# NORM ATTAINING BILINEAR FORMS ON SPACES OF CONTINUOUS FUNCTIONS

*by* J. ALAMINOS, Y. S. CHOI, S. G. KIM and R. PAYÁ

(Received 6 January, 1997)

**Abstract.** We show that continuous bilinear forms on spaces of continuous functions can be approximated by norm attaining bilinear forms.

**1. Introduction.** In the early sixties E. Bishop and R. Phelps showed that the set of norm attaining functionals on a Banach space is dense in the dual space. Shortly after, attention was paid to possible extensions of this result to more general settings, especially bounded linear operators between Banach spaces. Very recently the problem of the denseness of norm attaining functions has moved to other types of mappings like multilinear forms or polynomials.

The first results appeared in a joint work of R. Aron, C. Finet and E. Werner [**2**], where they showed that the Radon-Nikodym property is sufficient for the denseness of norm attaining multilinear forms. For spaces with a shrinking monotone basis the Dunford-Pettis property is also sufficient (see [**6**]). The first example of a Banach space not satisfying the denseness of the norm attaining multilinear forms, namely a predual of a Lorentz sequence space, was found in [**1**] (see also [**13**]). Very recently the second author [**5**] furnished a more appealing counterexample by showing that the norm attaining bilinear forms on $L_1[0, 1]$ are not dense. The study of this kind of problem in "classical" Banach spaces is far from being complete and we try to fill here one of the gaps by discussing spaces of continuous functions.

Recall that continuous bilinear forms on a Banach space can be identified with bounded linear operators from the space into its dual. Under this identification norm attaining bilinear forms become norm attaining operators, but easy examples show that an operator may attain its norm while the corresponding bilinear form does not. Therefore, the denseness of norm attaining operators is a necessary condition for the denseness of norm attaining bilinear forms, but it is not sufficient (see [**10**]).

Concerning spaces of real valued continuous functions, W. Schachermayer [**17**] proved that any weakly compact operator from a $C(K)$ space to an arbitrary real Banach space can be approximated by norm attaining operators. In particular, this applies when the range space is the dual of $C(K)$, but this time every bounded linear operator is weakly compact [**11**]. This is the first step towards our main result, the denseness of norm attaining bilinear forms on spaces of continuous functions. Actually the proof is easy and consists of one further application of Schachermayer's arguments. To get a formally more general result, we deal with the space $C_0(L)$ of continuous functions vanishing at infinity on a locally compact Hausdorff space $L$. Moreover, our proof works in the complex as well as in the real case. The

Research of the first and fourth authors supported by D.G.I.C.Y.T. Project No. PB93-1142. Research of the second author supported by KOSEF grant 961-0102-014-2.

arguments we use to deal with bilinear forms do not work in the $N$-linear case $N > 2$, unless $L$ is scattered. Thus, for scattered $L$ we are able to show the denseness of norm attaining $N$-linear forms on $C_0(L)$. We conclude the paper with some remarks on norm attaining multilinear mappings between spaces of continuous functions and a short discussion of the numerical radius of such mappings.

**2. Main results.** Throughout $L$ will denote a locally compact Hausdorff space and $C_0(L)$ will be the Banach space of real or complex valued continuous functions on $L$ vanishing at infinity, with its natural supremum norm. We denote by $M(L)$ the dual of $C_0(L)$, that is, the Banach space of real or complex regular Borel measures on $L$, endowed with the variation norm. The following lemma was proved by W. Schachermayer [**17**, Lemma 3.1] in the real case. We give an alternative proof which works in the complex case as well.

LEMMA 1. *Let $W$ be a weakly compact subset of $M(L)$, $\mu_0 \in M(L)$ and $\varepsilon > 0$. Then there is a bounded linear operator $S : M(L) \to M(L)$, $\|S\| \leq 1$ such that*

(1) *there is $f_0 \in C_0(L)$, $\|f_0\| = 1$ with $\|\mu_0\| = \langle f_0, S\mu_0 \rangle$,*
(2) $\|S\mu - \mu\| < \varepsilon$ for all $\mu \in W$

