ON MAXIMAL SUBSYSTEMS OF ROOT SYSTEMS

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1. Introduction. Let g be a semisimple Lie algebra over an algebraically closed field K of characteristic 0. Let h be a Cartan subalgebra of g and let Δ be the root system of g with respect to h.

Definition 1.1. A subset Δ_1 of Δ is called a subsystem of Δ if Δ_1 satisfies the following two conditions:

(i) if $\alpha \in \Delta_1$, then $-\alpha \in \Delta_1$.

(ii) if $\alpha, \beta \in \Delta_1$, and if $\alpha + \beta \in \Delta$, then $\alpha + \beta \in \Delta_1$.

A subsystem Δ_1 is called maximal if Δ_1 is a proper subset of Δ and Δ_1 is not properly contained in any proper subsystem of Δ.

The purpose of this paper is to give a detailed study of the maximal subsystems of Δ. We study maximal subsystems Δ_1 of Δ from the point of view of how Δ extends Δ_1. Some of the results of this paper overlap those of Borel and de Siebenthal **(1)**. Our techniques, however, are different.

In §3 we introduce the concept of the characteristic of a maximal subsystem Δ_1 of Δ. It turns out to be a prime or 0 depending only on Δ and Δ_1. In §4 we give another characterization of the characteristic of a maximal subsystem of Δ. (We apologize to the reader for overworking the word characteristic, but we feel that the word is apt in this case.) Theorem 3.1 is our main theorem on maximal subsystems of connected root systems.

In §5 we sketch a proof of the statement: If Δ_1 and Δ_2 are two maximal subsystems of a connected root system Δ and if Δ_1 and Δ_2 have the same structure and characteristic, then there is a rotation σ of Δ such that $\sigma\Delta_1 = \Delta_2$. The proof of this result depends, to some extent, on case-by-case considerations.

In §6 we give a sketch of how the results of this paper may be used to classify the real forms of a complex semi-simple Lie algebra. The techniques of §6 are similar to those of S. Murakami **(5)**, and were discovered simultaneously in **(7)**.

2. $l - 1$ maximal subsystems of Δ. Let g, h, and Δ be as in §1. Let M be a module over a ring S, and let A be a subset of M. In this paper we shall use the notation $\{A\}_S$ for the submodule of M generated by A over S.

For each $X \in g$, let adX be the linear map of g into g given by $adX \cdot Y = [X, Y]$ ($[\ldots, \ldots]$ is the product in g). Let $\langle X, Y\rangle = \text{trace}\,(adX\, adY)$. Then

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it is well known that $\langle \ldots, \ldots \rangle$ is a non-degenerate bilinear form on $g \times g$ and on $h \times h$. Let h^* be the dual of h. If $\lambda \in h^*$ we define $H_\lambda \in h$ by $\langle H_\lambda, H \rangle = \lambda(H)$ for each $H \in h$. On h^* we define the bilinear form $\langle \ldots, \ldots \rangle$ by setting $\langle \lambda, \mu \rangle = \langle H_\lambda, H_\mu \rangle$, $\lambda, \mu \in h^*$. It is known that $\langle \ldots, \ldots \rangle$ takes on rational values and is positive definite on $\{\Delta\}_Q$ (where Q is the field of rational numbers).

Set $l = \dim \{\Delta\}_Q$ $(=\dim_K h)$.

Definition 2.1. A subsystem Δ_1 of Δ is called l maximal if Δ_1 is maximal and $\dim \{\Delta_1\}_Q = l$.

A subsystem Δ_1 of Δ is called $l - 1$ maximal if $\dim \{\Delta_1\}_Q = l - 1$ and if whenever Δ_1 is properly contained in a subsystem Δ_2 of Δ, then $\dim\{\Delta_2\}_Q = l$.

The following lemma can be found in either **(1)** or **(2)**; we include a proof for the sake of completeness.

LEMMA 2.1. *Let $\beta_1, \ldots, \beta_s$ be elements of Δ and let $\gamma = \beta_1 + \ldots + \beta_s$. If $\gamma \neq 0$ and $\gamma \in \Delta$, then $\gamma - \beta_j \in \Delta$ for some j, $1 \leqslant j \leqslant s$.*

Proof. Assume that $\gamma - \beta_i \notin \Delta$ for $i = 1, \ldots, S$. Since

$$2\langle \gamma, \beta_i \rangle / \langle \beta_i, \beta_i \rangle = p_i - r_i,$$

where p_i is the largest non-negative integer such that $\gamma - p_i \beta_i$ is a root and r_i is the largest non-negative integer such that $\gamma + r_i \beta_i$ is a root, we see that

$$2\langle \gamma, \beta_i \rangle / \langle \beta_i, \beta_i \rangle = -r_i \leqslant 0.$$

But this implies that

$$\langle \gamma, \gamma \rangle = \sum_{i=1}^{S} \langle \gamma, \beta_i \rangle \leqslant 0.$$

Since $\langle \gamma, \gamma \rangle > 0$, we have a contradiction.

Using Lemma 2.1 we prove

LEMMA 2.2. *Let Δ_1 be a subsystem of Δ. If*

$$\beta = \sum_{i=1}^{S} m_i \beta_i,$$

where $\beta_i \in \Delta_1$ and $m_i \in Z$, $i = 1, \ldots, S$ (Z is the ring of integers) and if $\beta \in \Delta$, then $\beta \in \Delta_1$.

Proof. We may assume that $m_i > 0$, $i = 1, \ldots, S$ (if $m_j < 0$ replace β_j by $-\beta_j \in \Delta_1$). We prove the lemma by induction on $m = \sum m_i$. If $m = 1$, then $\beta = \beta_1 \in \Delta_1$. Assume that the result is true for $m = k$. If $m = k + 1$, then we apply Lemma 2.1 to see that $\beta - \beta_j \in \Delta$ for some β_j such that $m_j \geqslant 1$. But

$$\beta - \beta_j = \sum \{m_i - \delta_{ij}\}\beta_i$$

(δ_{ij} is the Kronecker delta). Since $\sum \{m_i - \delta_{ij}\} = k$, $\beta - \beta_j \in \Delta_1$ by the inductive hypothesis. By the definition of a subsystem we know that

$$\beta = (\beta - \beta_j) + \beta_j \in \Delta_1.$$

Definition 2.2. Let Δ_1 be a subsystem of Δ (not necessarily a proper subsystem). A subset π_1 of Δ_1 is called a fundamental system for Δ_1 if

(i) the elements of π_1 are linearly independent,

(ii) if $\alpha \in \Delta_1$, then $\alpha = \sum_{\gamma \in \pi_1} m_\gamma \gamma$ where $m_\gamma \in Z$ for all $\gamma \in \pi_1$ and the m_γ are all either non-negative or non-positive (i.e., $\alpha = \pm \sum_{\gamma \in \pi_1} | m_\gamma | \gamma$).

Dynkin's method of constructing fundamental systems for Δ_1 is as follows: Let $\succ$ be a linear order on $\{\Delta_1\}_Q$. Let π_1 be the set of positive roots in Δ_1 that cannot be written as a sum of two positive roots in Δ_1. Such roots are called simple with respect to $\succ$. π_1 is a fundamental system for Δ_1. (For details see, for example, Jacobson **(4)**.)

The following result gives a relationship between fundamental systems of $l - 1$ maximal subsystems of Δ and fundamental systems of Δ.

LEMMA 2.3. *Let Δ_1 be an $l - 1$ maximal subsystem of Δ. If π_1 is a fundamental system for Δ_1, then there is a fundamental system π for Δ such that $\pi_1 \subset \pi$.*

Proof. Let $\pi_1 = \{\beta_1, \ldots, \beta_{l-1}\}$. Let $\mu \in \{\Delta\}_Q$ be such that μ is linearly independent of π_1. Order $\{\Delta\}_Q$ lexicographically with respect to the ordered basis $\{\mu, \beta_1, \ldots, \beta_{l-1}\}$. We show that with respect to this order on $\{\Delta\}_Q$, $\beta_1, \ldots, \beta_{l-1}$ are simple in Δ.

If $\beta_i = \gamma + \delta$, $\gamma, \delta \in \Delta$, and γ, δ are positive, then

$$\gamma = r\mu + \sum_{j=1}^{l-1} r_j \beta_j, \qquad \delta = s\mu + \sum_{j=1}^{l-1} s_j \beta_j,$$

where $r, s \geqslant 0$. $\gamma + \delta = \beta_i$ implies that $r + s = 0$ and thus $r = s = 0$. Let $\tilde{\Delta}$ be the subsystem of Δ generated by $\{\gamma, \delta, \Delta_1\}$ (i.e., $\{\gamma, \delta, \Delta_1\}_Q \cap \Delta$). $\mathrm{Dim}\{\tilde{\Delta}\}_Q = \dim\{\Delta_1\}_Q = l - 1$ implies that $\tilde{\Delta} = \Delta_1$ by definition of $l - 1$ maximality. Thus $\gamma, \delta \in \Delta_1$. But the elements of π_1 are simple in Δ_1 with respect to the above order restricted to Δ_1. This is a contradiction.

Thus if π is the set of simple roots in Δ with respect to the above order, $\pi_1 \subset \pi$.

Let π be a fundamental system for Δ and let π_1 be any subset of π containing $l - 1$ elements. Let Δ_1 be the root system in Δ generated by π_1. Clearly Δ_1 is $l - 1$ maximal. Furthermore, Lemma 2.3 asserts that every $l - 1$ maximal subsystem is obtained in this manner. Since the Weyl group acts simply transitively on the fundamental systems of Δ, we obtain the immediate

COROLLARY TO LEMMA 2.3. *Let π be a fixed fundamental system for Δ. Let Δ_1 be an $l - 1$ maximal subsystem in Δ. There is an element σ of the Weyl group of Δ such that $\sigma\Delta_1 \cap \pi$ is a fundamental system for $\sigma\Delta_1$.*

3. l maximal subsystems of Δ. Let Δ_1 be a maximal subsystem of Δ. We wish to determine a relationship between a fundamental system of Δ_1 and one of Δ. If $\dim \{\Delta_1\}_Q = l - 1$, then Lemma 2.3 (and its proof) gives a method of

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constructing a fundamental system for Δ that extends a fundamental system for Δ_1. If $\dim \{\Delta_1\}_Q = l$, then the situation is more complicated.

Let us assume for the remainder of this section that Δ_1 is an l maximal subsystem of Δ and that $\pi_1 = \{\beta_1, \ldots, \beta_l\}$ is a fundamental system for Δ_1.

LEMMA 3.1. *Let $\succ$ be the lexicographic order on $\{\Delta\}_Q$ with respect to the ordered basis $\{\beta_1, \ldots, \beta_l\}$. Let γ be the smallest positive root in $\Delta - \Delta_1$. Then*

(1) *γ is simple in Δ with respect to $\succ$;*

(2) *if*

$$\gamma = \sum_{i=r}^{l} m_i \beta_i$$

with $m_r > 0$, then for each $\mu \in \Delta - \Delta_1$,

$$\mu = \sum_{i=1}^{l} n_i \beta_i$$

with $n_r \neq 0$;

(3) *$\beta_{r+1}, \ldots, \beta_l$ are simple in Δ with respect to $\succ$.*

Proof. If γ were not simple in Δ with respect to $\succ$, then $\gamma = \delta + \rho$, $\delta, \rho \succ 0$, $\delta, \rho \in \Delta$. Since $\gamma \in \Delta - \Delta_1$, one of δ or ρ must be in $\Delta - \Delta_1$ (since Δ_1 is a subsystem of Δ). This implies that, say, $\delta \in \Delta - \Delta_1$. But then $\delta \succ 0$ and $\delta \prec \gamma$. This is a contradiction and thus γ is simple with respect to $\succ$.

