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SEMILATTICES WITH A TRANSITIVE AUTOMORPHISM GROUP

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Abstract

If L is any semilattice, let T_L denote the Munn semigroup of L, and Aut (L) the automorphism group of L.

We show that every semilattice L can be isomorphically embedded as a convex subsemilattice in a semilattice L' which has a transitive automorphism group in such a way that

(i) every partial isomorphism α of L can be extended to an automorphism $\bar{\alpha}$ of L',

(ii) every partial isomorphism $\alpha\colon eL \to fL$ of L can be extended to a partial isomorphism $\alpha_{L'}\colon eL' \to fL'$ of L' such that $T_L \to T_{L'}$, $\alpha \to \alpha_{L'}$ embeds T_L isomorphically in $T_{L'}$,

(iii) every automorphism γ of L can be extended to an automorphism $\gamma_{L'}$ of L' such that Aut $(L) \to$ Aut (L'), $\gamma \to \gamma_L$ embeds Aut (L) isomorphically in Aut (L').

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We shall give a procedure for embedding any semilattice in a semilattice which has a transitive automorphism group. The particular procedure which will be given here is very likely to be relevant for the construction of inverse semigroups. The reader may consult McAlister (1974a, b, 1978) and O'Carroll (1976) for a further motivation of our embedding theorem.

We follow the terminology and the notation of Birkhoff (1967) and Howie (1976). If $L, \wedge$ is any semilattice, and $e \in L$, then the principal ideal of L which is generated by e will be denoted by eL. If e and f are elements of L such that $\alpha\colon eL \to fL$ is an isomorphism, then α will be called a partial isomorphism of L. The partial isomorphisms of L form an inverse subsemigroup T_L of the symmetric

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inverse semigroup on the set L. This semigroup T_L will be called the Munn semigroup of L (Howie (1976), Munn (1966)). We define the relation $\mathscr{U}_L$ on L by $\mathscr{U}_L = \{(e,f) \in L \times L \mid eL \cong fL\}$. The automorphism group of L will be denoted by $\operatorname{Aut}(L)$.

The following result summarizes Lemmas 1, 2 and 4 of Pastijn (preprint).

LEMMA 1. *Any semilattice [lattice] L can be isomorphically embedded as a filter and as a subsemilattice in a semilattice [lattice] K in such a way that*

(i) $L \times L \subseteq \mathscr{U}_K$,

(ii) *every partial isomorphism $\alpha: eL \to fL$ of L can be extended to a partial isomorphism $\alpha_K: eK \to fK$ of K such that the mapping*

$$\psi: T_L \to T_K, \quad \alpha \to \alpha_K$$

embeds T_L isomorphically in T_K,

(iii) *every automorphism γ of L can be extended to an automorphism γ_K of K such that the mapping*

$$\xi: \operatorname{Aut}(L) \to \operatorname{Aut}(K), \quad \gamma \to \gamma_K,$$

embeds $\operatorname{Aut}(L)$ *isomorphically in* $\operatorname{Aut}(K)$.

We now start with the embedding procedure. Let $L, \wedge$ be any semilattice. Let L^1 be the semilattice arising from L by the adjunction of an identity element 1, unless L had already an identity element, in which case $L = L^1$. For any $\alpha \in T_{L^1}$, with $\alpha: e_\alpha L^1 \to f_\alpha L^1$, let L_α be an isomorphic copy of L^1 which contains $f_\alpha L^1$ as a principal ideal, and $\hat{\alpha}: L^1 \to L_\alpha$ an isomorphism of L^1 onto L_α which extends the partial isomorphism α. We can always suppose that for any $\alpha, \beta \in T_{L^1}$, with $\alpha \neq \beta$, we have $L_\alpha \cap L_\beta = (f_\alpha \wedge f_\beta) L^1$. In particular, if α is an automorphism of L^1, then $\alpha, \beta \in T_{L^1}$ and $L_\alpha = L^1$. Let $L^{(1)} = (\bigcup_{\alpha \in T_{L^1}} L_\alpha) \cup \{1^{(1)}\}$. On $L^{(1)}$ we define a partial order $\leqslant$ by the following:

$$x \leqslant 1^{(1)} \quad \text{for all } x \in L^{(1)}$$

and

$$x \leqslant y, \quad x, y \in L^{(1)}, x \neq 1^{(1)} \neq y,$$

if and only if $x, y \in L_\alpha$ for some $\alpha \in T_{L^1}$ and $x \leqslant y$ in L_α.