*Proof.* By a well known characterization of weak compactness in $M(L)$ (see [**8**, Lemma VI.2.13] for example) the measures in $W$ are uniformly absolutely continuous with respect to some positive regular Borel measure $\lambda$, so there exists $\delta > 0$ such that, for any Borel set $E \subseteq L$,

$$\lambda(E) < \delta \Rightarrow |\mu|(E) < \varepsilon/2, \forall \mu \in W$$

where $|\mu|$ denotes the variation of the measure $\mu$. Let us consider the polar decomposition $\mu_0 = g_0|\mu_0|$, where $g_0$ is the sign of $\mu_0$, a measureable real or complex function with $|g_0| = 1$. By Lusin's Theorem (see [**16**, Theorem 2.23], for example) and the inner regularity of $\lambda$, we may find a compact set $K \subset L$ such that the restriction of $g_0$ to $K$ is continuous and $\lambda(L \backslash K) < \delta$, so $|\mu|(L \backslash K) < \varepsilon/2$ for any $\mu \in W$.

Now take $t_0 \in L \backslash K$ and define the operator $S$ by

$$S(\nu) = \nu_{|K} + \left( \int_{L \backslash K} \bar{g}_0 d\nu \right) \delta_{t_0} \qquad (\nu \in M(L)),$$

where $\nu_{|K}$ is the restriction of $\nu$ to $K$, $\delta_{t_0}$ is the Dirac measure at $t_0$ and $\bar{g}_0$ denotes complex conjugation ($\bar{g}_0 = g_0$ in the real case).

It is easy to check that $\|S\| \leq 1$. Moreover, for $\mu \in W$ we have

$$\left\| S(\mu) - \mu \right\| \leq \left\| \mu_{|L \backslash K} \right\| + \left| \int_{L \backslash K} \bar{g}_0 d\mu \right| \left\| \delta_{t_0} \right\| \leq 2|\mu|(L \backslash K) < \varepsilon.$$

Finally, if $f_0 \in C_0(L)$ with $\|f_0\| = 1$ agrees with $\bar{g}_0$ on $K$ and satisfies $f_0(t_0) = 1$, then

$$\langle f_0, S(\mu_0) \rangle = \int_K f_0 d\mu_0 + |\mu_0|(L \backslash K) f_0(t_0) = |\mu_0|(K) + |\mu_0|(L \backslash K) = \|\mu_0\|,$$

as required. Note that in case $K = L$ the lemma is trivial; we simply take for $S$ the identity operator and $f_0 = \bar{g}_0$. $\square$

As a consequence of the above lemma, the result in [**17**, Theorem B] can be extended to our slightly more general context, with exactly the same proof.

THEOREM 2. *Every weakly compact operator from $C_0(L)$ into an arbitrary (real or complex) Banach space can be approximated by norm attaining weakly compact operators.*

Now we intend to apply the preceding results to obtain the denseness of norm attaining bilinear forms on spaces of continuous functions. We shall use the following notation. Given Banach spaces $X$ and $Y$ and a natural number $N$, we denote by $\mathcal{L}(^N X, Y)$ the space of all continuous $N$-linear mappings from $X^N$ into $Y$. We say that $\varphi \in \mathcal{L}(^N X, Y)$ attains its norm if there is $x_0 \in (B_X)^N$ (the cartesian product of $N$-times the closed unit ball of $X$) such that

$$\|\varphi(x_0)\| = \|\varphi\| := \sup\{\|\varphi(x)\| : x \in (B_X)^N\},$$

and we denote by $\mathcal{NA}(^N X, Y)$ the set of norm attaining $N$-linear mappings. When $Y$ is the scalar field, we simply omit it. Thus $\mathcal{L}(^N X)$ is the Banach space of all continuous $N$-linear forms on $X$ and $\mathcal{NA}(^N X)$ is the subset of norm attaining forms. Recall that $\mathcal{L}(^{N+1} X)$ can be identified with the space $\mathcal{L}(X, \mathcal{L}(^N X))$ of all bounded linear operators from $X$ into $\mathcal{L}(^N X)$, the $(N+1)-$ linear form $\varphi$ corresponding to the operator $\tilde{\varphi}$ given by

$$[\tilde{\varphi}(x)](x_1, x_2, \ldots, x_N) = \varphi(x, x_1, x_2, \ldots, x_N) \qquad (x, x_1, x_2, \ldots, x_N \in X).$$

It is clear that $\tilde{\varphi}$ attains its norm if $\varphi$ does, but the converse is far from true, even in the case $N = 1$ (see [**10**]).