Assume that β_j is not simple in Δ with respect to $\succ$ for some $j > r$. Then, as above, $\beta_j = \delta + \rho$, $\delta, \rho \succ 0$, $\delta, \rho \in \Delta$. If δ and ρ were in Δ_1, then β_j would not be simple in Δ_1 with respect to $\succ$. Thus at least one of δ, ρ is in $\Delta - \Delta_1$. Assume that $\delta \in \Delta - \Delta_1$. Since $\delta \succ 0$, $\rho \succ 0$, $\delta + \rho = \beta_j$, and $j > r$, we must have $\delta = \sum_{i>r} m_i \beta_i$, $\rho = \sum_{i>r} n_i \beta_i$. In particular, this implies that $\delta \succ 0$ and $\delta \prec \gamma$, which is impossible. Thus β_j is simple in Δ with respect to $\succ$ for $j > r$.

Let $V = \{\gamma, \beta_{r+1}, \ldots, \beta_l\}_Q$ and let $\hat{\Delta} = \Delta \cap V$. Then $\hat{\Delta}$ is a subsystem of Δ and $\{\gamma, \beta_{r+1}, \ldots, \beta_l\}$ is the set of simple roots of $\hat{\Delta}$ with respect to $\succ$ restricted to V. Since $\beta_r \in \hat{\Delta}$ and $\beta_r \succ 0$, we deduce that

$$\beta_r = t\gamma + \textstyle\sum_{i>r} t_i \beta_i,$$

$t > 0$ and $t_i \geqslant 0$, $t_i, t \in Z$, $i = r+1, \ldots, l$. By maximality of Δ_1 in Δ we see that $\{\Delta_1, \gamma\}_Z \supset \Delta$. By the above expression for β_r in terms of $\{\gamma, \beta_{r+1}, \ldots, \beta_l\}$, we see that

$$\{\gamma, \beta_1, \ldots, \beta_{r-1}, \beta_{r+1}, \ldots, \beta_l\}_Z \supset \Delta.$$

If $\alpha \in \Delta - \Delta_1$, then $\alpha = s\gamma + \sum_{i\neq r} s_i \beta_i$, $s_i, s \in Z$, $i = 1, \ldots, r-1, r+1, \ldots, l$. If $s = 0$, then by Lemma 1.2 $\alpha \in \Delta_1$. Thus $s \neq 0$. Using the above expression for β_r we see that

$$\gamma = (1/t)\beta_r - \textstyle\sum_{i>r} (t_i/t)\beta_i.$$

Hence

$$\alpha = (s/t)\beta_r + \textstyle\sum_{i\neq r} r_i \beta_i$$

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(for some $r_i \in Q$) with $s/t \neq 0$. We have thus completed the proof of Lemma 3.1.

Definition 3.1. Let Δ_1 be an l maximal subsystem of Δ. Let $\pi_1 = \{\beta_1, \ldots, \beta_l\}$ be a fundamental system for Δ_1. An element $\beta_r \in \pi_1$ is called deletable if for each $\alpha \in \Delta - \Delta_1$, $\alpha = \sum m_i \beta_i$ and $m_r \neq 0$.

If β_r is deletable in π_1, then let $>$ be the lexicographic order on $\{\Delta\}_Q$ given by the ordered basis $\{\beta_r, \beta_1, \ldots, \beta_{r-1}, \beta_{r+1}, \ldots, \beta_l\}$. Lemma 3.1 tells us that with respect to this order $\beta_1, \ldots, \beta_{r-1}, \beta_{r+1}, \ldots, \beta_l$ are simple and if γ is the smallest positive element in $\Delta - \Delta_1$ with respect to $>$, then $\{\gamma, \beta_1, \ldots, \beta_{r-1}, \beta_{r+1}, \ldots, \beta_l\}$ is a fundamental system for Δ. In the course of the proof of Lemma 3.1 we saw that

$$\gamma = (1/t)\beta_r - \textstyle\sum_{i\neq r} (t_i/t)\beta_i$$

where $t > 0$, $t_i \geqslant 0$, and t, t_i are integers $i = 1, \ldots, r-1, r+1, \ldots, l$. In the following proposition, we shall show that t is actually a prime that depends only on Δ_1 and Δ.

PROPOSITION 3.1. *Let Δ_1 be an l maximal subsystem of Δ. Let $\pi_1 = \{\beta_1, \ldots, \beta_l\}$ be a fundamental system for Δ_1. If β_r is deletable and if $\{\gamma, \pi_1 - \{\beta_r\}\}$ is the fundamental system of Δ constructed above, then there is a prime p such that*

$$\Delta_1 = \{\alpha \in \Delta |\, \alpha = s\gamma + \textstyle\sum_{i\neq r} s_i \beta_i,\ p|s\}.$$

Furthermore, the prime p is the same for each deletable element of π_1 and every fundamental system of Δ_1.

Proof. In the remarks above we see that

$$\gamma = (1/t)\beta_r - \textstyle\sum_{i\neq r} (t_i/t)\beta_i$$

where t is a positive integer and t_i is a non-negative integer for $i \neq r$.

By Lemma 2.2, $\Delta_1 = \{\alpha \in \Delta|\, \alpha = m\gamma + \sum_{i\neq r} m_i \beta_i$ and $t|m\}$. Thus in order to prove the first part of the proposition we need only show that t is a prime. Assume the contrary; then $t = q\cdot s$, where q and s are integers >1.

Let $\Delta^q = \{\alpha \in \Delta|\, \alpha = m\gamma + \sum_{i\neq r} m_i \beta_i$ and $q|m\}$. Then Δ^q is a subsystem of Δ and $\Delta^q \supset \Delta_1$. Thus (by maximality of Δ_1) either $\Delta^q = \Delta_1$ or $\Delta^q = \Delta$. If $\Delta^q = \Delta$ and if $\alpha \in \Delta$, then

$$\alpha = k\cdot q\gamma + \textstyle\sum_{i\neq r} m_i \beta_i$$

with k an integer. Since $\gamma \in \Delta$, we must have $q = 1$, a contradiction. Assume that $\Delta^q = \Delta_1$. Since $q \leqslant t$, there is an element $\alpha \in \Delta$ such that

$$\alpha = q\gamma + \textstyle\sum_{i\neq r} q_i \beta_i.$$

In fact, if we relabel $\gamma, \beta_1, \ldots, \beta_{r-1}, \beta_{r+1}, \ldots, \beta_l$ as $\alpha_1, \ldots, \alpha_l$, then every element $\mu \in \Delta$ is of the form

$$\sum_{j=1}^{k} \alpha_{i_j}, \quad \text{where} \sum_{j=1}^{s} \alpha_{i_j} \in \Delta \text{ for } 1 \leqslant s \leqslant k.$$

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But since $\beta \in \Delta_1$, $\beta = t\gamma + \sum_{i>r} t_i \beta_i$ and also

$$\beta = \sum_{j=1}^{k} \alpha_{i_j} \qquad \text{where } \sum_{j=1}^{s} \alpha_{i_j} \in \Delta,\ 1 \leqslant s \leqslant k;$$

hence there is an element $\alpha \in \Delta$ such that $\alpha = q\gamma + \sum_{i \neq r} q_i \beta_i$. But since $\Delta^q = \Delta_1$, this implies $t|q$. Since we know $q|t$, this implies that $q = t$. This contradicts the definition of q and s and thus t is a prime. Set $t = p$.

In the course of the proof we have actually shown that if $\alpha \in \Delta - \Delta_1$, then $\alpha = \sum (m_i/p)\beta_i$ where the m_i are integers. We use this fact to prove the unicity of p. Assume that β_s is a deletable element of π_1 and that $\{\rho, \pi_1 - \{\beta_s\}\}$ is the simple system constructed as above. Then by the proof of the first part of the proposition

$$\rho = (1/q)\beta_s - \sum_{i \neq s} (s_i/q)\beta_i$$

with s_i non-negative integers and q a prime. Since we also know that $\rho = \sum (m_i/p)\beta_i$, $m_i \in Z$, we must have $(m_s/p) = (1/q)$ and thus $m_s q = p$. Since p and q are primes, this implies that $m_s = 1$ and $q = p$. Thus p depends only on π_1 and not the particular deletable element of π_1.

Let π_1 and π_2 be fundamental systems of Δ_1 and let p_1 and p_2 be the corresponding primes as above. Let σ be the element of the Weyl group of Δ_1 such that $\sigma\pi_1 = \pi_2$. σ is a linear isometry of $\{\Delta\}_Q$ and $\sigma\Delta = \Delta$ implies that σ is in the Weyl group of Δ. In particular, $\sigma(\Delta - \Delta_1) = \Delta - \Delta_1$. Let $\alpha \in \Delta - \Delta_1$. Then $\alpha = \sum (m_i/p_i)\beta_i$, $\sigma\alpha = \sum (m_i/p_1)(\sigma\beta_i)$ where $\pi_1 = \{\beta_1, \ldots, \beta_l\}$. Setting $\pi_2 = \{\gamma_1, \ldots, \gamma_l\}$ and noting that if $\delta \in \Delta - \Delta_1$, then $\delta = \sum (q_i/p_2)\gamma_i$ q_i an integer $i = 1, \ldots, l$. If $\delta = \sigma\alpha$, then we see easily that $p_1 = p_2$. Thus p depends only on Δ_1.

In the course of the proof of Proposition 2.1 we have shown that if $\pi_1 = \{\beta_1, \ldots, \beta_l\}$ is a fundamental system of Δ_1 and if $\alpha \in \Delta - \Delta_1$, then $\alpha = (1/p) \sum m_i \beta_i$ with $m_i \in Z$ and p a prime independent of the particular π_1. That is, if we consider the lattice $\{\Delta_1\}_Z$, then $(1/p)\{\Delta_1\}_Z \supset \Delta$. And p is the only prime such that this inclusion holds. We shall call p the characteristics of Δ_1 in Δ. If Δ_1 is an $l-1$ maximal, maximal subsystem we say that it has characteristic 0 in Δ. In the next section we shall study this concept.

We conclude this section by giving a complete characterization of maximal subsystems of Δ in the case when Δ is connected (i.e. g is simple).

