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A REMARK ON PROJECTIVE MODULES

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Let R denote the field of real numbers and let A be the ring of regular functions on R^n, that is the localization of $R[T_1, \ldots, T_n]$ with respect to the set of all polynomials vanishing nowhere on R^n. Let X be an algebraic subset of R^n and let $I(X)$ be the ideal of A of all functions vanishing on X. Assume that X is compact and nonsingular and $k = \text{codim } X = 1$, 2, 4 or 8. We prove here that if the $A/I(X)$-module $I(X)/I(X)^2$ can be generated by k elements, then there exist a projective A-module P of rank k and a homomorphism from P onto $I(X)$.

1. Introduction

Let R denote the field of real numbers and let A be the ring of all functions $f : R^n \to R$ such that $f = \phi/\psi$ for some polynomial functions $\phi, \psi : R^n \to R$ with ψ vanishing nowhere. In other words, A is (isomorphic to) the localization of the polynomial ring $R[T_1, \ldots, T_n]$ with respect to the set consisting of all polynomials vanishing nowhere on R^n. Given a subset X of R^n, we denote by $I(X)$ the ideal of A of all functions vanishing on X.

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In this note we prove the following.

THEOREM. *Let* X *be a nonsingular algebraic subset of* R^n *of codimension* k. *Assume that the* $A/I(X)$-*module* $I(X)/I(X)^2$ *can be generated by* k *elements. If* $k = 1, 2, 4$ *or* 8 *and* X *is compact, then there exist a finitely generated projective* A-*module* P *of rank* k *and a surjective homomorphism* $h: P \to I(X)$.

For $k = 1$ or 2 some better results are known. Indeed, since A is a factorial ring (being a localization of $R[T_1, \ldots, T_n]$), the ideal $I(X)$ is principal if $k = 1$, without the compactness assumption. If $k = 2$, then the ideal $I(X)$ is a complete intersection (see [4]) and one can even drop the compactness assumption for $\dim X = 1$ (see [8]). Moreover, the theorem holds true if $k = 2$ but X is not necessarily compact (see for example, [7, Theorem 3.1]).

It is unknown whether all finitely generated projective A-modules of rank greater than one are free (proofs of the Serre conjecture concerning finitely generated projective modules over polynomial rings do not seem to extend to this case, see [10],[12]). Therefore the theorem does not allow us to conclude that the ideal $I(X)$ is a complete intersection for $k = 2, 4$ or 8 (see the remark above for $k = 2$).

The author does not know whether the theorem remains true for $k = 4$ or 8 if one drops the compactness assumption or replaces R by another, say real closed, field.

2. Proof of the Theorem

Our terminology and notions concerning real algebraic geometry are consistent with those of [2], [3] and [13]. In particular, A is the ring of regular functions on R^n (see [3, Chapter 3] or [11]). Also recall that an algebraic vector bundle ξ over an affine real algebraic variety X is said to be strongly algebraic if there exists an algebraic bundle η over X such that $\xi \oplus \eta$ is algebraically isomorphic to a product vector bundle $X \times R^m$ (see [2], [3, Chapter 12] and [13]).

EXAMPLE 1. The real projective space RP^n with its standard structure of an abstract real algebraic variety is an affine variety (see [3, Theorem 3.4.4] or [1, p. 432]). Moreover, every C^∞ R-vector bundle over RP^n is C^∞ isomorphic to a strongly algebraic vector bundle (see [3, Example 12.3.7(c)]). Indeed, let ξ be a C^∞ R-vector bundle over RP^n. Then ξ is stably equivalent to the canonical line bundle γ^n over RP^n or to the direct sum of several copies of γ^n [6, p. 223, Theorem 12.7]. Obviously, γ^n is strongly algebraic and hence ξ is stably equivalent to a strongly algebraic vector bundle. It follows that ξ is C^∞ isomorphic to a strongly algebraic vector bundle (see [2, p. 109]).

The next technical result is proved in [13, Proposition 2].

LEMMA 2. *Let* X *be an affine nonsingular real algebraic variety and let* ξ *be a strongly algebraic vector bundle over* X. *Assume that* X *is compact in the Euclidean topology. If* s *is a* C^∞ *section of* ξ *vanishing on a closed nonsingular algebraic subvariety* Y *of* X, *then there exists an algebraic section* u *of* ξ *which is arbitrarily close to* s *in the* C^∞ *topology and vanishes on* Y.

The last auxiliary result is the following.