THEOREM 3. *Every continuous bilinear form on $C_0(L)$ can be approximated by norm attaining bilinear forms.*

*Proof.* Let $\varphi \in \mathcal{L}(^2 C_0(L))$ and $\varepsilon > 0$ be given. Since every bounded linear operator from $C_0(L)$ into $C_0(L)^*$ is weakly compact [**11**], we can use the above theorem to find $\psi \in \mathcal{L}(^2 C_0(L))$ such that $\|\psi - \varphi\| < \varepsilon/2$ and the operator $\tilde{\psi}$ attains its norm, that is $\|\tilde{\psi}\| = \|\tilde{\psi}(g_0)\|$ for some $g_0 \in B_{C_0(L)}$. Let $W$ be the closure of $\tilde{\psi}(B_{C_0(L)})$ in $M(L)$. Then $W$ is weakly compact and by Lemma 1 there exists a bounded linear operator $S : M(L) \to M(L)$, $\|S\| = 1$ such that

(1) there is $f_0 \in B_{C_0(L)}$ such that $\|\tilde{\psi}(g_0)\| = \langle f_0, S\tilde{\psi}(g_0)\rangle$,
(2) $\|S\mu - \mu\| < \varepsilon/2$ for all $\mu \in W$.

It follows from (2) that $\|S\tilde{\psi} - \tilde{\psi}\| \leq \varepsilon/2$. Therefore, if $\chi \in \mathcal{L}(^2 C_0(L))$ is such that $\tilde{\chi} = S\tilde{\psi}$, we have

$$\|\chi - \varphi\| = \|S\tilde{\psi} - \tilde{\varphi}\| \leq \|S\tilde{\psi} - \tilde{\psi}\| + \|\tilde{\psi} - \tilde{\varphi}\| < \varepsilon,$$

and

$$|\chi(g_0, f_0)| = \langle f_0, S\tilde{\psi}(g_0)\rangle = \|\tilde{\psi}(g_0)\| = \|\tilde{\psi}\| \geq \|S\tilde{\psi}\| = \|\chi\|,$$

which shows that $\chi$ attains its norm. $\square$

In general, the denseness of norm attaining bilinear forms on a Banach space does not imply the denseness of norm attaining $N$-linear forms for $N > 2$ (see [**13**]). Actually we do not know if the $N$-linear version of the above theorem is also true. Nevertheless, our proof for bilinear forms will work for $N$-linear forms provided that the corresponding operators are weakly compact. We next give a characterization of those locally compact spaces $L$ such that every bounded linear operator from $C_0(L)$ into $\mathcal{L}(^N C_0(L))$ is weakly compact. Recall that $L$ is said to be *scattered* if every nonempty subset of $L$ has a (relatively) isolated point. If $L$ is scattered then $M(L)$ can be identified with $l_1(\Gamma)$ for some set $\Gamma$ and $C_0(L)$ is an Asplund space (see [**7**, Lemma VI.8.1]). On the other hand, if $L$ is not scattered, then $C_0(L)$ contains $l_1$. Part of the following proposition may be known but we could not find a direct reference, so we indicate a proof.

PROPOSITION 4. *The following statements are equivalent.*

(1) *For some $N \geq 2$, every bounded linear operator from $C_0(L)$ into $\mathcal{L}(^N C_0(L))$ is weakly compact.*

(2) *Every bounded linear operator from $C_0(L)$ into $\mathcal{L}(^2 C_0(L))$ is weakly compact.*

(3) *$L$ is scattered.*

(4) *For all $N \in \mathbb{N}$, every bounded linear operator from $C_0(L)$ into $\mathcal{L}(^N C_0(L))$ is compact.*

*Proof.* (1)$\Rightarrow$(2). This follows from the fact that $\mathcal{L}(^2 X)$ can be isometrically embedded into $\mathcal{L}(^N X)$ for any Banach space $X$ and any $N \geq 2$.

