gaussian processes*, Math. Scand., **22** (1968), 209–223.

18. Kolmogorov, A. N., *Grundbegriffe der Wahrscheinlichkeitsrechnung*, Springer, Berlin, 1933.

19. Lévy, P., *Théorie de l'Addition des Variables Aléatoires*, Gauthier-Villars, Paris, 1937.

20. Lévy, P., *Processus Stochastiques et Mouvement Brownien*, Gauthier-Villars, Paris, 1948.

21. McKean, H. P. Jr., *Stochastic Integrals*, Academic Press, New York, 1969.

22. Paley, R. E. A. C. and Wiener, N., *Fourier Transforms in the complex domain*, Amer. Math. Soc. Colloq. Publ., Vol. **19,** Amer. Math. Soc., Providence, R.I., 1934.

23. Paley, R. E. A. C., Wiener, N. and Zygmund, A., *Note on random functions*, Math. Z., **37** (1959), 647–668.

24. Smoluchowski, M., Über Brownsche Molekular-bewegung, Ann. Phys., **48** (1915), 110–120.

25. Strassen, V., *An invariance principle for the law of interated logarithm*, Z. Wahrscheinlichkeitstheorie verw. Gebiete, **3,** (1964), 211–226.

26. Wiener, N., *The mean of a functional of arbitrary elements*, Ann. of Math., **22** (1920), 66–72.

27. Wiener, N., *Differential space*, J. Math and Phys., **2** (1923), 132–174.

28. Wiener, N., *Generalized harmonic analysis*, Acta Math. **55** (1930), 117–258.

29. Wiener, N., *I am a mathematician. The later life of a prodigy*, Doubleday, New York, 1956.

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