LEMMA 2.

(i) *$L^{(1)}$ is a semilattice which contains L^1 as a subsemilattice and as a principal ideal.*

(ii) *$L^{(1)}$ is a lattice if and only if L^1 is a lattice.*

(iii) *Every partial isomorphism α of L can be extended to the partial isomorphisms $\hat{\alpha}$ and $(\alpha^{-1})^{\wedge -1}$ of $L^{(1)}$ such that $L^1 = \operatorname{dom} \hat{\alpha}$ and $L^1 = \operatorname{im} (\alpha^{-1})^{\wedge -1}$.*

(iv) *Every automorphism* γ *of* L *can be extended to an automorphism* γ_1 *of* $L^{(1)}$ *in such a way that the mapping*

$$\xi\colon \operatorname{Aut}(L) \to \operatorname{Aut}(L^{(1)}), \quad \gamma \to \gamma_1,$$

embeds $\operatorname{Aut}(L)$ *isomorphically in* $\operatorname{Aut}(L^{(1)})$.

PROOF. (i) Let α and β be any elements of T_{L^1}, and let $x \in L_\alpha$, $y \in L_\beta$. If $\alpha = \beta$, then the greatest lower bound $\inf(x,y)$ of x and y in $L^{(1)}$ equals $x \wedge y$, where the meet is to be taken in $L_\alpha = L_\beta$. In case $\alpha \neq \beta$, let $x \wedge (f_\alpha \wedge f_\beta)$ be the meet of x and $f_\alpha \wedge f_\beta$ in L_α, let $y \wedge (f_\alpha \wedge f_\beta)$ be the meet of y and $f_\alpha \wedge f_\beta$ in L_β, and let

$$(x \wedge (f_\alpha \wedge f_\beta)) \wedge (y \wedge (f_\alpha \wedge f_\beta))$$

be the meet of $x \wedge (f_\alpha \wedge f_\beta)$ and $y \wedge (f_\alpha \wedge f_\beta)$ in $(f_\alpha \wedge f_\beta)L^1$: it is easy to see that $(x \wedge (f_\alpha \wedge f_\beta)) \wedge (y \wedge (f_\alpha \wedge f_\beta))$ is then the greatest lower bound $\inf(x,y)$ of x and y in $L^{(1)}$. We conclude that $L^{(1)}, \leqslant$ is an inf-semilattice. Since the operation 'inf' on $L^{(1)}$ extends the meet operations on the L_α, $\alpha \in T_{L^1}$, there is no harm in denoting $\inf(x,y)$ by $x \wedge y$ for all $x, y \in L^{(1)}$. From this it follows in particular that L is a subsemilattice of $L^{(1)}$. Furthermore, $L^1 = 1L^{(1)}$ is a principal ideal of $L^{(1)}$.

(ii) If $L^{(1)}$ is a lattice, then every principal ideal of $L^{(1)}$ is a sublattice of $L^{(1)}$. Hence L^1 is then a sublattice of $L^{(1)}$.

Let us suppose that L^1 is a lattice, and let $x, y \in L^{(1)}$. The least upper bound $\sup(x,y)$ of x and y in $L^{(1)}$ equals the join $x \vee y$ of x and y in L_α if $x, y \in L_\alpha$, $\alpha \in T_{L^1}$, and $\sup(x,y) = 1^{(1)}$ otherwise. $L^{(1)}$ then becomes a lattice, and the operation 'sup' on $L^{(1)}$ extends the join operations on the L_α, $\alpha \in T_{L^1}$; therefore we can denote $\sup(x,y)$ by $x \vee y$ for all $x, y \in L^{(1)}$.

(iii) Trivial from our construction.

(iv) Let γ be any automorphism of L. The automorphism γ can be extended in a trivial way to an automorphism γ' of L^1. The automorphism γ_1 of $L^{(1)}$ which is defined by

$$1^{(1)}\gamma_1 = 1^{(1)}$$

and

$$x\gamma_1 = x\hat{\alpha}^{-1}(\alpha\gamma')^\wedge \quad \text{if } x \in L_\alpha, \quad \alpha \in T_{L^1},$$

extends γ and the mapping $\xi\colon \operatorname{Aut}(L) \to \operatorname{Aut}(L^{(1)})$, $\gamma \to \gamma_1$, is injective. Furthermore, if $\gamma, \delta \in \operatorname{Aut}(L)$, $x \in L_\alpha$, $\alpha \in T_{L^1}$, then

$$\begin{aligned} x\gamma_1\delta_1 &= (x\hat{\alpha}^{-1}(\alpha\gamma')^\wedge)\,\delta_1 \\ &= x\hat{\alpha}^{-1}(\alpha\gamma')^\wedge(\alpha\gamma')^{\wedge -1}(\alpha\gamma'\delta')^\wedge \\ &= x\hat{\alpha}^{-1}(\alpha(\gamma\delta)')^\wedge \\ &= x(\gamma\delta)_1. \end{aligned}$$