THEOREM 3.1. *Let $\pi = \{\alpha_1, \ldots, \alpha_l\}$ be a fundamental system for Δ, and let $\beta = \sum m_i \alpha_i$ be the largest root in Δ with respect to π. Let p be a prime, and set $\Delta_1 = \{\alpha \in \Delta |\ \alpha = \sum n_i \alpha_i,\ n_1 \equiv 0 \pmod p\}$. Then*:

(1) *Δ_1 is l-maximal if and only if $m_1 \geqslant p$.*

(2) *Let Δ_2 be an arbitrary l-maximal subsystem of Δ with characteristic p; then there is an element σ of the Weyl group of Δ such that $\sigma\Delta_2 = \Delta_1$ (with a possible relabelling of $\{\alpha_1, \ldots, \alpha_l\}$).*

(3) *A maximal subsystem* Δ_2 *is* $l-1$ *maximal if and only if there is a* σ *in the Weyl group of* Δ *such that* $\sigma\Delta_2 = \Delta_1$ *and* $m_1 = 1$ (*after a possible relabelling of* $\{\alpha_1, \ldots, \alpha_l\}$).

Proof. (1) If $p > m_1$, then $\dim \{\Delta_1\}_Q = l-1$ and thus Δ_1 cannot be l-maximal. If $m_1 \geqslant p$, then there is $\alpha \in \Delta_1$, $\alpha = \sum n_i \alpha_i$ and $n_1 = p$. Let γ be the smallest of such α's in Δ_1 (with respect to the given order of π). Then $\{\gamma, \pi - \{\alpha_1\}\}$ is a fundamental system for Δ_1. In fact, we show that with respect to the lexicographic order $\{\alpha_1, \ldots, \alpha_l\}$ on $\{\Delta_1\}_Q$, $\gamma, \alpha_2, \ldots, \alpha_l$ are simple in Δ_1. Clearly $\alpha_2, \ldots, \alpha_l$ are simple. Suppose $\gamma = \delta + \mu$, $\delta > 0$, $\mu > 0$, $\delta, \mu \in \Delta_1$. Then $\delta < \gamma$ and $\mu < \gamma$, contradicting the definition of γ. Thus γ is simple and $\{\gamma, \alpha_2, \ldots, \alpha_l\}$ is a fundamental system for Δ_1. We can now show that Δ_1 is maximal in Δ.

Suppose $\rho \in \Delta - \Delta_1$, $\rho = \sum s_i \alpha_i$, $s_i \in Z$, and $s_1 \not\equiv 0 \pmod p$. If $\delta \in \Delta$ is arbitrary, then $\delta = \sum t_i \alpha_i$, $t_i \in Z$. Since $s_1 \not\equiv 0 \pmod p$, there are integers u and v such that $t_1 = us_1 + vp$. Thus $u\rho + v\gamma = \sum q_i \alpha_i$ with $q_1 = t_1$. Hence

$$\delta = u\rho + v\gamma + \sum_{i=2}^{l} (t_i - q_i)\alpha_i.$$

Thus we have shown that $\{\rho, \Delta_1\}_Z \supset \Delta$. Suppose that $\Delta \supset \tilde{\Delta} \supset \Delta_1$. If If $\tilde{\Delta} - \Delta_1 \neq \emptyset$, then there is a $\rho \in \tilde{\Delta} - \Delta_1$. By the above arguments $\{\rho, \Delta_1\}_Z \supset \Delta$, and thus by Lemma 2.2 we have $\Delta = \tilde{\Delta}$. We have thus concluded a proof of (1).

(2) By Proposition 3.1 we know that there is a fundamental system $\tilde{\pi} = \{\gamma_1, \ldots, \gamma_l\}$ of Δ such that $\Delta_2 = \{\alpha \in \Delta \mid \alpha = \sum r_i \gamma_i, r_1 \equiv 0 \pmod p\}$. Let σ be the element of the Weyl group of Δ such that $\sigma\tilde{\pi} = \pi$. Relabel $\{\alpha_1, \ldots, \alpha_l\}$ such that $\alpha_i = \sigma\gamma_i$ for $i = 1, \ldots, l$. Then $\sigma\Delta_2 = \Delta_1$, as above. Thus concludes the proof of (2).

(3) Suppose Δ_2 is an $l-1$ maximal, maximal subsystem of Δ. By the corollary to Lemma 2.3 there is an element σ of the Weyl group of Δ such that $\sigma\Delta_2 \cap \pi$ is a fundamental system for $\sigma\Delta_2$. Relabel $\{\alpha_1, \ldots, \alpha_l\}$ (if necessary) so that $\sigma\Delta_2 \cap \pi = \{\alpha_2, \ldots, \alpha_l\}$. Suppose that $m_1 > 1$. Then, if we consider the subsystem $\Delta_3 = \{\alpha \in \Delta \mid \alpha = \sum k_i \alpha_i, k_1 \equiv 0 \pmod{m_1}\}$, $\Delta_3 \neq \Delta$ since $\alpha_1 \notin \Delta_3$. And $\Delta_3 \supset \sigma\Delta_2$. Thus $\sigma^{-1}\Delta_3 \supset \Delta_2$ and hence Δ_2 is not maximal. This contradiction implies that $m_1 = 1$.

The proof of the Theorem 3.1 is now complete.

4. The characteristic of an l maximal subsystem of Δ. Let Δ_1 be an l maximal subsystem of Δ with characteristic p. g has a root space decomposition $g = h + \sum_{\alpha \in \Delta} g_\alpha$. Set $g_1 = h + \sum_{\alpha \in \Delta_1} g_\alpha$. Since Δ_1 is l maximal, g_1 is a semisimple Lie algebra over K. Set $P_1 = \sum_{\alpha \in \Delta - \Delta_1} g_\alpha$, $g = g_1 \oplus P_1$ (vector space direct sum).

If $\alpha \in \Delta_1$ and $\beta \in \Delta - \Delta_1$ and if $\alpha + \beta \in \Delta$, then by the definition of subsystem, $\alpha + \beta \in \Delta - \Delta_1$. Thus $[X, Y] \in P_1$ if $X \in g_1$, $Y \in P_1$. Thus we can define a representation (ρ, P_1) of g_1 as follows:

$$\rho(X)\cdot Y = \mathrm{ad}(X)Y, \qquad X \in g_1,\ Y \in P_1.$$

Recall that if $X, Y \in g$, then $\mathrm{ad}(X)Y = [X, Y]$.

We shall show that the number of irreducible components of this representation is exactly $p - 1$. To this end fix $\pi_1 = \{\beta_1, \ldots, \beta_l\}$ a fundamental system for Δ_1. We assume that β_1 is deletable (see Definition 3.1). Order $\{\Delta\}_Q$ lexicographically with respect to the ordered basis $\{\beta_1, \ldots, \beta_l\}$ of $\{\Delta\}_Q$. Let γ be the smallest positive root in $\Delta - \Delta_1$ with respect to this order. Proposition 2.1 asserts that

$$\Delta_1 = \{\alpha \in \Delta |\ \alpha = s\gamma + \textstyle\sum_{i>1} s_i\beta_i,\ s \equiv 0 \bmod (p)\}.$$

Let

$$M^r = \{\alpha \in \Delta |\ \alpha = s\gamma + \textstyle\sum_{i>1} s_i\beta_i,\ s \equiv r \pmod p\}, \qquad r = 1, \ldots, p-1.$$

Then $M^r \neq \emptyset$ and $\Delta_1 \cup M^1 \cup \ldots \cup M^{p-1} = \Delta$. If we set $V^i = \sum_{\alpha \in M^i} g_\alpha$, then $[g_1, V^i] \subset V^i$. If we denote by (ρ_i, V^i) the subrepresentation of g_1 obtained from (ρ, P_1) by restricting ρ to V^i, then we clearly have $P_1 = V^1 \oplus \ldots \oplus V^{p-1}$ (direct sum) and $\rho = \rho_1 \oplus \ldots \oplus \rho_{p-1}$.

Using the above order, we denote by γ^i the largest element of M^i. $\langle\gamma^i, \beta_j\rangle \geqslant 0$ for $j = 1, \ldots, l$. Thus γ^i is a dominant integral element of $\{\Delta_1\}_Q$ with respect to π_1. Let (ρ_i^+, W^i) be the irreducible subrepresentation of (ρ_i, V^i) corresponding to the highest weight γ^i. We shall show that $(\rho_i^+, W^i) = (\rho_i, V^i)$.

Let M_+^r be the set of weights of (ρ_r^+, W^r) (clearly $M_+^r \subset \Delta$).

LEMMA 4.1. *If* $\alpha \in M^r$ *and if* $\alpha = \gamma^r - \sum m_i\beta_i$, *where* $m_i \in Z$, $i = 1, \ldots, l$, *then* $\alpha \in M_+^r$.

Proof. Let F_i be a non-zero root vector in g corresponding to $-\beta_i$, $i = 1, \ldots, l$. Let X_r be non-zero in g_{γ^r}.

We prove Lemma 4.1 by induction on $\sum |m_i| = m$. If $m = 1$, then $\alpha = \gamma^r - \beta_j$ $(\gamma^r + \beta_j \notin \Delta)$. Since $\alpha \in \Delta$, we have $[F_j, X_r] \neq 0$ and $[F_j, X_r] \in g_\alpha$. But $\rho_i^+(F_j)X_r = [F_j, X_r]$ and thus $g_\alpha \subset W^r$. And hence $\alpha \in M_+^r$. Thus the result is true for $m = 1$.

Assume that the result is true for $m = k$. Assume that $m = k + 1$. By Lemma 2.1, either $\alpha - \gamma^r \in \Delta$ or $\alpha + \epsilon_j\beta_j \in \Delta$ for some j, where $|m_j| \geqslant 1$ and $\epsilon_j = m_j/|m_j|$. If $\alpha - \gamma^r \in \Delta$, then $\sum m_j\beta_j \in \Delta$ and thus by Lemma 2.2 we have $\sum m_i\beta_i \in \Delta_1$. Let $\delta = \sum m_i\beta_i$ and let X be a non-zero root vector in g corresponding to $-\delta$. Then, as above, $\rho_r^+(X)X_r \neq 0$; thus $\alpha \in M_+^r$. If $\alpha + \epsilon_j\beta_j \in \Delta$, then

$$\alpha + \epsilon_j\beta_j = \gamma^r - \textstyle\sum \epsilon_i(|m_i| - \delta_{ij})\beta_i.$$

Since $\alpha + \epsilon_j\beta_j \in M^r$ and $\sum (|m_i| - \delta_{ij}) = k$, we see by the inductive hypothesis that $\alpha + \epsilon_j\beta_j \in M_+^r$. Now, as above, $\rho_r^+(Y)Z$ is non-zero in g_α $(Y \in g_{-\epsilon_j\beta_j},\ Y \neq 0$, and $Z \in g_{\alpha+\epsilon_j\beta_j},\ Z \neq 0)$; thus $\alpha \in M_+^r$.

LEMMA 4.2. *Let* $\alpha, \beta \in M^r$. *If* $\alpha = \sum (m_i/p)\beta_i$ *and* $\beta = \sum (n_i/p)\beta_i$ *with* $m_i, n_i \in Z$, $i = 1, \ldots, l$, *then* $m_i \equiv n_i \pmod p$, $i = 1, \ldots, l$.

Proof. $M^r = \{\alpha \in \Delta | \alpha = s\gamma + \sum_{i>1} s_i \beta_i, s \equiv r \pmod p\}$ (notice that s, s_i in the above are all integers of the same sign).