(2)$\Rightarrow$(3). If $L$ is infinite (otherwise there is nothing to prove) the identity operator from $c_0$ into $l_\infty$ factors through $C_0(L)$. Hence there is a (bounded linear) operator from $C_0(L)$ into $l_\infty$ which is not weakly compact and (2) implies that $\mathcal{L}(^2 C_0(L)) = \mathcal{L}(C_0(L), M(L))$ cannot contain (an isomorphic copy of) $l_\infty$. Then we can apply a result by G. Emmanuele [**9**, theorem 4] to obtain that every operator from $C_0(L)$ into $M(L)$ is compact. If follows that $C_0(L)$ cannot contain $l_1$ (see [**12**, Corollary 5], for example) so $L$ is scattered.

(3)$\Rightarrow$(4). If $L$ is scattered, $C_0(L)^* = l_1(\Gamma)$ has the Schur property. Hence every operator from $C_0(L)$ into it is compact, which is (4) for $N = 1$. Note also that $C_0(L)^*$ does not contain $c_0$. Assume by induction that $\mathcal{L}(^N C_0(L))$ does not contain $c_0$ and that every operator from $C_0(L)$ into $\mathcal{L}(^N C_0(L))$ is compact. By a theorem of N. Kalton [**14**, Theorem 4], $\mathcal{L}(^{N+1} C_0(L)) = \mathcal{L}(C_0(L), \mathcal{L}(^N C_0(L)))$ does not contain $l_\infty$ and, being a dual space, it cannot contain $c_0$. Now we can use a theorem of Pelczynski (see [**8**, Theorem VI.15]) to obtain that every operator from $C_0(L)$ into $\mathcal{L}(^{N+1} C_0(L))$ is weakly compact. By the Brace–Grothendieck Theorem (see [**8**, pp. 177]) every such operator takes weakly Cauchy sequences into norm convergent sequences. Since $C_0(L)$ does not contain $l_1$, (4) follows from Rosenthal's $l_1$-theorem.

(4)$\Rightarrow$(1). This is trivial. □

For $t_1, t_2, \ldots, t_N \in L$, consider the $N$-linear form

$$[\delta_{t_1} \otimes \delta_{t_2} \otimes \ldots \otimes \delta_{t_N}](f_1, f_2, \ldots, f_N) = \prod_{k=1}^{N} f_k(t_k)$$

and let $\mathcal{F}$ be the linear subspace of $\mathcal{L}(^N C_0(L))$ generated by these $N$-linear forms. An easy compactness argument shows that $\mathcal{F} \subseteq \mathcal{NA}(^N C_0(L))$. Now, if $L$ is scattered, it follows easily from assertion (4) in the above proposition that $\mathcal{F}$ is dense in $\mathcal{L}(^N C_0(L))$. Therefore, we have

COROLLARY 5. *If $L$ is scattered, $\mathcal{NA}(^N C_0(L))$ is dense in $\mathcal{L}(^N C_0(L))$ for every $N \in \mathbb{N}$.*

The above results also give some information on norm attaining vector-valued multilinear mappings on spaces of continuous functions. Let us recall the definition of the so-called property ($\beta$), which was introduced by J. Lindenstrauss [**15**] as a sufficient condition on the range space for the denseness of norm attaining operators. A Banach space $Y$ has property ($\beta$) if there are a set $\{(y_\alpha, y_\alpha^*) : \alpha \in \Gamma\} \subset Y \times Y^*$ and a constant $\lambda < 1$ such that

(i) $\|y_\alpha\| = \|y_\alpha^*\| = y_\alpha^*(y_\alpha) = 1$ for every $\alpha \in \Gamma$,
(ii) $|y_\alpha^*(y_\beta)| \leq \lambda$ for $\alpha, \beta \in \Gamma, \alpha \neq \beta$.
(iii) $\|y\| = \sup\{|y_\alpha^*(y)| : \alpha \in \Gamma\}$ for every $y \in Y$.

It is easy to show that $C_0(L)$ has property ($\beta$) if and only if $L$ has a dense set of isolated points, and this is clearly the case when $L$ is scattered. Moreover, it was shown in [**6**, Theorem 2.1] that $\mathcal{NA}(^N X, Y)$ is dense in $\mathcal{L}(^N X, Y)$ whenever $\mathcal{NA}(^N X)$ is dense in $\mathcal{L}(^N X)$ and $Y$ has property ($\beta$). Therefore, from Theorem 3 and Corollary 5 we obtain.