Thus ξ is an isomorphism of $\operatorname{Aut}(L)$ into $\operatorname{Aut}(L^{(1)})$.

In McAlister (1974b), Theorem 4.2 it was proved that every semilattice L can be embedded as an ideal in a partially ordered set $\mathscr{X}$ in such a way that every partial isomorphism of L can be extended to an automorphism of $\mathscr{X}$. Slight refinements for this result may be obtained from McAlister (1976), Section 3. Corollary 4.4 of McAlister (1978) states that in Theorem 4.2 of McAlister (1974b) $\mathscr{X}$ can be chosen to be a semilattice. Using quite different techniques, this result was also obtained in Section 2 of Meakin and Pastijn (preprint). We now present a theorem which is an application of our Lemma 2, and which is a refinement for the results mentioned here. It should be remarked that the following theorem could also be obtained using Lemma 1.2 of O'Carroll (1976), together with the results of McAlister (1974a, b). Our construction, however, has the advantage that the proofs are purely semilattice-theoretic and independent of any knowledge about the construction of E-unitary inverse semigroups.

THEOREM 3. *Every semilattice* [*lattice*] L *can be embedded isomorphically as an ideal in a semilattice* [*lattice*] L' *in such a way that*

(i) *every partial isomorphism* α *of* L' *can be extended to an automorphism* $\bar{\alpha}$ *of* L',

(ii) *every automorphism* γ *of* L *can be extended to an automorphism* γ' *of* L' *in such a way that the mapping*

$$\xi\colon \operatorname{Aut}(L) \to \operatorname{Aut}(L'), \quad \gamma \to \gamma',$$

embeds $\operatorname{Aut}(L)$ *isomorphically in* $\operatorname{Aut}(L')$.

PROOF. Let us consider the sequence of semilattices

$$L = L^{(0)}, L^{(1)}, L^{(2)}, \ldots, L^{(j)}, L^{(j+1)}, \ldots$$

where for each j, $L^{(j+1)}$ is constructed from $L^{(j)}$ in the way described before Lemma 2. It is then clear from Lemma 2(i) that $L' = \bigcup_{j=0}^{\infty} L^{(j)}$ is a semilattice which contains L as an ideal. From Lemma 2(ii) it follows that L' is a lattice if L is a lattice. Let $x, y \in L'$, and $\alpha\colon xL' \to yL'$ a partial isomorphism of L'. There exists a j such that $x, y \in L^{(j)}$, and then $xL' = xL^{(j)}$, $yL' = yL^{(j)}$ since $L^{(j)}$ must be an ideal of L'. Hence α is a partial isomorphism of $L^{(j)}$. From Lemma 2(iii) it now follows that we can consider a sequence

$$\alpha = \alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(k)}, \alpha^{(k+1)}, \ldots$$

in which for every k $\alpha^{(k+1)}$ is a partial isomorphism of $L^{(j+k+1)}$ which extends the partial isomorphism $\alpha^{(k)}$ of $L^{(j+k)}$ such that

$$L^{(j+k-1)} = \operatorname{dom} \alpha^{(k)} \quad \text{if } k \text{ is odd}$$

and

$$L^{(j+k-1)} = \operatorname{im} \alpha^{(k)} \quad \text{if } k \text{ is even}, \quad k \geqslant 2.$$

It follows that $\bigcup_{k=0}^{\infty} \alpha^{(k)} = \bar{\alpha}$ is an automorphism of L' which extends the partial isomorphism α. Thus condition (i) is satisfied.

(ii) follows from Lemma 2(iv).

Note that every partial isomorphism of L is also a partial isomorphism of L'. Therefore Theorem 3(i) guarantees that every partial isomorphism of L can be extended to an automorphism of L'.