Assume that $\gamma = \sum (q_i/p)\beta_i$ (notice that $q_1 = 1$).

$$\alpha = s\gamma + \sum_{i>1} s_i \beta_i, \qquad s, s_i \in Z, s \equiv r \pmod p,$$

and

$$\beta = t\gamma + \sum_{i>1} t_i \beta_i, \qquad t, t_i \in Z, t \equiv r \pmod p.$$

Hence

$$\alpha = s/p\beta_1 + \sum (sq_i + ps_i)/p\beta_i \quad \text{and} \quad \beta = t/p\beta_1 + \sum (tq_i + pt_i)/p\beta_i.$$

$m_1 = s$, $m_i = sq_i + ps_i$, $n_1 = t$, $n_i = tq_i + pt_i$, $i + 2, \ldots, l$. Clearly $m_1 \equiv n_1 \pmod p$ (by definition of M^r), $m_i \equiv sq_i \pmod p$, and $n_i \equiv tq_i \pmod p$, and hence $m_i \equiv rq_i \pmod p$ and $n_i \equiv rq_i \pmod p$. Thus $m_i \equiv n_i \pmod p$.

Using Lemmas 4.1 and 4.2, we can now prove the main result of this section:

THEOREM 4.1. *Let* p, g_1, P_1, ρ *be as defined in the beginning of this section. Then the number of irreducible components of the representation* (ρ, P_1) *is exactly* $p - 1$.

Proof. Using the above notation we need only to show that $M^r = M_+^r$. Now Lemma 4.2 says that if $\alpha, \beta \in M^r$, then $\alpha - \beta = \sum r_i \beta_i$ with $r_i \in Z$. In particular, this says that for each $\alpha \in M^r$, $\gamma^r - \alpha = \sum r_i \beta_i$ with $r_i \in Z$, $i = 1, \ldots, l$. And this says that $\alpha = \gamma^r - \sum r_i \beta_i$ with $r_i \in Z, i = 1, \ldots, l$. Lemma 4.1 implies that $\alpha \in M_+^r$. Thus $M^r \subset M_+^r$. Since $M_+^r \subset M_r$, we have shown that $M_+^r = M^r$. And the theorem is proved.

Restricting Theorem 4.1 to the case $p = 2$, we have

COROLLARY TO THEOREM 4.1. *If* g_1, ρ, P_1 *are as above and* $p = 2$, *then* (ρ, P_1) *is an irreducible representation of* g_1.

5. Conjugacy theorems. Let g, h, Δ, and l be as in the preceding sections. Let $\tilde{W}(\Delta)$ be the group of all rotations of Δ. Let $W(\Delta)$ be the (normal) subgroup generated by the Weyl reflections of Δ.

Definition 5.1. If Δ_1 and Δ_2 are two subsets of Δ, then they are said to be conjugate if there is a $\sigma \in \tilde{W}(\Delta)$ such that $\sigma\Delta_1 = \Delta_2$.

The main purpose of this section is to sketch a proof of

THEOREM 5.1. *Let* Δ_1 *and* Δ_2 *be maximal subsystems of* Δ *and assume that they have the same structure. Assume that* Δ *is connected* (i.e., g *is simple*). *Then*

(1) *if* Δ_1, Δ_2 *are* l *maximal with the same characteristic, they are conjugate*;

(2) *if* Δ_1, Δ_2 *are* $l - 1$ *maximal, they are conjugate.*

We know of no proof of this theorem that is completely independent of the classification of simple Lie algebras over an algebraically closed field of characteristic 0.

A proof of Theorem 5.1 (1) for the case $p = 2$ and Theorem 5.1 (2) can be found in **(7)**. The proof is essentially a case-by-case check using weak forms of the results of this paper.

The following lemma can be proved using the techniques of Dynkin's classification of complex simple Lie algebras (that is, the use of 1, 2, 3, 4, and 6, pp. 130–131 in Jacobson **(4)**).

LEMMA 5.1. *Assume $\mathfrak{g}$ is simple and $\mathfrak{g}$ is not B_2, G_2, or F_4. Let π be a fundamental system for Δ. If π_1 and π_2 are subsets of π that contain $l - 1$ elements and if π_1 and π_2 have the same diagram, then there is a $\tau \in \tilde{W}(\Delta)$ such that $\tau\pi = \pi$ and $\tau\pi_1 = \pi_2$.*

Using Lemma 5.1 we can prove

PROPOSITION 5.1. *Let Δ_1 and Δ_2 be $l - 1$ maximal subsystems of Δ. If $\Delta \neq B_2$, G_2, or F_4 and if Δ_1 and Δ_2 have the same structure, then Δ_1 and Δ_2 are conjugate.*

Proof. Let π be a fixed fundamental system for Δ. By Lemma 2.3 we know that there are fundamental systems π_1 and π_2 of Δ such that

$$|\Delta_1 \cap \pi_1| = |\Delta_2 \cap \pi_2| = l - 1$$

($|A|$ means the cardinality of A).

Let $\sigma_1, \sigma_2 \in W(\Delta)$ be such that $\sigma_i \pi_i = \pi$, $i = 1, 2$. Then $\sigma_1(\Delta_1 \cap \pi_1)$ and $\sigma_2(\Delta_2 \cap \pi_2)$ are subsets of π containing $l - 1$ elements and having the same Dynkin diagram. Thus there is a $\tau \in \tilde{W}(\Delta)$ such that

$$\tau\sigma_1(\pi_1 \cap \Delta_1) = \sigma_2(\pi_2 \cap \Delta_2).$$

Thus

$$\sigma_2^{-1}\tau\sigma_1(\pi_1 \cap \Delta_1) = \pi_2 \cap \Delta_2.$$

This implies that $\sigma_2^{-1}\tau\sigma_1\Delta_1 = \Delta_2$. And the proposition is proved.

Lemma 5.1 and Proposition 5.1 do not extend to the case $\mathfrak{g} = B_2, G_2$, or F_4. Consider, for example,

$$B_2 = \underset{\alpha_1}{\circ} \Rightarrow \underset{\alpha_2}{\circ}\ .$$

If $\Delta_1 = \{\alpha_1, -\alpha_1\}$, $\Delta_2 = \{\alpha_2, -\alpha_2\}$. Then if there were a $\tau \in \tilde{W}(\Delta)$ such that $\tau\Delta_1 = \Delta_2$, then $\tau\alpha_1 = \pm\alpha_2$, and we would have $\langle\tau\alpha_1, \tau\alpha_1\rangle \neq \langle\alpha_1, \alpha_1\rangle$. Thus τ would not be an isometry, which is impossible.

Using Proposition 5.1 we can prove (2) of Theorem 5.1.

Proof of Theorem 5.1 (2). If $g = G_2$ or F_4, then if Δ_1 is maximal, it is l maximal. If $g = B_2$, then a fundamental system of Δ is

$$\pi = \underset{\alpha_1}{\circ} \Rightarrow \underset{\alpha_2}{\circ}\,.$$

The subset $\{\alpha_1\}$ of π corresponds to a 1-maximal subsystem in Δ that is also maximal in Δ. The subset $\{\alpha_2\}$ does not. Thus every 1-maximal, maximal subsystem of Δ is conjugate to $\{\alpha_1, -\alpha_1\}$. All other cases of Theorem 5.1 (2) are taken care of by Proposition 5.1.

To prove Part (1) of Theorem 5.1 we shall need

LEMMA 5.2. *Let Δ_1 and Δ_2 be two l maximal subsystems of Δ of characteristic p such that Δ_1 and Δ_2 contain respectively $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$, $l - 1$ maximal subsystems of Δ. If $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$ are conjugate, then Δ_1 and Δ_2 are conjugate.*

Proof. Let $\sigma \in \tilde{W}(\Delta)$ be such that $\sigma\tilde{\Delta}_1 = \tilde{\Delta}_2$. Then $\sigma\Delta_1 \cap \Delta_2 \supset \tilde{\Delta}_2$. Let π be a fundamental system of Δ such that $|\pi \cap \Delta_2| = l - 1$. Let $\pi = \{\alpha_1, \ldots, \alpha_l\}$ and $\pi \cap \tilde{\Delta}_2 = \{\alpha_2, \ldots, \alpha_l\}$. By the definition of characteristic,

$$\sigma\Delta_1 = \{\alpha \in \Delta | \ \alpha = \sum s_i \alpha_i, s_1 \equiv 0 \ (\mathrm{mod}\ p)\},$$
$$\Delta_2 = \{\alpha \in \Delta | \ \alpha = \sum s_i \alpha_i, s_1 \equiv 0 \ (\mathrm{mod}\ p)\}.$$

Thus $\Delta_1 = \Delta_2$, which was to be proved.

COROLLARY TO LEMMA 5.2. *If $g \neq B_2, G_2$, or F_4 and if Δ_1 and Δ_2 are l maximal subsystems of characteristic p in Δ such that $\Delta_1 \supset \tilde{\Delta}_1$, $\Delta_2 \supset \tilde{\Delta}_2$ where $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$ are $l - 1$ maximal subsystems of Δ with the same structure, then Δ_1 and Δ_2 are conjugate.*

In order to prove Theorem 5.1 (1) we must still study B_2, G_2, and F_4 individually. The proofs for B_2 and G_2 are simple (using the same technique as the proof of Theorem 5.1 (2) for B_2). For F_4 the proof is slightly more difficult.

In the general case for $g \neq B_2$, G_2, or F_4 the proof goes as follows.

Let π_1 and π_2 be fundamental systems for Δ_1 and Δ_2. (We assume that Δ_1 and Δ_2 have the same structure and characteristic.) Write out the diagrams of π_1 and π_2. If there is a deletable element (see §2) $\beta_i \in \pi_i$, $i = 1, 2$, such that $\{\pi_1 - \beta_1\}$ and $\{\pi_2 - \beta_2\}$ have the same structure, then by the corollary to Lemma 6.2 Δ_1 and Δ_2 are conjugate. (This condition is in fact necessary and sufficient.) We are now left with a case-by-case determination which is a rather straightforward computation.

The case $p = 2$ of Theorem 5.1 has interesting applications (see §6) and in this case the proof is much simpler. We can use the corollary to Theorem 4.1 and study the irreducible representations as in §4 to give a proof.

LEMMA 5.2. *Suppose Δ_1 and Δ_2 are l maximal subsystems of Δ of characteristic* 2 *with the same structure. Suppose that π_1 and π_2 are fundamental systems of Δ_1 and Δ_2 respectively and that β and γ are the respective highest weights of the representa-*

tions defined in §4 *corresponding to* Δ_1 *and* Δ_2 *with respect to* π_1 *and* π_2. *Finally suppose that there is an ordering of* π_1 (*of* π_2) *in a Dynkin diagram such that* $\pi_1 = \{\beta_1, \ldots, \beta_l\}$ $(\pi_2 = \{\gamma_l, \ldots, \gamma_l\})$ *and* $\beta = \sum (k_i/2)\beta_i$, $\gamma = \sum (k_i/2)\gamma_i$. *Then* Δ_1 *and* Δ_2 *are conjugate.* (The diagrams above differ only in labels.)

Proof. Let σ be the linear map of $\{\Delta\}_Q$ to $\{\Delta\}_Q$ such that $\sigma\beta_i = \gamma_i$, $i = 1, \ldots, l$. Clearly σ is an isometry with respect to $\langle \ldots, \ldots \rangle$.