LEMMA 3. *Let* A *be a closed* C^∞ *submanifold of a* C^∞ *manifold* M. *Assume that the normal vector bundle of* A *in* M *is trivial. If* $\operatorname{codim} A = 1,2,4$ *or* 8, *then there exist a* C^∞ R*-vector bundle* ξ *over* M *and a* C^∞ *section* s *of* ξ *such that rank* $\xi = \operatorname{codim} A$, s *is transverse to the zero section of* ξ *and the set of zeros* $s^{-1}(0)$ *of* s *is equal to* A.

Proof. Let $k = \operatorname{codim} A$ and let S^k be the unit k-dimensional sphere. Since the normal vector bundle of A is trivial, there exist a C^∞ map $f : M \to S^k$ and a regular value y of f such that $f^{-1}(y) = A$ (see [9]). If $k = 1,2,4$ or 8, then one can find a C^∞ R-vector bundle γ over S^k and a C^∞ section u of γ such that rank $\gamma = k$, u is transverse to the zero section of γ and $u^{-1}(0) = \{y\}$ (the construction of γ and u is easily available if one identifies S^1, S^2, S^4 and S^8 with the projective line over the reals, complexes, quaternions and Cayley numbers, respectively). It suffices to set $\xi = f^*\gamma$ and $s = f^*u$, where, as usual, $f^*\gamma$ denotes the pull-back vector bundle and f^*u denotes the pull-back section.

Proof of the Theorem. We identify R^n with a subset of RP^n via the map which sends $(x_1,\dots,x_n)$ to $[1,x_1,\dots,x_n]$. Let Y be the Zariski closure of X in RP^n. Then $Y = X \cup X'$, where X' is contained in $RP^n \setminus R^n$. Notice that X is a C^∞ submanifold of RP^n and the normal vector bundle of X is trivial. It follows from Lemma 3 that there exist a C^∞ vector bundle ξ over RP^n and a C^∞ section s of ξ such that rank $\xi = k$, s is transverse to the zero section of ξ and $s^{-1}(0) = X$. By Example 1, we can assume that ξ is a strongly algebraic vector bundle.

Let $\mathrm{Sing}(Y)$ be the set of singular points of Y. By the Hironaka theorem [5], there exist a nonsingular real algebraic variety V and a real algebraic morphism $\pi : V \to RP^n$ such that π isomorphically transforms $V \setminus \pi^{-1}(\mathrm{Sing}(Y))$ onto $RP^n \setminus \mathrm{Sing}(Y)$ and the Zariski closure Z of $\pi^{-1}(Y \setminus \mathrm{Sing}(Y))$ in V is nonsingular. Moreover, since π is the composition of finitely many algebraic blowing-ups, it is a proper map in the Euclidean topology (in particular, V is compact) and V is an *affine* real algebraic vareity. Notice that $Z = Z_1 \cup Z_2$, where $Z_1 = \pi^{-1}(X)$ and Z_2 is a Zariski closed subset of V disjoint from Z_1. Since Z and Z_2 are both Zariski closed, Z is nonsingular and $\dim Z = \dim Z_2$, it follows that Z_1 is Zariski closed in V (see [1, Lemma 1.6]) and, of course, nonsingular.

Clearly, the pull-back vector bundle $\pi^*\xi$ over V is strongly algebraic and the pull-back section π^*s is of class C^∞ and transverse to the zero section of $\pi^*\xi$ and $(\pi^*s)^{-1}(0) = Z_1$. By Lemma 2, there exists an algebraic section ν of $\pi^*\xi$ arbitrarily close to s in the C^∞ topology and vanishing on Z_1. Thus we can assume that ν is transverse to the zero section of $\pi^*\xi$ and $\nu^{-1}(0) = Z_1$.

Let η be the restriction of ξ to R^n and let $\rho = (\pi|\pi^{-1}(R^n))^{-1}$. Then $\eta = \rho^*(\pi^*\xi|\pi^{-1}(R^n))$ and $u = \rho^*\nu$ is an algebraic section of η which is transverse to the zero section of η and satisfies $X = u^{-1}(0)$.

Let Q be the A-module of all algebraic sections of η. It follows from the definition of a strongly algebraic vector bundle that Q is a finitely generated projective module of rank k (see also [3, Proposition 12.1.11]) and hence so is the module $P = \mathrm{Hom}(Q,A)$. Since u is transverse to the zero section of η and $u^{-1}(0) = X$, one easily sees that for every α in P, the element $\alpha(u)$ belongs to $I(X)$ and all elements of this form generate $I(X)$. To conclude the proof, we define $h : P \to I(X)$ by $h(\alpha) = \alpha(u)$ for α in P.

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