Recall that a convex subsemilattice [sublattice] of a semilattice [lattice] is the intersection of an ideal and a filter.

THEOREM 4. *Every semilattice [lattice] L can be isomorphically embedded as a convex subsemilattice [sublattice] in a semilattice [lattice] L' which has a transitive automorphism group in such a way that*

(i) *every partial isomorphism α of L can be extended to an automorphism $\bar{\alpha}$ of L',*

(ii) *every partial isomorphism $\alpha: eL \to fL$ of L can be extended to a partial isomorphism $\alpha_{L'}: eL' \to fL'$ of L' such that the mapping*

$$\psi: T_L \to T_{L'}, \quad \alpha \to \alpha_{L'},$$

embeds T_L isomorphically in $T_{L'}$,

(iii) *every automorphism γ of L can be extended to an automorphism $\gamma_{L'}$ of L' such that the mapping*

$$\xi: \mathrm{Aut}(L) \to \mathrm{Aut}(L'), \quad \gamma \to \gamma_{L'}$$

embeds $\mathrm{Aut}(L)$ *isomorphically in* $\mathrm{Aut}(L')$.

PROOF. Let us consider the sequence of semilattices

$$L = L_0, K_0, L_1, K_1, \ldots, L_j, K_j, L_{j+1}, K_{j+1}, \ldots,$$

where for each j K_j is constructed from L_j in the way prescribed by Lemma 1, and where for each j L_{j+1} is constructed from K_j in the same way as $L^{(1)}$ was constructed from L (see before Lemma 2). It is then clear from Lemma 1 and Lemma 2(i) that $L' = \bigcup_{j=0}^{\infty} L_j = \bigcup_{j=0}^{\infty} K_j$ is a semilattice which contains L as a convex subsemilattice. From Lemma 1 and Lemma 2(i) it follows that L' is a lattice if L is a lattice. Let $\alpha^{(j)}$ be a partial isomorphism of L_j for some $j \in \mathbf{N}$. From Lemma 1(ii) and from Lemma 2(iii) it now follows that we can consider a sequence

$$\alpha^{(j)}, \alpha^{(j)}_{K_j}, \alpha^{(j+1)}, \alpha^{(j+1)}_{K_{j+1}}, \ldots,$$

where for any $k \geqslant j$, $\alpha^{(k)}_{K_k}$ is a partial isomorphism of K_k which extends the partial isomorphism $\alpha^{(k)}$ of $L^{(k)}$, and $\alpha^{(k+1)}$ a partial isomorphism of $L^{(k+1)}$ which extends the partial isomorphism $\alpha^{(k)}_{K_k}$ of K_k such that

$$K^1_k = \operatorname{dom} \alpha^{(k+1)} \quad \text{if } k \text{ is odd}$$

and

$$K^1_k = \operatorname{im} \alpha^{(k+1)} \quad \text{if } k \text{ is even.}$$

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It follows that $\bigcup_{k=j}^{\infty} \alpha^{(k)}$ is an automorphism of L' which extends $\alpha^{(j)}$. In particular, every partial isomorphism α of L can be extended to an automorphism $\tilde{\alpha}$ of L, and so (i) is satisfied.

Let us consider any two elements $x, y \in L'$. There exists a j such that $x, y \in L_{j-1}$. By Lemma 1(i) we know that $(x, y) \in \mathscr{U}_{K_{j-1}}$, and so there exists a partial isomorphism $\alpha_{K_{j-1}}$ of K_{j-1} and L_j which maps x onto y. Since $\alpha_{K_{j-1}}$ can be extended to an automorphism of L' we can now conclude that $\mathrm{Aut}(L')$ acts transitively on L'.

We know that for any j K_j is an ideal of L_{j+1}; consequently T_{K_j} is a subsemigroup of $T_{L_{j+1}}$. From this fact, and from Lemma 1(ii) we conclude that condition (ii) is satisfied.

From Lemma 1(iii) and from Lemma 2(iv) it follows that condition (iii) is satisfied. This concludes the proof of the theorem.

Note that a lattice L' which has a transitive automorphism group must *a fortiori* be uniform, and $T_{L'}$ must be bisimple. That any semilattice can be isomorphically embedded as a subsemilattice in a uniform semilattice also follows from Reilly (1965).

From Pastijn (preprint) and our present construction it actually follows that in Theorem 4 L is embedded as an open interval in L'. If L has an identity and a zero, then L is of course embedded as a closed interval in L'.

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