If we can show that $\sigma\Delta = \Delta$, then Δ_1 and Δ_2 will be conjugate. Every element of $\Delta - \Delta_1$ can be written in the form

$$\alpha = \beta - \beta_{i_1} - \beta_{i_2} - \ldots - \beta_{i_r}$$

where $\beta - \beta_{i_1} - \ldots - \beta_{i_k} \in \Delta - \Delta_1$ for $1 \leqslant k \leqslant r$ and $\beta_{i_j} \in \pi_1, j = 1, \ldots, r$. Let us say α is in the rth level. We shall show that $\sigma(\Delta - \Delta_1) \subset \Delta$ by induction on r.

If α is in the first level, then $\alpha = \beta - \beta_{i_1}$. Thus $\langle \beta, \beta_{i_1} \rangle > 0$ $(\beta + \beta_{i_1} \notin \Delta)$ and hence $\sigma\alpha = \gamma - \gamma_{i_1}$. But $\langle \gamma, \gamma_{i_1} \rangle = \langle \beta, \beta_{i_1} \rangle > 0$. Hence $\gamma - \gamma_{i_1} \in \Delta$. And hence $\sigma\alpha \in \Delta$. Suppose true for all α with level less than r. We shall show that the result is true for r. Let α be on the rth level. Then $\alpha + \beta_j \in \Delta + \Delta_1$ for some $1 \leqslant j \leqslant l$ and $\alpha + \beta_j$ is on the $(r-1)$th level. Consider the β_j string containing $\alpha + \beta_j$. That is

$$(\alpha + \beta_j) - k\beta_j, \ldots, \alpha + \beta_j, (\alpha + \beta_j) + \beta_j, \ldots, (\alpha + \beta_j) + s\beta_j.$$

Then $2\langle \alpha + \beta_j, \beta_j \rangle / \langle \beta_j, \beta_j \rangle = k - s$. But σ is an isometry and $\sigma((\alpha + \beta_j) + t\beta_j) \in \Delta$ for $t = 0, 1, \ldots, s$ by the inductive hypothesis. Thus

$$\sigma(\alpha + \beta_j) - \sigma\beta_j, \ldots, \sigma(\alpha + \beta_j) - k\sigma\beta_j \in \Delta$$

and in particular $\sigma\alpha \in \Delta$. Thus the lemma is proved.

We complete the proof of Theorem 5.1 in the special case of characteristic 2 by including a table of the possible conjugacy classes of l maximal subsystems of the root system of each simple Lie algebra, and the corresponding highest weight. In Table I, the column g corresponds to the class of the simple Lie algebra g under consideration. The column π corresponds to a fundamental system for the root system under consideration. The next columns refer to the subsystem

$$\Delta_k = \{\alpha \in \Delta | \ \alpha = \sum m_i \alpha_i \text{ where } m_k \equiv 0 \ (\text{mod } 2)\}.$$

If in the column β_k there is a " $-$," then we know that Δ_k is not l maximal. If there is an expression that corresponds to a root in Δ in the column β_k, then this means that $\{\beta_k, \pi - \{\alpha_k\}\}$ is a fundamental system for Δ_k. The column π_k gives a Dynkin diagram $\pi_k = \{\beta_k, \pi - \{\alpha_k\}\}$. The column g_k corresponds to the subalgebra of g that corresponds to Δ_k as defined at the beginning of §4. The column γ_k is a list of the highest weights of the representations corresponding to g_k in g defined in §4. We only include an expression for γ_k in the case when Δ_k is l maximal.

TABLE I

g	π	β_k	π_k	γ_k	g_k
A_l	$\alpha_1 - \alpha_2 \cdots - \alpha_l$		$\alpha_1 - \alpha_2 \cdots - \alpha_{k-1}$ $\alpha_{k+1} - \alpha_{k+2} \cdots - \alpha_l$		A_{k-1}
B_l	$\alpha_1 - \alpha_2 \cdots \alpha_{l-1} \Leftarrow \alpha_l$	$k = 1$	$\alpha_2 - \alpha_3 \cdots \alpha_{l-1} \Leftarrow \alpha_l$		B_{l-1}
		$k > 1$ $\alpha_{k-1} + 2\sum_{i=k}^{l} \alpha_i$	β_k $\alpha_1 - \alpha_2 \cdots \alpha_{k-2} - \alpha_{k-1}$ $\alpha_{k+1} - \cdots \Leftarrow \alpha_l$	$\frac{1}{2}\beta_k + \frac{1}{2}\alpha_{k-1} + \sum_{i \neq k, k-1} \alpha_i$	$D_k \times B_{l-k}$
C_l	$\alpha_1 - \alpha_2 \cdots \alpha_{l-1} \Rightarrow \alpha_l$	$k = l$	$\alpha_1 - \alpha_2 \cdots \alpha_{l-1}$		A_{l-1}
		$k < l$ $2\sum_{i=k}^{l-1} \alpha_i + \alpha_l$	$\alpha_1 - \alpha_2 \cdots \alpha_{k-1} \Rightarrow \beta_k$ $\alpha_{k+1} - \cdots \alpha_{l-1} \Rightarrow \alpha_l$	$\frac{1}{2}(\beta_k + \alpha_l) + \sum_{i \neq k, l} \alpha_i$	$C_k \times C_{l-1}$
D_l	α_l $\alpha_1 - \alpha_2 \cdots \alpha_{l-2} - \alpha_{l-1}$	$k = l$	$\alpha_1 - \alpha_2 - \cdots - \alpha_{l-1}$		A_{l-1}
		$k = 1$	$\alpha_2 - \alpha_3 - \cdots - \alpha_{l-2} - \alpha_{l-1}$ α_l		D_{l-1}
		$1 < k \leqslant l - 2$ $\alpha_{k-1} + 2\sum_{k}^{l-2} \alpha_i + \alpha_{l-1} + \alpha_l$	β_k α_l $\alpha_1 - \alpha_2 \cdots \alpha_{k-2} - \alpha_{k-1}$ $\alpha_{k-1} - \cdots - \alpha_{l-2} - \alpha_{l-1}$	$\frac{1}{2}(\beta_k + \alpha_{k-1} + \alpha_l + \alpha_{l-1}) + \sum_{i \neq k, k-1, l, l-1} \alpha_i$	$D_k \times D_{l-k}$
E_6	α_6 α_3 α_2 α_4 α_1 α_5	$k = 1$	α_6 $\alpha_2 - \alpha_3 - \alpha_4 - \alpha_5$		D_5
		$k = 2$ $\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4 + \alpha_6$	$\beta_2 - \alpha_5 - \alpha_4 - \alpha_3 - \alpha_6$ α_1	$\frac{1}{2}\beta_2 + \alpha_5 + \frac{3}{2}\alpha_4 + \alpha_3 + \frac{1}{2}\alpha_6 + \frac{1}{2}\alpha_1$	$A_5 \times A_1$
		$k = 3$ $\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_6$	$\alpha_2 - \alpha_1 - \beta_3 - \alpha_5 - \alpha_4$ α_6	$\frac{1}{2}\alpha_2 + \alpha_1 + \frac{3}{2}\beta_3 + \alpha_5 + \frac{1}{2}\alpha_4 + \frac{1}{2}\alpha_6$	$A_5 \times A_1$
		$k = 6$ $\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6$	$\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5$ β_6	$\frac{1}{2}\alpha_1 + \alpha_2 + \frac{3}{2}\alpha_3 + \alpha_4 + \frac{1}{2}\alpha_5 + \frac{1}{2}\beta_6$	$A_5 \times A_1$
E_7	α_7 α_3 α_2 α_4 α_1 α_5 α_6	$k = 1$ $2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + 2\alpha_7$	α_7 $\alpha_6 - \alpha_5 - \alpha_4 - \alpha_3 - \alpha_2$ β_1	$\frac{1}{2}\alpha_6 + \alpha_5 + \frac{3}{2}\alpha_4 + 2\alpha_3 + \frac{3}{2}\alpha_2 + \alpha_7 + \frac{1}{2}\beta_1$	$D_6 \times A_1$
		$k = 2$ $\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_7$	α_6 $\alpha_7 - \alpha_3 - \alpha_4 - \alpha_5$ α_1 β_2	$\frac{1}{2}\alpha_7 + \alpha_3 + \frac{3}{2}\alpha_4 + 2\alpha_5 + \frac{3}{2}\beta_2 + \alpha_6 + \frac{1}{2}\alpha_1$	$D_6 \times A_1$
		$k = 3$ $\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_7$	α_6 $\alpha_2 - \alpha_1 - \beta_3 - \alpha_5$ α_7 α_4	$\frac{1}{2}\alpha_2 + \alpha_1 + \frac{3}{2}\beta_3 + 2\alpha_5 + \frac{3}{2}\alpha_7 + \alpha_6 + \frac{1}{2}\alpha_7$	$D_6 \times A_1$
		$k = 4$ $\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_6 + \alpha_7$	$\alpha_5 - \alpha_6 - \beta_4 - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_7$	$\frac{1}{2}\alpha_5 + \alpha_6 + \frac{3}{2}\beta_4 + 2\alpha_1 + \frac{3}{2}\alpha_2 + \alpha_3 + \frac{1}{2}\alpha_7$	A_7
		$k = 5$ $\alpha_2 + 2(\alpha_3 + \alpha_4 + \alpha_5) + \alpha_6 + \alpha_7$	α_4 $\beta_5 - \alpha_1 - \alpha_2 - \alpha_3$ α_6 α_7	$\frac{1}{2}\beta_5 + \alpha_1 + \frac{3}{2}\alpha_2 + 2\alpha_3 + \frac{3}{2}\alpha_7 + \alpha_4 + \frac{1}{2}\alpha_6$	$D_6 \times A_1$
		$k = 6$	α_3 $\alpha_1 - \alpha_2 - \alpha_4 - \alpha_5$ α_7		E_6
		$k = 7$ $\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_7$	$\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5 - \alpha_6 - \beta_7$	$\frac{1}{2}\alpha_1 + \alpha_2 + \frac{3}{2}\alpha_3 + 2\alpha_4 + \frac{3}{2}\alpha_5 + \alpha_6 + \frac{1}{2}\beta_7$	A_7