COROLLARY 6. *If $L$ has a dense set of isolated points, then $\mathcal{NA}(^2 C_0(L), C_0(L))$ is dense in $\mathcal{L}(^2 C_0(L), C_0(L))$. If $L$ is scattered, then $\mathcal{NA}(^N C_0(L), C_0(L))$ is dense in $\mathcal{L}(^N C_0(L), C_0(L))$ for every $N \in \mathbb{N}$.*

We conclude this paper with some observations on the numerical radius of a multilinear mapping. For a wide discussion of the linear case we refer the reader to the books by F. F. Bonsall and J. Duncan ([**3**], [**4**]). In [**6**] the numerical radius $v(\varphi)$ of an $N$-linear mapping $\varphi \in \mathcal{L}(^N X, X)$ is defined by

$$v(\varphi) = \sup\{|x^*(\varphi(x_1, x_2, \dots, x_N))| : \|x^*\| = \|x_k\| = x^*(x_k) = 1, \forall k = 1, \dots, N\},$$

and we say that $\varphi$ attains its numerical radius if the above supremum is actually a maximum. We will next show that in the case $X = C_0(L)$ there is no difference between the norm and the numerical radius (see [**4**, Theorem 32.5] for the linear case). We need the following elementary fact.

LEMMA 7. *Given $t_0 \in L$, any function $f \in B_{C_0(L)}$ can be written as*

$$f = \alpha g + (1 - \alpha)h$$

*where $0 \leq \alpha \leq 1, g, h \in B_{C_0(L)}$ and $|g(t_0)| = |h(t_0)| = 1$.*

*Proof.* Assume that $s_0 := f(t_0)$ satisfies $|s_0| < 1$ (otherwise there is nothing to prove). Consider the unit ball $\mathbb{D}$ in the scalar field, and let $p, q : \mathbb{D} \to \mathbb{D}$ be continuous functions satisfying that $|p(s_0)| = |q(s_0)| = 1$ and $s = \alpha p(s) + (1 - \alpha)q(s)$ for every $s \in \mathbb{D}$ and some fixed $\alpha$ with $0 < \alpha < 1$. The construction of $p$ and $q$ is easy; note that $\alpha$ is uniquely determined by $s_0$ in the real case, while one can take $\alpha = 1/2$ independently of $s_0$ in the complex case. Now let $\rho : L \to [0, 1]$ be a continuous function with compact support and $\rho(t_0) = 1$. Then it is enough to take $g(t) = \rho(t)p(f(t)) + (1 - \rho(t))f(t)$ and $h(t) = \rho(t)q(f(t)) + (1 - \rho(t))f(t)$ for every $t \in L$. □

PROPOSITION 8. *For every $\varphi \in \mathcal{L}(^N C_0(L), C_0(L))$, we have $v(\varphi) = \|\varphi\|$. Moreover, $\varphi$ attains its numerical radius if and only if $\varphi$ attains its norm.*

*Proof.* Assume without loss of generality that $\|\varphi\| = 1$, fix $\varepsilon > 0$ and let $f_1, f_2, \ldots, f_N \in B_{C_0(L)}$ be such that $\|F\| \geq 1 - \varepsilon$ where $F := \varphi(f_1, f_2, \ldots, f_N)$. Now choose $t_0 \in L$ such that $|F(t_0)| \geq 1 - \varepsilon$ and apply the above lemma to decompose each function $f_k$ in the form

$$f_k = \alpha_k g_k + (1 - \alpha_k) h_k$$

where $0 \leq \alpha_k \leq 1$, $g_k, h_k \in B_{C_0(L)}$ and $|g_k(t_0)| = 1$ for $k = 1, 2, \ldots, N$. Then we can use the $N$-linearity of $\varphi$ to write $F$ as a convex combination of the form

$$F = \varphi(f_1, f_2, \ldots, f_N) = \sum_{j=1}^{2^N} \beta_j \varphi\left(f_1^j, f_2^j, \ldots, f_N^j\right)$$

where $f_k^j \in B_{C_0(L)}$ and $|f_k^j(t_0)| = 1$ for all $k = 1, 2, \ldots, N$ and $j = 1, 2, \ldots, 2^N$. We must clearly have $\left|\left[\varphi\left(f_1^j, f_2^j, \ldots, f_N^j\right)\right](t_0)\right| \geq 1 - \varepsilon$ for some $j$. Thus, we may assume from the very beginning that $|f_k(t_0)| = 1$ for $k = 1, 2, \ldots, N$. By rotations, we may actually arrange that $f_k(t_0) = 1$. Then

$$v(\varphi) \geq |\delta_{t_0}(\varphi(f_1, f_2, \ldots, f_N))| \geq 1 - \varepsilon,$$

which shows that $v(\varphi) \geq \|\varphi\|$, the reverse inequality being clear.