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TABLE I (*Continued*)

g	π	β_k	π_k	γ_k	g_k
E_8	α_8; α_5; α_4 α_6; α_3 α_7; α_2; α_1	$k=1$ $2\alpha_1+3\alpha_2+4\alpha_3+5\alpha_4+6\alpha_5+4\alpha_6+2\alpha_7$	α_8 $\alpha_2\ \alpha_3\ \alpha_4\ \alpha_5\ \alpha_6\ \alpha_7\ \beta_1$	$\frac{3}{2}\alpha_2+2\alpha_3+\frac{5}{2}\alpha_4+3\alpha_5+2\alpha_6+\alpha_7+\frac{3}{2}\alpha_8+\frac{1}{2}\beta_1$	$E_7\times A_1$
		$k=2$ $\alpha_1+2\sum_{i=2}^{5}\alpha_i+\alpha_6+\alpha_8$	α_8 $\beta_2\ \alpha_7\ \alpha_6\ \alpha_5\ \alpha_4\ \alpha_3\ \alpha_1$	$\frac{3}{2}\beta_2+2\alpha_7+\frac{5}{2}\alpha_6+3\alpha_5+2\alpha_4+\alpha_3+\frac{3}{2}\alpha_3+\frac{1}{2}\alpha_1$	$E_7\times A_1$
		$k=3$ $\alpha_2+2\sum_{i=3}^{5}\alpha_i+\alpha_6+\alpha_8$	α_8 $\alpha_2\ \alpha_1\ \beta_3\ \alpha_7\ \alpha_6\ \alpha_5\ \alpha_4$	$\frac{1}{2}\alpha_2+\alpha_1+\frac{3}{2}\beta_3+2\alpha_7+\frac{5}{2}\alpha_6+3\alpha_5+2\alpha_4+\frac{3}{2}\alpha_8$	D_8
		$k=4$ $\alpha_3+2\alpha_4+2\alpha_5+\alpha_6+\alpha_3.$	α_3 $\alpha_8\ \alpha_5\ \alpha_6\ \alpha_7\ \beta_4\ \alpha_2\ \alpha_1$	$\frac{1}{2}\alpha_8+\alpha_5+\frac{3}{2}\alpha_6+2\alpha_7+\frac{5}{2}\beta_4+3\alpha_2+2\alpha_1+\frac{3}{2}\alpha_3$	D_8
		$k=5$ $\alpha_4+2\alpha_5+\alpha_6+\alpha_8$	α_4 $\alpha_6\ \alpha_7\ \beta_5\ \alpha_3\ \alpha_2\ \alpha_1\ \alpha_8$	$\frac{3}{2}\alpha_6+2\alpha_7+\frac{5}{2}\beta_5+3\alpha_3+2\alpha_2+\alpha_1+\frac{3}{2}\alpha_4+\frac{1}{2}\alpha_8$	$E_7\times A_1$
		$k=6$ $\alpha_7+2\alpha_6+2\alpha_5+\alpha_4+\alpha_3$	β_6 $\alpha_8\ \alpha_5\ \alpha_4\ \alpha_3\ \alpha_2\ \alpha_1\ \alpha_7$	$\frac{3}{2}\alpha_8+2\alpha_5+\frac{5}{2}\alpha_4+3\alpha_3+2\alpha_2+\alpha_1+\frac{3}{2}\beta_6+\frac{1}{2}\alpha_7$	$E_7\times A_1$
		$k=7$ $\alpha_2+2(\alpha_3+\alpha_7+\alpha_8)+3(\alpha_5+\alpha_6)+4\alpha_5$	α_6 $\beta_7\ \alpha_1\ \alpha_2\ \alpha_3\ \alpha_4\ \alpha_5\ \alpha_8$	$\frac{1}{2}\beta_7+\alpha_1+\frac{3}{2}\alpha_2+\frac{5}{2}\alpha_4+3\alpha_5+3\alpha_8+\frac{3}{2}\alpha_6$	D_8
		$k=8$ $\alpha_7+2(\alpha_6+\alpha_4+\alpha_8)+3\alpha_5+\alpha_3$	β_8 $\alpha_7\ \alpha_6\ \alpha_5\ \alpha_4\ \alpha_3\ \alpha_2\ \alpha_1$	$\frac{1}{2}\alpha_7+\alpha_6+\frac{3}{2}\alpha_5+2\alpha_4+\frac{5}{2}\alpha_3+2\alpha_2+2\alpha_1+\frac{3}{2}\beta_8$	D_8
F_4	$\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4$	$k=1$ $2\alpha_1+2\alpha_2+\alpha_3$	$\beta_1\ \alpha_4\ \alpha_3\ \alpha_2$	$\frac{1}{2}\beta_1+\alpha_4+\frac{3}{2}\alpha_3+2\alpha_2$	B_4
		$k=2$ $2\alpha_2+\alpha_3$	$\alpha_3\ \alpha_4\ \beta_2\ \alpha_1$	$\frac{1}{2}\alpha_3+\alpha_4+\frac{3}{2}\beta_2+2\alpha_1$	B_4
		$k=3$ $2\alpha_2+2\alpha_3+\alpha_4$	$\alpha_2\ \alpha_1\ \beta_3\ \alpha_4$	$\alpha_2+2\alpha_1+\frac{3}{2}\beta_2+\frac{1}{2}\alpha_4$	$C_3\times A_1$
		$k=4$ $2\alpha_1+4\alpha_2+3\alpha_3+2\alpha_4$	$\alpha_1\ \alpha_2\ \alpha_3\ \beta_4$	$\alpha_1+2\alpha_2+\frac{3}{2}\alpha_3+\frac{1}{2}\beta_4$	$C_3\times A_1$
G_2	$\alpha_1\ \alpha_2$	$k=1$ $2\alpha_1+3\alpha_2$	$\alpha_2\ \beta_1$	$\frac{3}{2}\alpha_2+\frac{1}{2}\beta_1$	$A_1\times A_1$
		$k=2$ $\alpha_1+2\alpha_2$	$\beta_2\ \alpha_1$	$\frac{3}{2}\beta_2+\frac{1}{2}\alpha_1$	$A_1\times A_1$

In the case A_l there are no l maximal subsystems. In the case B_l all of the classes l maximal subsystems with characteristic 2 as shown in Table I are distinct. Thus Theorem 3.1 (2) contains Theorem 5.1 (b) in the case B_l. In case C_l, Δ_k has the same structure as Δ_r if and only if $r+k=l$. And in this case Lemma 5.3 proves Theorem 5.1 (b). In the case D_l we have the same situation as in C_l. In the case F_4 we see that Δ_1 and Δ_2 are conjugate and Δ_3 and Δ_4 are conjugate by Lemma 5.3. In case G_2 we have Δ_1 conjugate to Δ_2. The tables and Lemma 5.3 complete the proof for E_l, $l=6,7,8$. We have thus proved Theorem 5.1 completely for $l-1$ maximal, maximal systems and l maximal systems of characteristic 2.

6. On the classification of real simple Lie algebras. The results stated without proof in this section can be found in **(7)** and for the most part in **(5)**. Let g be a complex simple Lie algebra, h a Cartan algebra with dim $h=l$, u a compact form of g such that $u\cap h$ is maximal abelian in u, Δ the root system of g with respect to h, and π a fundamental system of Δ.

A fundamental result of E. Cartan states that up to isomorphism every real form g_0 of g can be found as follows:

Let A be an involutive automorphism of u (i.e. $A^2 = 1$). Let $u_A{}^+ = \{X \in u | AX = X\}$ and $u_A{}^- = \{X \in u | AX = -X\}$. Then

$$g_A = u_A{}^+ + \sqrt{(-1)}.u_A{}^- = g_0.$$

Furthermore, if A and B are involutive automorphisms of u, then g_A and g_B are isomorphic if and only if there is an automorphism C of u such that $C^{-1}AC = B$.

An algebraic proof of this result can be found in **(7)**.

Definition 6.1. Let A and B be involutive automorphisms of u. A and B are said to be equivalent (written $A \equiv B$) if there is an automorphism C of u such that $CAC^{-1} = B$.

Thus, to classify all real forms of g up to isomorphism we need only classify all involutive automorphisms of u up to equivalence.

Let $I(\pi)$ be the set of rotations σ of Δ such that $\sigma^2 = 1$ and $\sigma\pi = \pi$. Let $\pi = \{\alpha_1, \ldots, \alpha_l\}$ and let X_i be a non-zero element of the root space with respect to h for α_i, Y_i be a non-zero element of the root space for $-\alpha_i$. For each $\tau \in I(\pi)$ let $i \to i'$ denote the corresponding permutation of $\{1, \ldots, l\}$. Define $TX_i = X_{i'}$, $TY_i = Y_{i'}$, $i = 1, \ldots, l$. Since $X_1, \ldots, X_l$, $Y_1, \ldots, Y_l$ generate g, T defines an involutive automorphism of g and of u. We call T the canonical automorphism of u associated with τ. For simplicity we use the same notation for $I(\pi)$ and the canonical automorphisms associated with $I(\pi)$.

THEOREM 6.1 (Gantmacher). *Every involutive automorphism of u is equivalent to an automorphism of the form $T\exp(\sqrt{(-1)}.\pi\,\mathrm{ad}(H))$ where $T \in I(\pi)$, $H \in \sqrt{(-1)}.(h \cap u)$ and $TH = H$.*

For a proof of Theorem 6.1, see **(7)**.

If $T\exp(\sqrt{(-1)}.\pi\,\mathrm{ad}(H))$ is involutive and $TH = H$, $T \in I(\pi)$, then since $T^2 = 1$, we must have $\exp(\sqrt{(-1)}.\pi\,\mathrm{ad}(H))$ is involutive. Since $H \in \sqrt{(-1)}.(h \cap u)$, this implies that $\alpha(H)$ is an integer for each $\alpha \in \Delta$. Set

$$\Delta_H{}^+ = \{\alpha \in \Delta | \alpha(H) \equiv 0 \pmod 2\}$$

and

$$\Delta_H{}^- = \Delta - \Delta_H{}^+ = \{\alpha \in \Delta | \alpha(H) \equiv 1 \pmod 2\}.$$

Since $TH = H$, we have $\tau\Delta_H{}^+ = \Delta_H{}^+$.

PROPOSITION 6.1. *Let T, H, and $\Delta_H{}^+$ be as above. Then $\Delta_H{}^+$ is a maximal subsystem of Δ. If $\Delta_H{}^+$ is l maximal, then it has characteristic 2. Conversely, every l maximal, subsystem of characteristic 2 or $l - 1$ maximal subsystem Δ_1 of Δ such that $\tau\Delta_1 = \Delta_1$ corresponds to an involutive automorphism $T\exp(\sqrt{(-1)}.\pi$ $\mathrm{ad}(H))$, where $T \in I(\pi)$, $TH = H$, and $H \in \sqrt{(-1)}.(h \cap u)$.*

To prove Proposition 6.1 we use the following lemma of Hano and Matsushima **(3)**.

LEMMA. *Suppose* Δ_1 *and* Δ_2 *are subsystems of* Δ *and suppose that* $\Delta_1 \cup \Delta_2 = \Delta$. *Then one of* Δ_1 or Δ_2 *is* Δ.

Proof of Proposition 6.1. We first show that Δ_H^+ is maximal in Δ. Let $\beta \in \Delta - \Delta_H^+$. Set $\tilde{\Delta} = \{\Delta_H^+, \beta\}_Z \cap \Delta$. Clearly $\tilde{\Delta}$ is a subsystem of Δ. Furthermore, if we show that $\tilde{\Delta} = \Delta$ for arbitrary β, Lemma 2.2 tells us that Δ_H^+ is maximal in Δ.

To this end we consider $\Delta - \tilde{\Delta} = \Delta'$. Notice that $\Delta' \subset \Delta_H^-$. Furthermore, we note that if $\gamma, \alpha \in \Delta_H^-$ and $\alpha + \gamma \in \Delta$, then $\alpha + \gamma \in \Delta_H^+$. We shall show that $\Delta' \cup \Delta_H^+$ is a subsystem of Δ. If $\alpha \in \Delta'$, clearly $-\alpha \in \Delta'$. If $\alpha \in \Delta'$ and $\gamma \in \Delta_H^+$, suppose that $\alpha + \gamma \in \tilde{\Delta}$. Then

$$\alpha + \gamma = \sum m_i \beta_i + m\beta, \qquad m_i \in Z,\ m \in Z.$$

But then $\alpha = -\gamma + \sum m_i \beta_i + m\beta$, which would then say that $a \in \tilde{\Delta}$, contrary to the definition of Δ'. Thus $\alpha + \gamma \in \Delta'$. Hence $\Delta_H^+ \cup \Delta'$ is a subsystem of Δ. But $\Delta = (\Delta_H^+ \cup \Delta') \cup \tilde{\Delta}$. And since $\beta \notin \Delta_H^+ \cup \Delta'$, Lemma 6.1 implies that $\tilde{\Delta} = \Delta$.

We now show that if Δ_H^+ is l maximal, it has characteristic 2 in Δ.

By Theorem 3.1 (2) there is a fundamental system for Δ, $\{\gamma_1, \ldots, \gamma_l\}$ such that $\gamma_2, \ldots, \gamma_l \in \Delta_H^+$. $\alpha \in \Delta_H^+$ if and only if $\alpha(H) \equiv 0 \pmod 2$. $\alpha \in \Delta$ implies that $\alpha = \sum m_i \gamma_i$, where the m_i's are all integers of the same sign. $\alpha(H) = \sum m_i \gamma_i(H)$. Since $\gamma_i(H) \equiv 0 \pmod 2$ for $i = 2, \ldots, l$, we have $\alpha(H) \equiv m_1 \gamma_1(H) \pmod 2$. $\gamma_1 \in \Delta - \Delta_H^+$; hence $\gamma_1(H) \equiv 1 \pmod 2$. Thus $\alpha \in \Delta_H^+$ if and only if $m_1 \equiv 0 \pmod 2$. Thus

$$\Delta_H^+ = \{\alpha \in \Delta \mid \alpha = \sum m_i \gamma_i,\ m_1 \equiv 0 \pmod 2\}.$$

But this says that Δ_H^+ has characteristic 2 or 0 in Δ.

Let Δ_1 be a maximal subsystem of Δ with characteristic 2 or 0 such that $\tau\Delta_1 = \Delta_1$. Let $\pi_1 = (\Delta - \Delta_1) \cap \pi$. Clearly $\pi_1 \neq \emptyset$, since if $\pi_1 = \emptyset$, then $\Delta_1 = \Delta$. Let us suppose that $\pi_1 = \{\alpha_1. \ldots, \alpha_s\}$. Let

$$\tilde{\Delta} = \left\{\alpha \in \Delta \,\middle|\, \alpha = \sum m_i \alpha_i,\ \sum_{i=1}^{s} m_i \equiv 0 \pmod 2\right\}.$$

Let $H_1, \ldots, H_l$ be the elements of h such that $\alpha_i(H_j) = \delta_{ij}$ (δ_{ij} the Kronecker delta). Since $\tau\Delta_1 = \Delta_1$, $\tau(\Delta - \Delta_1) = \Delta - \Delta_1$, and $\tau\pi = \pi$, this implies that $\tau\pi_1 = \pi_1$. Hence if we set $H = H_1 + \ldots + H_s$, we have $TH = H$. Furthermore, $\tilde{\Delta} = \Delta_H^+$. And $\tau\Delta_H^+ = \Delta_H^+$. If we show that $\tilde{\Delta} = \Delta_1$ we shall have completed the proof of Proposition 6.1. Let $\{\gamma_1, \ldots, \gamma_l\}$ be a fundamental system for Δ such that

$$\Delta_1 = \{\alpha \in \Delta \mid \alpha = \sum m_i \gamma_i,\ m_1 \equiv 0 \pmod 2\}.$$

$\{\gamma_1, \ldots, \gamma_l\}$ exists by Theorem 3.1 (2). Let $\tilde{H}_1, \ldots, \tilde{H}_l$ in h be defined by $\gamma_i(\tilde{H}_j) = \delta_{ij}$. Then

$$\Delta_1 = \{\alpha \in \Delta |\ \alpha(\tilde{H}_1) \equiv 0 \pmod 2\}.$$

Now $\alpha_i(\tilde{H}_1) \equiv 1 \pmod 2$, $i = 1, \ldots, s$. Hence if $\alpha \in \Delta_1$, then $\alpha = \sum m_i \alpha_i$ and

$$\alpha(\tilde{H}_1) = \sum m_i \alpha_i(\tilde{H}_1) \equiv \sum_{i=1}^{s} m_i \alpha_i(\tilde{H}_1) \equiv \sum_{i=1}^{s} m_i \pmod 2.$$

Since $\alpha(\tilde{H}_1) \equiv 0 \pmod 2$, this implies that $\Delta_1 \subset \tilde{\Delta}$ since $\tilde{\Delta} \neq \Delta$. This says that $\Delta_1 = \Delta_H{}^+$.

We next give a more root-theoretic criterion for congruence.

PROPOSITION 6.2. *Let* $A = \exp(\sqrt{(-1)}.\pi\, \mathrm{ad}(H))$ *and* $B = \exp(\sqrt{(-1)}.\pi\, \mathrm{ad}(H'))$ *be two involutive automorphisms of* u, *with* $\sqrt{(-1)}.H$, $\sqrt{(-1)}.H' \in h \cap u$. $A \equiv B$ *if and only if* $\Delta_H{}^+$ and $\Delta_{H'}{}^+$ *are conjugate.*

Proof. Suppose that $\sigma \in \tilde{W}(\Delta)$ is such that $\sigma\Delta_{H'}{}^+ = \Delta_H{}^+$. There is an automorphism S of u such that $Sh = h$ and $\sigma\alpha(H) = \alpha(S^{-1}H)$ for each $H \in h$. Consider

$$\begin{aligned} S^{-1}AS &= S^{-1}\exp(\sqrt{(-1)}.\pi\, \mathrm{ad}(H))S = \exp(S^{-1}\sqrt{(-1)}.\pi\, \mathrm{ad}(H)S) \\ &= \exp(\sqrt{(-1)}.\pi\, \mathrm{ad}(S^{-1}H)) = \exp(\sqrt{(-1)}.\pi\, \mathrm{ad}(\tilde{H})). \end{aligned}$$

But

$$\begin{aligned} \Delta_{\tilde{H}}{}^+ &= \{\alpha \in \Delta |\ \alpha(S^{-1}H) \equiv 0 \pmod 2\} \\ &= \{\alpha \in \sigma |\ \sigma\alpha(H) \equiv 0 \pmod 2\} = \Delta_{H'}{}^+. \end{aligned}$$

Thus $S^{-1}AS = B$. Hence $A \equiv B$.

Let us suppose that $A \equiv B$; then there is an S in the automorphism group of u such that $S^{-1}AS = B$. Now let U denote the connected component of the identity in the group G of automorphisms of u; then

$$G = \bigcup_{i=1}^{s} T_i U,$$

the components of G with respect to U, and we may assume that $T_j h = h$. Thus $S = T_j R, R \in U$. Now then $S^{-1}AS = R^{-1}T_j{}^{-1}AT_jR$. Hence $T_j{}^{-1}AT_j = RBR^{-1}$. But $T_j{}^{-1}AT_j$ and B are contained in the maximal torus $\exp(\mathrm{ad}(h \cap u))$ in U and thus there is an element of the normalizer of this torus, V, such that $T_j{}^{-1}AT_j = VBV^{-1}$ (see Séminaire "Sophus Lie" **(6)**, Exposé 23). Thus $V^{-1}T_j{}^{-1}AT_jV = B$. And $T_jVh = h$. Let $\tilde{S} = T_jV$. Let $\tilde{\sigma}$ be the element of $\tilde{W}(\Delta)$ defined by $\tilde{\sigma}\alpha(X) = \alpha(\tilde{S}^{-1}X)$ for all $X \in h$. Then $\tilde{\sigma}\Delta_{H'}{}^+ = \Delta_H{}^+$.

Combining Propositions 6.1 and 6.2 and Table I we have thus given a complete classification of all equivalence classes of inner involutive automorphisms.

LEMMA 6.1. *Let* $A = T\exp(\sqrt{(-1)}.\pi\, \mathrm{ad}(H))$ *be an involutive automorphism of* u, *where* $T \in I(\pi)$ *and* $TH = H$, $\sqrt{(-1)}.H \in u \cap h$. *Then* $A \equiv T\exp(\sqrt{(-1)}.\pi\, \mathrm{ad}(\tilde{H}))$, *where* $T\tilde{H} = \tilde{H}$ *and if* $\alpha_i \in \pi - \Delta_{\tilde{H}}{}^+$, *then* $\tau\alpha_i = \alpha_i$.

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Proof. Let $H_1, \ldots, H_l$ in h be defined by $\alpha_i(H_j) = \delta_{ij}$. Let $\pi_1 = \pi - \Delta_H^+$. Assume that $\pi_1 = \{\alpha_1, \ldots, \alpha_s\}$. In the course of the proof of Proposition 6.1 we showed that $\Delta_H^+ = \Delta_{H'}^+$, where $H' = H_1 + H_2 + \ldots + H_s$ and $TH' = H'$. Hence $A = T\exp(\sqrt{(-1)}.\pi\,\mathrm{ad}(H'))$. Suppose $\tau\alpha_j \neq \alpha_j$ for some j between 1 and s. Let $\tau\alpha_j = \alpha_{j'}$ (note that $1 \leqslant j' \leqslant s$). Set $\hat{H} = (H_j - H_{j'})$. Then $T\hat{H} = -\hat{H}$. Since $T^2 = 1$, this implies that $\hat{H} = (T-1)H''$, where $H'' \in \sqrt{(-1)}.h \cap u$. Now

$$H' + \hat{H} = 2H_j + \sum_{i \neq h,h'}^{s} H_i.$$

Thus $\exp(\sqrt{(-1)}.\pi\,\mathrm{ad}(H' + \hat{H})) = \exp(\sqrt{(-1)}.\pi\,\mathrm{ad}(\tilde{H}))$, where

$$\tilde{H} = \sum_{i \neq j,j'}^{s} H_i.$$

Now

$$\exp(\pi\sqrt{(-1)}.\mathrm{ad}(H''))T\exp(\pi\sqrt{(-1)}.\mathrm{ad}(H'))\exp(-\pi\sqrt{(-1)}.\mathrm{ad}(H''))$$
$$= T\exp(\pi\sqrt{(-1)}.\mathrm{ad}(H' + (T-1)H'')) = T\exp(\pi\sqrt{(-1)}.\mathrm{ad}(H' + \hat{H}))$$
$$= T\exp(\pi\sqrt{(-1)}.\mathrm{ad}(\tilde{H})).$$

Setting $S = \exp(\pi\sqrt{(-1)}.\mathrm{ad}(H''))$, we have thus shown that

$$SAS^{-1} = T\exp(\pi\sqrt{(-1)}.\mathrm{ad}(\tilde{H}))$$

with $\pi - \Delta_{\tilde{H}}^+ = \pi - \Delta_H^+ - \{\alpha_j, \tau\alpha_j\}$. Lemma 6.1 now follows by induction.