For the second part of the statement, it is obvious that $\varphi$ attains its norm if it attains its numerical radius. For the converse, just note that, if $\varphi$ attains its norm, the above argument works with $\varepsilon = 0$. $\square$

## REFERENCES


**1.** M. D. Acosta, F. J. Aguirre and R. Paya, There is no bilinear Bishop-Phelps theorem, *Israel J. Math.* **93** (1996), 221–227.

**2.** R. M. Aron, C. Finet and E. Werner, Norm-attaining n-linear forms and the Radon-Nikodym property, in *Proc. 2nd Conf. on Function Spaces* (K. Jarosz, ed. ), L. N. Pure and Appl. Math., Marcel Dekker (1995), pp. 19–28.

**3.** F. F. Bonsall and J. Duncan, *Numerical ranges of operators on normed spaces and of elements of normed algebras*, London Math. Soc. Lect. Notes Ser. **2** (Cambridge Univ. Press, 1971).

**4.** F. F. Bonsall and J. Duncan, *Numerical ranges II*, London Math. Soc. Lect. Notes Ser. **10** (Cambridge Univ. Press, 1973).

**5.** Y. S. Choi, Norm attaining bilinear forms on $L^1[0, 1]$, *J. Math. Anal. Appl.* **211** (1997), 295–300.

**6.** Y. S. Choi and S. G. Kim, Norm or numerical radius attaining multilinear mappings and polynomials, *J. London Math. Soc.* **54** (1996), 135–147.

**7.** R. Deville, G. Godefroy and V. Zizler, *Smoothness and renormings in Banach spaces*, Pitman Monographs and Surveys in Pure and Appl. Math. **64** (Longman Sc. & Tech., Essex, 1993).

**8.** J. Diestel and J. J. Uhl Jr., *Vector measures*, Math. Surveys **15** (Amer. Math. Soc., Providence R.I. 1977).

**9.** G. Emmanuele, A remark on the containment of $c_0$ in spaces of compact operators, *Math. Proc. Camb. Phil. Soc.* **111** (1992), 331–335.

**10.** C. Finet and R. Payá, Norm attaining operators from $L_1$ into $L_\infty$, *Israel J. Math.* (to appear).

**11.** A. Grothendieck, Sur les applications lineaires faiblement compactes d'espaces du type C(K), *Canad. J. Math.* **5** (1953), 129–173.

**12.** J. M. Gutierrez, Weakly continuous functions on Banach spaces not containing $l_1$, *Proc. Amer. Math. Soc.* **1993** no. 1 (1993), 147–152.

**13.** M. Jiménez-Sevilla and R. Payá, Norm attaining multilinear forms and polynomials on preduals of Lorentz sequence spaces, *Studia Math.* **127** (1998), 99–112.

14. N. Kalton, Spaces of compact operators, *Math. Ann.* **208** (1974), 267–278.

15. J. Lindenstrauss, On operators which attain their norm, *Israel J. Math.* **1** (1963), 139–148.

16. W. Rudin, *Real and complex analysis* (McGraw-Hill, New York, 1966).

17. W. Schachermayer, Norm attaining operators on some classical Banach spaces, *Pac. J. Math.* **105**, no. 2 (1983) 427–438.

J. Alaminos and R. Payá
Departamento de Análisis Matemático
Facultad de Ciencias
Universidad de Granada
18071 Granada
Spain
E-mail: alaminos@goliat.ugr.es and rpaya@goliat.ugr.es

Y. S. Choi and S. G. Kim
Department of Mathematics
Pohang University of Science and Technology
Pohang
Korea
E-mail: mathchoi@euclid.postech.ac.kr and skim@euclid.postech.ac.kr