LEMMA 6.2. *If $\tau \in I(\pi)$ and $\tilde{\pi} = \{\alpha \in \pi |\ \tau\alpha = \alpha\}$, then $\tilde{\pi}$ is connected.*

Proof. If $\alpha, \beta \in I(\pi)$ and $\alpha, \beta_1, \ldots, \beta_r, \beta$ is a chain connecting α and β in π (i.e., $\langle \alpha, \beta_1 \rangle \neq 0$, $\langle \beta_i, \beta_{i+1} \rangle \neq 0$, $i = 1, \ldots, r-1$, $\langle \beta_r, \beta \rangle \neq 0$), then $\tau\alpha, \tau\beta_1, \ldots, \tau\beta_r, \tau\beta$ is a chain connecting α and β in π ($\tau\alpha = \alpha$, $\tau\beta = \beta$). If then for some $1 \leqslant k \leqslant r$ we have $\tau\beta_k \neq \beta_k$, then we would have a "cycle" in π, and this is impossible (see Jacobson (**4**, p. 130) for the pertinent definitions and theorems). Thus $\tau\beta_k = \beta_k$, $k = 1, \ldots, r$. Hence $\tilde{\pi}$ is connected.

Let $\tilde{\Delta}$ be the subsystem of Δ generated by $\tilde{\pi}$. We now prove

LEMMA 6.3. *Let $T\exp(\sqrt{(-1)}.\pi\,\mathrm{ad}(H))$ be an involutive automorphism of u such that $T \in I(\pi)$, $\sqrt{(-1)}.H \in u \cap h$, and $TH = H$. Let $H_1, \ldots, H_l$ be the elements in h defined by $\alpha_i(H_j) = \delta_{ij}$, where $\pi = \{\alpha_1, \ldots, \alpha_l\}$. Then if $H \neq 0$,*

$$T\exp(\sqrt{(-1)}.\pi\,\mathrm{ad}(H)) \equiv T\exp(\sqrt{(-1)}.\pi\,\mathrm{ad}(H_k)),$$

where $\tau\alpha_k = \alpha_k$.

Proof. Let $\{\alpha_{i_1}, \ldots, \alpha_{i_r}\} = \pi - \Delta_H^+$. By Lemma 6.2 we may assume that $\alpha_{i_j} \in \tilde{\pi}$, $j = 1, \ldots, r$. Furthermore,

$$\tilde{\Delta} \cap \Delta_H^+ = \{\alpha \in \tilde{\Delta} |\ \alpha(H) \equiv 0 \pmod 2\}$$

and thus by Proposition 6.1, $\tilde{\Delta} \cap \Delta_H^+$ is a maximal subsystem of $\tilde{\Delta}$ of characteristic 0 or 2. Theorem 3.1 (2) says that there is an $\alpha_{i_k} \in \tilde{\pi}$ and a $\sigma \in W(\tilde{\Delta})$ such that

$$\sigma(\tilde{\Delta} \cap \Delta_H^+) = \{\alpha \in \tilde{\Delta} \mid \alpha = \sum m_{ij}\, \alpha_{ij},\, m_{ik} \equiv 0 \pmod 2\}.$$

Since σ is a product of Weyl reflections, σ can be considered an element of the Weyl group of Δ. Now $\tilde{\pi} \cap \pi - (\sigma\Delta_H^+) = \alpha_{i_k}$. For simplicity, set $i_k = k$. And thus by Lemma 6.1 we have

$$T \exp(\sqrt{(-1)}.\pi \operatorname{ad}(H)) \equiv T \exp(\sqrt{(-1)}.\pi \operatorname{ad}(H_k)).$$

PROPOSITION 6.2. $T \exp(\sqrt{(-1)}.\pi \operatorname{ad}(H_k)) \equiv T \exp(\sqrt{(-1)}.\pi \operatorname{ad}(H_r))$ *(where $\alpha_k, \alpha_r \in \tilde{\pi}$ and H_k, H_r are as in Lemma* 6.1) *if $\Delta_{H_k}{}^+ \cap \tilde{\Delta}$ is conjugate to $\Delta_{H_r}{}^+ \cap \tilde{\Delta}$ in $\tilde{\Delta}$.*

TABLE II

g	π	τ	A	$\tilde{\pi}_A$	$g_A{}^+$
A_l, $l = 2p$	o—o···o—o; $\alpha_1\ \alpha_2\ \alpha_{l-1}\ \alpha_l$	$\tau\alpha_i = \alpha_{l+1-i}$	T	o—o···o$\Leftarrow$o; $\tilde{\alpha}_1\ \tilde{\alpha}_2\ \tilde{\alpha}_{p-1}\ \tilde{\alpha}_p$	B_p
A_l, $l = 2p+1$	o—o···o—o; $\alpha_1\ \alpha_2\ \alpha_{l-1}\ \alpha_l$	$\tau\alpha_i = \alpha_{l+1-i}$	T	o—o···o$\Rightarrow$o; $\tilde{\alpha}_1\ \tilde{\alpha}_2\ \tilde{\alpha}_p\ \tilde{\alpha}_{p+1}$	C_{p+1}
			$Te^{\operatorname{ad}\pi i H_{p+1}}$	o—o···o$\tilde{\alpha}_{p-1}$ branching to o $\tilde{\alpha}_p$ and o $\tilde{\alpha}_p + \tilde{\alpha}_{p+1}$; $\tilde{\alpha}_1\ \tilde{\alpha}_2$	D_{p+1}
D_l, $l > 4$	o—o···o—o with o α_l attached; $\alpha_1\ \alpha_2\ \alpha_{l-2}\ \alpha_{l-1}$	$\tau\alpha_{l-1} = \alpha_l$	T	o—o···o$\Leftarrow$o; $\tilde{\alpha}_1\ \tilde{\alpha}_2\ \tilde{\alpha}_{l-2}\ \tilde{\alpha}_{l-1}$	B_{l-1}
				$\beta_k = \sum_k^l \tilde{\alpha}_i$	
			$Te^{\operatorname{ad}\pi i H_k}$	o—o···o $\Leftarrow$o o—···o $\Leftarrow$o; $\tilde{\alpha}_1\ \tilde{\alpha}_2\ \tilde{\alpha}_{k-1}\ \beta_k\ \tilde{\alpha}_{k+1}\ \tilde{\alpha}_{l-2}\ \tilde{\alpha}_l$	$B_k \times B_{l-k-1}$
D_4	o—o—o with o α_4 attached; $\alpha_1\ \alpha_2\ \alpha_3$	$\tau_1\alpha_1 = \alpha_4$	T_j $j = 1, 2$	o—o$\Leftarrow$o; $\tilde{\alpha}_{j+2}\ \tilde{\alpha}_2\ \tilde{\alpha}_1$; $j = 1, 2$	B_3
		$\tau_2\alpha_1 = \alpha_3$	T_3	o—o$\Leftarrow$o; $\tilde{\alpha}_1\ \tilde{\alpha}_2\ \tilde{\alpha}_3$	B_3
		$\tau_3\alpha_3 = \alpha_4$	$T_j e^{\operatorname{ad}\pi i H_2}$ $j = 1, 2$	o$\Leftarrow$o o; $\tilde{\alpha}_{j+2}\ \tilde{\alpha}_2+\tilde{\alpha}_1\ \tilde{\alpha}_1$; $j = 1, 2$	$B_2 \times A_1$
			$T_3 e^{\operatorname{ad}\pi i H_2}$	o$\Leftarrow$o o; $\tilde{\alpha}_1\ \tilde{\alpha}_2+\tilde{\alpha}_3\ \tilde{\alpha}_3$	$B_2 \times A_1$
E_6	o—o—o—o—o with o α_6 attached to α_3; $\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4\ \alpha_5$	$\tau\alpha_1 = \alpha_5$	T	o—o$\Rightarrow$o—o; $\tilde{\alpha}_1\ \tilde{\alpha}_2\ \tilde{\alpha}_3\ \tilde{\alpha}_6$	F_4
		$\tau\alpha_2 = \alpha_4$	$Te^{\operatorname{ad}\pi i H_3}$	o$\Leftarrow$o—o—o; $\tilde{\alpha}_6\ \tilde{\alpha}_2+\tilde{\alpha}_3\ \tilde{\alpha}_1\ \tilde{\alpha}_2$	C_4
			$Te^{\operatorname{ad}\pi i H_6}$	o—o—o$\Rightarrow$o; $\tilde{\alpha}_2+\tilde{\alpha}_3+\tilde{\alpha}_6\ \tilde{\alpha}_1\ \tilde{\alpha}_2\ \tilde{\alpha}_3$	C_4

Note. In the above table $\tilde{\alpha}_i = \alpha_i|_{h_A{}^+}$, where $h_A{}^+ = \{H \in h \mid AH = H\}$; $g_A{}^+ = \{X \in h \mid AX = X\}$; and $\tilde{\pi}_A$ is a fundamental system for $\tilde{\Delta}$, the set of roots of $g_A{}^+$ with respect to $h_A{}^+$.

Proof. If $\Delta_{H_k}{}^+ \cap \tilde{\Delta}$ is conjugate to $\Delta_{H_r}{}^+ \cap \tilde{\Delta}$ in $\tilde{\Delta}$, then the techniques of the proof of Lemma 6.3 will show that $T \exp(\sqrt{(-1)}.\pi \operatorname{ad}(H_k)) \equiv T \exp(\sqrt{(-1)}.\pi \operatorname{ad}(H_r))$.

Using Table II, we can now complete the classification of real forms of g. We first note that $I(\pi) \neq \{1\}$ only for A_l, D_l, E_6.

For A_l we notice that if l is even and if $\tau \in I(\pi) - \{1\}$, then $\tau\alpha_k \neq \alpha_k$ for any $\alpha_k \in \pi$. Thus Lemma 6.1 implies that up to equivalence the only involutive outer automorphism is T. If l is odd, $l = 2p + 1$, then the only fixed point of τ is α_{p+1}; thus up to equivalence we need only consider T and $T \exp(\sqrt{(-1)}.\pi \operatorname{ad}(H_{p+1}))$ and these two automorphisms cannot be equivalent because Table II shows that they have non-isomorphic fixed point sets.

In the case of D_l for $l > 4$, $\tau \in I(\pi) - \{1\}$, then

$$\tilde{\pi} = \{\alpha \in \pi | \tau\alpha = \alpha\} = \{\alpha_1, \ldots, \alpha_{l-2}\};$$

thus Lemma 6.4 and the classification of inner involutive automorphisms say that $T \exp(\sqrt{(-1)}.\operatorname{ad}(H_i)) \equiv T \exp(\sqrt{(-1)}.\operatorname{ad}(H_{l-1-i}))$ for $1 \leqslant i \leqslant l - 1$. Table II shows that every outer involutive automorphism is then conjugate to T or one of the $T \exp(\sqrt{(-1)}.\operatorname{ad}(H_i))$, where $i = 1, \ldots, s$ and $s = p + 1$ if $l = 2p + s$, $s = p$ if $l = 2p + 2$.

E_6 and D_4 are handled similarly.

For the techniques of calculation of the tables the reader may consult the appendix to **(7)** or **(5)**.

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